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MEASURE AND INTEGRATION ON NONORIENTABLE KLEIN SURFACES

Ilie Barza[1 §], Dorin Ghisa[2]
[1]Karlstad University
Dept. of Engineering Sciences, Physics and Mathematics
S-651 88-Karlstad, SWEDEN
e-mail: Ilie.Barza@kau.se
[2]York University, Glendon College
Dept. of Mathematics
2275-Bayview Avenue, Toronto, CANADA, M4N 3M6
e-mail: dghisa@yorku.ca

Abstract: Potential theory on nonorientable surfaces should appear as a natural extension of the classical potential theory on Riemann surfaces. In this paper a step towards such an extention is made. A relationship between nonorientable Klein surfaces and their orientable double covers is established, in terms of measures they support, as well as of the integrals with respect to those measures.

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[§]Correspondence author

1. Introduction

Klein surfaces are the most general two-dimentional real manifolds which support harmonic functions, and consequently, on which a potential theory can be built. There is a well known relationship between Klein surfaces and (bordered, or border free) Riemann surfaces (see for example [6], [7]). Namely, if **R** is such a Riemann surface on which a fixed point free antianalytic involution **h** (called **symmetry** in the sense of Klein) exists, then $\mathbf{S}=\mathbf{R}/\langle\mathbf{h}\rangle$, where $\langle\ \mathbf{h}\ \rangle$ is the two element group generated by **h**, carries a unique dianalytic structure, which makes the canonical projection $\pi:\mathbf{R}\rightarrow\mathbf{S}$ a dianalytic function. Vice-versa, to every nonorientable Klein surface **S** a unique (up to a conformal isomorphism) **symmetric Riemann surface** (**R**,**h**) can be associated, such that $\mathbf{R}/\ \langle\mathbf{h}\ \rangle$ and **S** are dianalytically equivalent. (**R**,**h**) is called **the orientable double cover of S**.The pair (**R**,**h**) will be called a symmetric Riemann surface **in the sense of Klein**, in order to distinguish it from other types of symmetric surfaces.

Potentials on Riemann surfaces are generated by measures (see [9], [10]). This makes interesting the question of the relationship between **R** and **S** in terms of the measures they support, as well as of the integrals with respect to those measures.

The model we are proposing is of a remarkable simplicity. In order to evaluate it properly, one should compare it with the standard model of G. de Rham based on differential forms of odd type applicable to analysis on arbitrary nonorientable manifolds (see G. de Rham, *Differentiable Manifolds,* Springer-Verlag, 1984). Here we preferred to use a direct approach, instead of particularizing the respective model to nonorientable Klein surfaces.

We obtained a simple formula (see Theorem 5) that gives a satisfactory answer to this question. The case of Lebesgue measure received a detailed treatment (see [12] for its use in the context of Riemann surfaces).

The paper is devided in seven sections. In Section 2 covering partitions of the symmetric Riemann surfaces are defined and a characterization of the symmetric subsets of these surfaces is given (Theorem 3).

In the third section two pull-back operators, which play an indispensable role in the integration theory on nonorientable surfaces, are introduced.The operators of **symmetrisation/antisymmetrisation** of the measures on symmetric Riemann surfaces are also defined. With the help of the pull-back and symmetrisation/antisymmetrisation operators we can give a representation theorem (Theorem 4) for the space of signed Borel measures on nonorientable surfaces.

In the forth section we define the Lebesgue measure on the real projective plane and on the Möbius strip.

In the fifth section we give some more examples of measures on the projective plane and on the Möbius strip. The last example can be generalised, in an obvious way, to arbitarary surfaces by using the Lebesgue measure on Riemann surfaces as defined, for example, by G.Springer (see [12]).

In the sixth section we present the integration theory on nonorientable surfaces in a quite general frame. The main result of this paper is Theorem 5, which gives **the working formula** (6).

2. Covering Partitions

In this paper we deal essentially with signed Borel measures. However, extentions to complex or vector-valued Borel measures are obvious. We also chose to ignore the case of measures on arbitrary σ-algebras of parts of $\mathbf{R}$ and $\mathbf{S}$. The reason for this last choice is that we need to make frequent references to the following two theorems proven in [10], whose more general forms are not available yet.

Theorem 1. *Let $\widetilde{\mu}$ be a Borel measure on $\mathbf{S}$. There is a unique Borel measure μ on $\mathbf{R}$ such that for every Borel set $B\subset\mathbf{R}$ on which π is injective $\mu(B) = \widetilde{\mu}(\pi(B))$.*

Theorem 2. *If $(\mathbf{R},\mathbf{h})$ is the orientable double cover of the nonorientable Klein surface $\mathbf{S}$, then there is a Borel set $A\subset\mathbf{R}$ such that:*

a) $A\cap\mathbf{h}A=\emptyset$;

b) $A\cup\mathbf{h}A=\mathbf{R}$;

c) *The canonical projection $\pi:\mathbf{R}\to\mathbf{S}$ is injective on both A and $\mathbf{h}A$.*

The couple (A, $\mathbf{h}$(A)) is called **a covering partition of R**. Covering partitions will be working tools throughout the paper. However, the final results will be, at any instance, independent on those particular partitions.

Let us denote by $\mathcal{B}(\mathbf{R})$ and $\mathcal{B}(\mathbf{S})$ the σ-algebra of Borel sets of $\mathbf{R}$ (respectively of $\mathbf{S}$) and by $\mathcal{B}^{(s)}(\mathbf{R})$ the **symmetric** Borel sets of $\mathbf{R}$, i.e. the sets X such that X = $\mathbf{h}$(X).

Theorem 3. *(i)*$\mathcal{B}^{(s)}(\boldsymbol{R}) = \{\ Z \cup \boldsymbol{h}(Z) \mid Z \in \mathcal{B}(\boldsymbol{R}),\ Z \cap\ \boldsymbol{h}(Z) = \emptyset\}$;

(ii) If X_1 , $X_2 \in \mathcal{B}^{(s)}(\boldsymbol{R})$ *and* $(A,\ \boldsymbol{h}(A))$ *is a partition of* $\boldsymbol{R}$ *with Borel sets, then the following affirmations are equivalent:*

(j) $X_1 = X_2$;

(jj) $X_1 \cap A = X_2 \cap A$;

(jjj) $X_1 \cap \boldsymbol{h}(A) = X_2 \cap \boldsymbol{h}(A)$.

Proof. (i) If $\mathrm{X} \in \mathcal{B}^{(\mathrm{s})}(\mathbf{R})$, then $\mathrm{X} \in \mathcal{B}(\mathbf{R})$ and $\mathrm{X} = \mathbf{h}(\mathrm{X})$. Consequently $\mathrm{X} = (\mathrm{X} \cap \mathrm{A}) \cup (\mathrm{X} \cap \mathbf{h}(\mathrm{A})) = (\mathrm{X} \cap \mathrm{A}) \cup [\mathbf{h}(\mathrm{X}) \cap \mathrm{h}(\mathrm{A})] = (\mathrm{X} \cap \mathrm{A}) \cup \mathbf{h}(\mathrm{X} \cap \mathrm{A})$ and $(\mathrm{X} \cap \mathrm{A}) \cap\ \mathbf{h}(\mathrm{X} \cap \mathrm{A}) \subseteq \mathrm{A} \cap \mathbf{h}(\mathrm{A}) = \emptyset$. Therefore $\mathrm{X} = \mathrm{Z} \cup \mathbf{h}(\mathrm{Z})$, where $\mathrm{Z} = \mathrm{X} \cap \mathrm{A} \in \mathcal{B}(\mathbf{R})$ and $\mathrm{Z} \cap \mathbf{h}(\mathrm{Z}) = \emptyset$.

Vice-versa, let $\mathrm{X} = \mathrm{Z} \cup \mathbf{h}(\mathrm{Z})$, $\mathrm{Z} \in \mathcal{B}(\mathbf{R})$ and $\mathrm{Z} \cap \mathbf{h}(\mathrm{Z}) = \emptyset$. Since $\mathbf{h}$ is obviously Borel measurable, $\mathbf{h}(\mathrm{Z}) \in \mathcal{B}(\mathbf{R})$ and consequently $\mathrm{X} = \mathrm{Z} \cup \mathbf{h}(\mathrm{Z}) \in \mathcal{B}(\mathbf{R})$. Moreover, since $\mathbf{h}$ is an involution, $\mathbf{h}(\mathrm{X}) = \mathbf{h}(\mathrm{Z} \cup \mathbf{h}(\mathrm{Z})) = \mathbf{h}(\mathrm{Z}) \cup \mathrm{Z} = \mathrm{X}$.

(ii) Suppose X_1, $\mathrm{X}_2 \in \mathcal{B}^{(\mathrm{s})}(\mathbf{R})$. The implications (j) $\Longrightarrow$ (jj) , (j) $\Longrightarrow$ (jjj) and (jj) &(jjj) $\Longrightarrow$ (j) are obvious. Let us show that (jj) $\Longrightarrow$ (jjj). We have $\mathrm{X}_1 = \mathbf{h}(\mathrm{X}_1)$, $\mathrm{X}_2 = \mathbf{h}(\mathrm{X}_2)$ and $\mathrm{X}_1 \cap \mathrm{A} = \mathrm{X}_2 \cap \mathrm{A}$. Then $\mathrm{X}_1 \cap \mathbf{h}(\mathrm{A}) = \mathbf{h}(\mathbf{h}(\mathrm{X}_1) \cap \mathrm{A}) = \mathbf{h}(\mathrm{X}_1 \cap \mathrm{A}) = \mathbf{h}(\mathrm{X}_2 \cap \mathrm{A}) = \mathbf{h}(\mathrm{X}_2) \cap \mathbf{h}(\mathrm{A}) = \mathrm{X}_2 \cap \mathbf{h}(\mathrm{A})$. Analogously it can be shown that (jjj) $\Longrightarrow$ (jj), which completely proves the theorem. □

3. The Pull-Back Operators $\mathbf{h}^*$ and π^*

Let us denote by $Mes(\mathcal{B}(\mathbf{R}))$ and $Mes(\mathcal{B}(\mathbf{S}))$ the sets of finite signed Borel measures on $\mathbf{R}$, respectively on $\mathbf{S}$. The elements of $Mes(\mathcal{B}(\mathbf{R}))$ and $Mes(\mathcal{B}(\mathbf{S}))$ are σ-additive bounded functions defined on $\mathcal{B}(\mathbf{R})$, respectively on $\mathcal{B}(\mathbf{S})$. Thus $\mathrm{Mes}(\mathcal{B}(\mathbf{R}))$ and $\mathrm{Mes}(\mathcal{B}(\mathbf{S}))$ endowed with the addition defined set-wise and the usual multiplication with scalars, are vector spaces. If $\mu \in \mathrm{Mes}(\mathcal{B}(\mathbf{R}))$, then $\mathbf{h}^*(\mu)$ is the signed Borel measure on $\mathbf{R}$ defined by

$$(\mathbf{h}^*(\mu))(Z) = \mu(\mathbf{h}(Z))$$

for every $\mathrm{Z} \in \mathcal{B}(\mathbf{R})$.

One can easily check that $\mathbf{h}^*$: $Mes(\mathcal{B}(\mathbf{R})) \rightarrow Mes(\mathcal{B}(\mathbf{R}))$ is a linear involution.

Let (A, $\mathbf{h}$(A)) be a covering partition of $\mathbf{R}$. The Theorems 1 and 2 imply that to every signed Borel measure $\widetilde{\mu}$ on $\mathbf{S}$ corresponds a unique $\mathbf{h}$-invariant signed Borel measure μ on $\mathbf{R}$ such that for B$\in \mathcal{B}(\mathbf{R})$,

$$\mu(B)=\mu(B\cap A)+\mu(B\cap \mathbf{h}(A))=\widetilde{\mu}(\pi(B\cap A))+\widetilde{\mu}(\pi(B\cap \mathbf{h}(A))).$$

This equality allows us to define the pull-back operator

$$\pi^{*}:Mes(\mathcal{B}(\mathbf{S}))\rightarrow Mes(\mathcal{B}(\mathbf{R}))$$

by

$$(\pi^{*}\widetilde{\mu})(Z):=\mu(Z):=\widetilde{\mu}(\pi(Z\cap A))+\widetilde{\mu}(\pi(Z\cap \mathbf{h}(A)) \tag{1}$$

for every $Z\in\mathcal{B}(\mathbf{R})$.

The following concepts are essential to the measure theory on nonorientable surfaces.

Definition 1. A signed measure $\mu\in Mes(\mathcal{B}(\mathbf{R}))$ is called $\mathbf{h}$-invariant (or $\mathbf{h}$-antiinvariant) iff $\mu(\mathbf{h}Z)=\mu(Z)$, (respectively $\mu(\mathbf{h}Z)=-\mu(Z)$) for every mesurable set $Z\subseteq\mathbf{R}$.

The measure μ appearing in Theorem 1 is obviously $\mathbf{h}$-invariant. We introduce the following notations:

$$\begin{cases} Mes^{(\mathrm{s})}(\mathcal{B}(\mathbf{R})):=\{\mu\in Mes(\mathcal{B}(\mathbf{R}))|\ \ \mathbf{h}^{*}(\mu)=\mu\} \\ Mes^{(\mathrm{a})}(\mathcal{B}(\mathbf{R})):=\{\mu\in Mes(\mathcal{B}(\mathbf{R}))|\ \ \mathbf{h}^{*}(\mu)=-\mu\}. \end{cases}$$

It is obvious, that $Mes^{(\mathrm{s})}(\mathcal{B}(\mathbf{R}))$ and $Mes^{(\mathrm{a})}(\mathcal{B}(\mathbf{R}))$ are linear subspaces of $Mes(\mathcal{B}(\mathbf{R}))$ and that $Mes^{(\mathrm{s})}(\mathcal{B}(\mathbf{R}))\cap Mes^{(\mathrm{a})}(\mathcal{B}(\mathbf{R}))=\{0\}$, where 0 is the null measure on **R.**

Theorem 4. *(i)* $Mes(\mathcal{B}(\boldsymbol{R}))=Mes^{(s)}(\mathcal{B}(\boldsymbol{R}))\bigoplus Mes^{(a)}(\mathcal{B}(\boldsymbol{R}))$;

(ii) $\pi^{*}(Mes(\mathcal{B}(\mathbf{S})))=Mes^{(s)}(\mathcal{B}(\boldsymbol{R}))$ *and the co-restriction* $\pi^{*}:Mes(\mathcal{B}(\mathbf{S}))\rightarrow Mes^{(s)}(\mathcal{B}(\boldsymbol{R}))$ *is an isomorphism of vector spaces.*

Proof. Let us define the symmetrisation operator (or the operator of **h**-invariance) $\mathcal{S}$ and the antisymmetrisation operator (or the operator of **h**-antiinvariance) $\mathcal{A}$ on $Mes(\mathcal{B}(\mathbf{R}))$ by:

$$\begin{cases} \mathcal{S}(\mu) = \frac{1}{2}(\mu + \mathbf{h}^*(\mu)) := \ \mu_{\mathrm{s}} \\ \\ \mathcal{A}(\mu) = \frac{1}{2}(\mu - \mathbf{h}^*(\mu)) := \ \mu_{\mathrm{a}}. \end{cases} \tag{2}$$

In other words,

$$\begin{cases} \mathcal{S} = \frac{1}{2}(\mathcal{I} + \mathbf{h}^*) \\ \\ \mathcal{A} = \frac{1}{2}(\mathcal{I} - \mathbf{h}^*)\,, \end{cases}$$

where $\mathcal{I}$ is the identity operator on $Mes(\mathcal{B}(\mathbf{R}))$. It can be easily checked that:

(j). $\mathcal{S}$ and $\mathcal{A}$ are linear operators,

$$\mathcal{S} \circ \mathcal{S} = \mathcal{S}, \quad \mathcal{A} \circ \mathcal{A} = \mathcal{A}, \quad \mathcal{S} \circ \mathcal{A} = \mathcal{A} \circ \mathcal{S} = 0$$

(the null operator), $\mathcal{S} + \mathcal{A} = \mathcal{I}$ and $\mathcal{S}$ - $\mathcal{A}$ = $\mathbf{h}^*$= an involution.

Thus ($\mathcal{S}, \mathcal{A}$) is **a pair of orthogonal projectors** of $Mes(\mathcal{B}(\mathbf{R}))$.

(jj). $\mu \in Mes^{(\mathrm{s})}(\mathcal{B}(\mathbf{R}))$ iff $\mathcal{S}$ (μ) $=\mu$ and $\mu \in Mes^{(\mathrm{a})}(\mathcal{B}(\mathbf{R}))$ iff $\mathcal{A}$ (μ) $=\mu$.

(jjj). $\mathcal{S}(Mes(\mathcal{B}(\mathbf{R}))) = Mes^{(\mathrm{s})}(\mathcal{B}(\mathbf{R}))$, $\mathcal{A}(Mes(\mathcal{B}(\mathbf{R})))$ $= Mes^{(\mathrm{a})}(\mathcal{B}(\mathbf{R}))$.

These last equalities and $\mathcal{S} + \mathcal{A} = \mathcal{I}$ completely proves (i).

To prove (ii), let us take $\widetilde{\mu} \in Mes(\mathcal{B}(\mathbf{S}))$ and let Z be an arbitrary element of $\mathcal{B}(\mathbf{R})$. For a covering partition (A, **h**(A)) of **R**, taking into account the fact that $\pi \circ \mathbf{h} = \pi$, we have:

$$\begin{aligned}(\pi^*(\widetilde{\mu}))(\mathbf{h}(Z)) &= \widetilde{\mu}(\pi(\mathbf{h}(Z) \cap A)) + \widetilde{\mu}(\pi(\mathbf{h}(Z) \cap \mathbf{h}(A)) \\ &= \widetilde{\mu}(\pi(Z \cap \mathbf{h}(A)) + \widetilde{\mu}(\pi(\mathbf{h}(Z \cap A)) \\ &= \widetilde{\mu}(\pi(Z \cap \mathbf{h}(A)) + \widetilde{\mu}(\pi(Z \cap A) = (\pi^*(\widetilde{\mu}))(Z),\end{aligned}$$

which shows that $\mathbf{h}*(\pi^*\ (\widetilde{\mu}))=\pi^*(\widetilde{\mu})$, or $\pi^*(\ \widetilde{\mu}) \in Mes^{(\mathrm{s})}(\mathcal{B}(\mathbf{R}))$. Hence $\pi^*(Mes(\mathcal{B}(\mathbf{S}))) \subseteq Mes^{(\mathrm{s})}(\mathcal{B}(\mathbf{R}))$.

We notice that we have implicitly proved that the following diagram commutes:

$$\begin{array}{ccc} & Mes(\mathcal{B}(\mathbf{S})) & \\ \pi^* \swarrow & & \searrow \pi^* \\ Mes(\mathcal{B}(\mathbf{R})) & \xrightarrow{\mathbf{h}^*} & Mes(\mathcal{B}(\mathbf{R})). \end{array}$$

In the proof of the injectivity of π^* we need the following consequence of (1):

If B is an $\mathbf{h}$-invariant Borel subset of $\mathbf{R}$ then

$$\mu(B) = 2\widetilde{\mu}(\pi(B \cap A)) = 2\widetilde{\mu}(\pi(B \cap \mathbf{h}A)). \tag{3}$$

Let $\mu \in Mes^{(\mathrm{s})}(\mathcal{B}(\mathbf{R}))$. Thus $\mu(\mathrm{Z})=\mu(\mathbf{h}(\mathrm{Z}))$ for every $\mathrm{Z}\in \mathcal{B}(\mathbf{R})$. Since $\pi(\mathrm{Z})=\pi(\mathbf{h}(\mathrm{Z}))$, we can define $\widetilde{\mu} \in Mes(\mathcal{B}(\mathbf{S}))$ by

$$\widetilde{\mu}(X) := \mu(\pi^{-1}(X) \cap A) = \mu(\pi^{-1}(X) \cap \mathbf{h}(A)), \tag{4}$$

for every $\mathrm{X} \in \mathcal{B}(\mathbf{S})$. Obviously, $\widetilde{\mu} \in Mes(\mathcal{B}(\mathbf{S}))$, and for $\mathrm{Z}\in \mathcal{B}(\mathbf{R})$ we have:

$$\begin{aligned}(\pi * (\widetilde{\mu}))(Z) &= \widetilde{\mu}(\pi(Z \cap A) + \widetilde{\mu}(\pi(Z \cap \mathbf{h}(A)) \\ &= \mu(Z \cap A) + \mu(Z \cap \mathbf{h}(A)) = \mu(Z) i.e. \pi * (\widetilde{\mu}) = \mu.\end{aligned}$$

Thus π^* is surjective.

Let now $\widetilde{\mu}_1$, $\widetilde{\mu}_2 \in Mes(\mathcal{B}(\mathbf{S}))$, $\widetilde{\mu}_1 \neq \widetilde{\mu}_2$ and let us denote $\mu_{\mathrm{k}}= \pi^*(\ \widetilde{\mu}_{\mathrm{k}})$ for $k = 1, 2$. There is $\mathrm{X}\in \mathcal{B}(\mathbf{S})$ such that $\widetilde{\mu}_1(\mathrm{X}) \neq \widetilde{\mu}_2(\mathrm{X})$. Then, since $\pi(A) = \pi(\mathbf{h}A) = \mathbf{S}$:

$$\begin{aligned}\mu_1(\pi^{-1}(X)) &= (\pi^*(\widetilde{\mu}_1))(\pi^{-1}(X)) = \widetilde{\mu}_1(\pi(\pi^{-1}(X) \cap A)) \\ &+ \widetilde{\mu}_1(\pi(\pi^{-1}(X) \cap \mathbf{h}A)) = \widetilde{\mu}_1(X \cap \pi(A)) + \widetilde{\mu}_1(X \cap \pi(\mathbf{h}(A))) \\ &= 2\widetilde{\mu}_1(X) \neq 2\widetilde{\mu}_2(X) = \mu_2(\pi^{-1}(X)).\end{aligned}$$

Thus $\widetilde{\mu}_1 \neq \widetilde{\mu}_2$ and, taking into account the linearity of π^*, this completes the proof of the theorem. □

According to this theorem, every finite signed Borel measure μ on $\mathbf{R}$ can be decomposed into the sum of an $\mathbf{h}$-invariant signed Borel measure μ_{s} and of an $\mathbf{h}$-antiinvariant signed Borel measure μ_{a}. Only the component μ_{s} "projects" into a signed Borel measure on $\mathbf{S}$, while μ_{a} is "ignored" by this projection. Particularly, every $\mathbf{h}$-antiinvariant finite signed Borel measure on $\mathbf{R}$ projects into the null measure on $\mathbf{S}$.

4. Lebesgue Measure on the Projective Plane and on the Möbius Srip

The conformal metric

$$d\sigma := \frac{1}{2}\left[1+\frac{1}{|z|^2}\right]|dz|$$

induces on $A_{\mathrm{r}} := \{z \in\mathbf{C}|\ 1/r \leq |z| \leq r\}$ a conformal structure making it a (bordered) Riemann surface (see, for example [1], [2] and [7]).

The mapping **h:**z→ $-1/\overline{z}$ of $\widehat{\mathbf{C}}$ on itself is a fixed point free antianalytic involution of $\widehat{\mathbf{C}}$ and its restriction to A_{r} is an antianalytic involution of A_{r}. Consequently:

$$\mathbf{P}^2 := \widehat{\mathbf{C}}\ /\langle\mathbf{h}\rangle \quad \text{and} \quad M_{\mathrm{r}} := A_{\mathrm{r}}/\langle\mathbf{h}\rangle$$

endowed with the dianalytic structures induced by their corresponding canonical projections, are Klein surfaces. Topologically, $\mathbf{P}^2$ is the real projective plane and M_{r} is a Möbius strip (see [8] and [11]).

The Lebesgue measure λ on $\mathbf{C}$ is not $\mathbf{h}$-invariant. Indeed, for the annuli

$$A_1 := \{z \in \mathbf{C}|1/r \leq |z| \leq 1\} \quad \text{and} \quad A_2 := \{z \in \mathbf{C}|1 \leq |z| \leq r\},$$

$\mathbf{h}(A_1) = A_2$, but

$$\lambda(A_1) = \pi(1-1/r^2) \neq \pi(r^2-1) = \lambda(A_2).$$

One defines $\lambda(\{\infty\}) = 0$. Thus λ becomes a (positive) measure on $\widehat{\mathbf{C}}$ and $\lambda(Z) = \lambda(Z\cap\mathbf{C})$ for every Borel subset of $\widehat{\mathbf{C}}$. To find an explicit form for the component λ_{s} of λ, given by the Theorem 4, let us use the following notations:

$$\mathrm{w} = u+iv = \mathbf{h}z = -1/\overline{z}, \quad z = x+iy. \tag{5}$$

Let us denote by $J_{\mathrm{w}}(x,y)$ the Jacobian of the transformation (5). Then for every Lebesgue measurable set $Z \subseteq\mathbf{C}$, we have:

$$\lambda(\mathbf{h}Z) = \iint\limits_{\mathbf{h}\mathrm{Z}} dudv = \iint\limits_{\mathrm{Z}} |J_{\mathrm{w}}(x,y)|\,dxdy = \iint\limits_{\mathrm{Z}} 1/\,|z|^4\,dxdy$$

Thus for every Lebesgue measurable set $Z \subseteq \widehat{\mathbf{C}}$, we have:

$$\lambda_{\mathrm{s}}(Z) = \frac{1}{2}\left[\lambda(Z) + \lambda(\mathbf{h}Z)\right] = \frac{1}{2}\left[\iint_{Z} dx\ dy + \iint_{Z} 1/\left|z\right|^4\ dx\ dy\right]$$
$$= \frac{1}{2}\iint_{Z}\left[1 + 1/\left|z\right|^4\right] dxdy,$$

which shows that λ_{s} is absolutely continuous with respect to λ and has the Radon-Nikodym derivative:

$$\left[\frac{d\lambda_{\mathrm{s}}}{d\lambda}\right](z) = \frac{1}{2}\left[1 + \frac{1}{|z|^4}\right].$$

We call the measure $\widetilde{\lambda} := (\pi^*)^{-1}(\lambda_{\mathrm{s}})$, where $\pi : \widehat{\mathbf{C}} \to \mathbf{P}^2$ is the canonical projection, **the Lebesgue measure on the projective plane**, and its restriction to M_{r}, **the Lebesgue measure on the Möbius strip**. The measure $\widetilde{\lambda}$ is defined on the σ-algebra $\mathcal{M}(\mathbf{P}^2)$ of subsets of $\mathbf{P}^2$ which are images by π of Lebesgue measurable subsets of $\widehat{\mathbf{C}}$. The explicit form of $\widetilde{\lambda}$ is given, according to (3) and (4), by the following formula:

$$\widetilde{\lambda}(\widetilde{Z}) = \frac{1}{2}\lambda_{\mathrm{s}}(\pi^{-1}(\widetilde{Z})) = \frac{1}{4}\iint_{Z}\left[1 + \frac{1}{|z|^4}\right] dxdy,$$

where $Z = \pi^{-1}(\widetilde{Z})$ and $\widetilde{Z} \in \mathcal{M}(\mathbf{P}^2)$. Particularly, if $\widetilde{Z} = M_{\mathrm{r}}$, we have:

$$\widetilde{\lambda}(\mathrm{M_r}) = \frac{1}{4}\iint_{\mathrm{A_r}}\left[1 + \frac{1}{|z|^4}\right] dxdy = \frac{1}{4}\int_{0}^{2\pi}\int_{1/\mathrm{r}}^{\mathrm{r}}(1 + 1/\rho^4)\rho\ d\rho\ d\theta$$
$$= \frac{\pi}{2}\left[r^2 - \frac{1}{r^2}\right] = \frac{1}{2}\lambda(A_{\mathrm{r}}).$$

Remark. For Borel subsets $\mathrm{Z} \subseteq \widehat{\mathbf{C}}$ with $\lambda(Z) = \lambda(\mathbf{h}Z) = +\infty$, the second equality in (2) leads to an indetermination. However, there will be no harm in stating that in such a case $\lambda(Z) - \lambda(\mathbf{h}Z) = 0$, in order to include also the Lebesgue measure in the domain of the operator $\mathcal{A}$ and to extend the use of the symmetrisation technique to $\mathbf{P}^2$.

5. Other Examples of Measures on the Projectiv Plane and on The Möbius strip

a). The **h**-invariant component of the Euclidean metric $ds^2 = dx^2 + dy^2$ on **C** (see [7]) is given by:

$$d\sigma^2 = \frac{1}{4}\left[1 + \frac{1}{|z|^2}\right]^2 ds^2.$$

This metric induces the following area element:

$$d\mathcal{A}(z) = \frac{1}{4}\left[1 + \frac{1}{|z|^2}\right]^2 dxdy.$$

This area element induces, in turn, the measure μ on **C** given by :

$$\mu(Z) = \frac{1}{4}\iint\limits_{\mathrm{Z}} \left[1 + \frac{1}{|z|^2}\right]^2 dxdy,$$

for every Lebesgue measurable set $Z \subseteq \mathbf{C}$. For $Z = A_{\mathrm{r}}$ we get:

$$\mu(A_{\mathrm{r}}) = \frac{\pi}{4}\left[r^2 - \frac{1}{r^2} + 4 \ln r\right].$$

The change of variable $w = \mathbf{h}z$ leads to $\mu(\mathbf{h}Z) = \mu(Z)$ for every Lebesgue measurable set $Z \subseteq \mathbf{C}$. Thus μ is **h**-invariant and it can be projected on $\mathbf{P}^2$. As a consequence we obtain, again via (3) and (4):

$$\widetilde{\mu}(\widetilde{Z}) = \frac{1}{2}\mu(Z) = \frac{1}{8}\iint\limits_{\mathrm{Z}} \left[1 + \frac{1}{|z|^2}\right]^2 dxdy\,,$$

where $Z = \pi^{-1}(\widetilde{Z})$. Particularly, when $\widetilde{Z} = M_{\mathrm{r}}$, we have:

$$\widetilde{\mu}(M_{\mathrm{r}}) = \frac{1}{2}\mu(A_{\mathrm{r}}) = \frac{\pi}{8}[r^2 - \frac{1}{\mathrm{r}^2} + 4ln\ r].$$

b) The natural Riemannian metric on $\widehat{\mathbf{C}}$ is the spheric metric

$$d\sigma^2 = \frac{|dz|^2}{(1 + |z|^2)^2} = \frac{dx^2 + dy^2}{(1 + |z|^2)^2}.$$

This metric is **h**-invariant. The area element induced by $d\sigma^2$ is

$$d\mathcal{A}(z) = \frac{dx\ dy}{(1+|z|^2)^2}\,,$$

which induces the measure ν on $\widehat{\mathbf{C}}$ given by:

$$\nu(Z) = \iint\limits_{Z} \frac{dx\ dy}{(1+|z|^2)^2} = \iint\limits_{\mathbf{C}\cap Z} \frac{dx\ dy}{(1+|z|^2)^2}$$

for any Borelian set $Z \subseteq \widehat{\mathbf{C}}$.

One sees that $\nu = \mathbf{h}^*(\nu)$, and thus ν can be projected on $\mathbf{P}^2$. The measure $\widetilde{\nu} := (\pi^*)^{-1}(\nu)$ is given explicitely by:

$$\widetilde{\nu}(\widetilde{Z}) = \frac{1}{2}\nu(\pi^{-1}(\widetilde{Z})) = \frac{1}{2}\iint\limits_{Z} \frac{dx\ dy}{(1+|z|^2)^2} = \frac{1}{2}\iint\limits_{\mathbf{C}\cap Z} \frac{dx\ dy}{(1+|z|^2)^2},$$

where $Z = \pi^{-1}(\widetilde{Z})$. When $\widetilde{Z} = M_{\mathrm{r}}$, one obtains:

$$\widetilde{\nu}(M_{\mathrm{r}}) = \frac{1}{2}\iint\limits_{A_{\mathrm{r}}} \frac{dx\ dy}{(1+|z|^2)^2} = \frac{1}{2}\int_0^{2\pi}\int_{1/\mathrm{r}}^{\mathrm{r}} \frac{\rho d\rho d\theta}{(1+\rho^2)^2} = \frac{\pi(r^2-1)}{2(r^2+1)}.$$

When $r \to \infty$ we get the spheric measure of $\mathbf{P}^2$. More exactly,

$$\widetilde{\nu}(\mathbf{P}^2) = \lim_{r\to\infty} \widetilde{\nu}(M_{\mathrm{r}}) = \frac{\pi}{2},$$

as expected.

c) Let $f \in L^1(\mathbf{C};\lambda)$ be an arbitrary Lebesgue integrable **real** function on $\mathbf{C}$. Let us consider the measure ν defined on the σ-algebra of Lebesgue measurable subsets of $\mathbf{C}$ by

$$\nu(A) := \iint\limits_{A} f(z)\ dx\ dy$$

for every Lebesgue measurable set A. Obviously, $\mathbf{h}A$ is also Lebesgue measurable and

$$\nu(\mathbf{h}A) = \iint\limits_{\mathbf{h}A} f(w) du dv = \iint\limits_{A} (f\circ\mathbf{h})(z)\frac{1}{|z|^4} dx dy.$$

Thus the **h**-invariant component of ν is given by

$$\nu_{\mathrm{s}}(A) = \frac{1}{2}\left[\nu(A) + \nu(\mathbf{h}A)\right] = \frac{1}{2}\iint\limits_{\mathrm{A}} \left[f(z) + \frac{1}{|z|^4}(f \circ \mathbf{h})(z)\right] dx\ dy.$$

If $f = f \circ \mathbf{h}$, one gets

$$\nu_{\mathrm{s}}(A) = \iint\limits_{\mathrm{A}} \frac{1}{2}\left[1 + \frac{1}{|z|^4}\right] f(z) dx dy.$$

Thus ν_{s} is absolutely continuous with respect to λ and has the Radon-Nilodym derivative

$$\left[\frac{d\nu_{\mathrm{s}}}{d\lambda}\right](z) = \frac{1}{2}\left[1 + \frac{1}{|z|^4}\right] f(z)$$

for every $z \in \mathbf{C}/\left\{0\right\}$.

The measure $\widetilde{\nu} := (\pi^*)^{-1}(\nu_{\mathrm{s}})$ acts on $\mathbf{P}^2$ in the following way. For every Borelian subset $\widetilde{Z}$ of $\mathbf{P}^2$,

$$\widetilde{\nu}(\widetilde{Z}) = \frac{1}{2}\nu_{\mathrm{s}}(Z) = \iint\limits_{\mathrm{Z}} \frac{1}{4}\left[1 + \frac{1}{|z|^4}\right]\ f(z)\ dx\ dy,$$

where $Z = \pi^{-1}(\widetilde{Z})$.

6. Integration on Nonorientable Klein Surfaces

Let **S** be a nonorientable Klein surface, (**R, h**) be its orientable double cover and $\pi : \mathbf{R} \to \mathbf{S}$ be the canonical projection.

Suppose that $\widetilde{\mu}$ and $\mu = \pi^*(\widetilde{\mu})$ are positive Borel measures on **S**, respectively on **R**, and that **E** is a Banach space or $[-\infty, +\infty]$. We denote by $\mathcal{F}(\mathbf{R})$ and $\mathcal{F}(\mathbf{S})$ the vector spaces of functions defined μ-a.e. on **R**, respectively $\widetilde{\mu}$-a.e. on **S** with range in **E**. If $\mathbf{E}=[-\infty, +\infty]$ we suppose that the elements of $\mathcal{F}(\mathbf{R})$ and $\mathcal{F}(\mathbf{S})$ are finite almost everywhere.

The pull-back maps induced by **h** and π, $\mathbf{h}^* : \mathcal{F}(\mathbf{R}) \to \mathcal{F}(\mathbf{R})$, respectively $\pi^* : \mathcal{F}(\mathbf{S}) \to \mathcal{F}(\mathbf{R})$ are defined by $\mathbf{h}^*(f) := f \circ \mathbf{h}$, respectively $\pi^*(\widetilde{f}) := \widetilde{f} \circ \pi$. It can be easily checked that $\mathbf{h}^*$ is an involution and a vector space automorphism, while π^* is a morphism of vector spaces, with $\mathbf{h}^* \circ \pi^* = \pi^*$.

The function $f \in \mathcal{F}(\mathbf{R})$ is called $\mathbf{h}$-invariant is $\mathbf{h}^*(f) := f$ and $\mathbf{h}$-antiinvariant is $\mathbf{h}^*(f) := -f$. We define

$$\begin{cases} \mathcal{F}_{\rm s}(\mathbf{R}) := \{f \in \mathcal{F}(\mathbf{R}) | \ \ \mathbf{h}^*(f) = f\} \\ \mathcal{F}_{\rm a}(\mathbf{R}) := \{f \in \mathcal{F}(\mathbf{R}) | \ \ \mathbf{h}^*(f) = -f\}. \end{cases}$$

It can be easily checked (see [2]) that

$$\mathcal{F}(\mathbf{R}) = \mathcal{F}_{\rm s}(\mathbf{R}) \bigoplus \mathcal{F}_{\rm a}(\mathbf{R}), \pi^*(\mathcal{F}(\mathbf{S})) = \mathcal{F}_{\rm s}(\mathbf{R})$$

and the co-restriction

$$\pi^* : \mathcal{F}(\mathbf{S}) \to \mathcal{F}_{\rm s}(\mathbf{R})$$

is a vector space isomorphism. For every $f \in \mathcal{F}(\mathbf{R})$ we have:

$$f_{\rm s} := \frac{1}{2}[f + \mathbf{h}^*(f)] \in \mathcal{F}_{\rm s}(\mathbf{R}), f_{\rm a} := \frac{1}{2}[f - \mathbf{h}^*(f)] \in \mathcal{F}_{\rm a}(\mathbf{R}) \text{ and } f = f_{\rm s} + f_{\rm a}.$$

The functions $f_{\rm s}$ and $f_{\rm a}$ are called the **invariant** (respectively, the **antiinvariant**) components of f.

If $(\mathbf{T},\mathcal{B}(\mathbf{T}), \upsilon)$ is one of the measure spaces $(\mathbf{R},\mathcal{B}(\mathbf{R}), \mu)$ or $(\mathbf{S},\mathcal{B}(\mathbf{S}), \widetilde{\mu})$, let us denote by $\mathcal{L}^{\rm p}(\mathbf{T},\upsilon)$ the set of Borel measurable functions $f : \mathbf{T} \to \mathbf{E}$ such that $|f|^{\rm p}$ is υ-integrable, for every p, $0 < p < \infty$, respectively $|f|$ is υ-a.e. bounded, for $p = \infty$. Here $|f(t)|$ stands for the norm of $f(t)$, if $\mathbf{E}$ is a Banach space, and for the absolute value, if $\mathbf{E} = [-\infty, +\infty]$. $\mathcal{L}^{\rm p}_{\rm s}(\mathbf{R},\mu)$ and $\mathcal{L}^{\rm p}_{\rm a}(\mathbf{R},\mu)$ are the subsets of $\mathcal{L}^{\rm p}(\mathbf{R},\mu)$ formed with $\mathbf{h}$-invariant, respectively $\mathbf{h}$-antiinvariant functions.

$\mathcal{L}^{\rm p}(\mathbf{R},\mu)$, $\mathcal{L}^{\rm p}_{\rm s}(\mathbf{R},\mu)$, $\mathcal{L}^{\rm p}_{\rm a}(\mathbf{R},\mu)$ and $\mathcal{L}^{\rm p}(\mathbf{S},\widetilde{\mu})$ are vector spaces with pointwise defined addition and multiplication with scalars.

If $\mathcal{L}^{\rm p}(\mathbf{T},\nu)$ is one of these vector spaces, then we denote as usual $||f||_{\rm p} = \left(\int\limits_{\mathbf{T}} |f|^{\rm p} \, d\nu\right)^{1/{\rm p}}$, for $1 \leq p < \infty$, respectively $||f||_\infty = ess \sup \{|f(t)| \ \ |t \in \mathbf{T}\}$, for $p = \infty$, and we notice that $||.||_{\rm p}$ are seminorms.

Theorem 5. *Let $\widetilde{\mu}$ be a positive Borel measure on $\mathbf{S}$ and $\mu = \pi^*(\widetilde{\mu})$. Then, for every p, $0 < p \leq \infty$, we have*

(i) $\widetilde{f} \in \mathcal{L}^{\rm p}(\mathbf{S},\widetilde{\mu})$, *if and only if* $\pi^*(\widetilde{f}) \in \mathcal{L}^{\rm p}(\mathbf{R},\mu)$;

(ii) $\mathcal{L}^{\rm p}(\mathbf{R},\mu) = \mathcal{L}^{\rm p}_{\rm s}(\mathbf{R},\mu) \bigoplus \mathcal{L}^{\rm p}_{\rm a}(\mathbf{R},\mu)$;

(iii)

$$\int_{\mathbf{S}} \widetilde{f} d\widetilde{\mu} = \int_{\mathbf{R}} \frac{1}{2}\pi^*(\widetilde{f})d\mu. \tag{6}$$

Moreover, for $1 \leq p \leq \infty$, *the co-restriction* $\pi^* : \mathcal{L}^{\mathrm{p}}(\mathbf{S},\widetilde{\mu}) \to \mathcal{L}^{\mathrm{p}}_{\mathrm{s}}(\mathbf{R},\mu)$ *is such that* $2^{-1/\mathrm{p}}\pi^*$ *is an isometry of* $\mathcal{L}^{\mathrm{p}}$*-spaces, if* $1 \leq p < \infty$ *and* π^* *is an isometry of* $\mathcal{L}\infty$*-spaces, i.e.*

$$\begin{cases} \left|\left|\widetilde{f}\right|\right|_{\mathrm{p}} = \left|\left|2^{-1/\mathrm{p}}\pi^*(\widetilde{f})\right|\right|_{\mathrm{p}} \text{ for } 1 \leq p < \infty, \\ \qquad\qquad \textit{respectively} \\ \left|\left|\widetilde{f}\right|\right|_{\infty} = \left|\left|\pi^*(\widetilde{f})\right|\right|_{\infty}. \end{cases} \tag{7}$$

(iv) *For* $\mathbf{E}=\mathbf{C}$, *the inner-product spaces* $L^2(\mathbf{S},\widetilde{\mu})$ *and* $L^2_{\mathrm{s}}(\mathbf{R},\mu)$ *are isomorphic, the isomorphism being given by the restriction to* $L^2(\mathbf{S},\widetilde{\mu})$ *of* $2^{-1/2}\pi^*$, *i.e. for arbitrary* $\widetilde{\mathbf{f}}, \widetilde{\mathbf{g}} \in L^2(\mathbf{S},\widetilde{\mu})$ *we have the following equality*:

$$\langle \widetilde{\mathbf{f}}, \widetilde{\mathbf{g}} \rangle_{L^2(\mathbf{S},\widetilde{\mu})} = \langle \frac{1}{\sqrt{2}}\pi^*(\widetilde{\mathbf{f}}), \frac{1}{\sqrt{2}}\pi^*(\widetilde{\mathbf{g}}) \rangle_{L^2_{\mathrm{s}}(\mathbf{R},\mu)}.$$

Proof. For every $\widetilde{E} \in \mathcal{B}(\mathbf{S})$, $E := \pi^{-1}(\widetilde{E}) \in \mathcal{B}(\mathbf{R})$ is an $\mathbf{h}$-invariant set. Let $(A,\mathbf{h}A)$ be a covering partition of $\mathbf{R}$, as in Theorem 2, and for an arbitrary $F \in \mathcal{B}(\mathbf{R})$, let us denote $F_1 = F \cap A$ and $F_2 = F \cap \mathbf{h}A$. A partition $\widetilde{\mathcal{E}} = \left\{\widetilde{E}_{\mathrm{i}}\right\}_{1 \leq i \leq n}$ of $\mathbf{S}$ and a **measurable step function**

$$\widetilde{t} = \sum_{\mathrm{i=1}}^{\mathrm{i=n}} \widetilde{\alpha}_{\mathrm{i}}\ \chi_{\widetilde{\mathrm{E}}_{\mathrm{i}}}$$

induce the partition $\mathcal{E}$ and the measurable step function t on $\mathbf{R}$ as follows:

$$\mathcal{E} = \{E_{\mathrm{i},1}\}_{1 \leq i \leq n} \cup \{E_{\mathrm{i},2}\}_{1 \leq i \leq n} \quad \text{and} \quad t = \sum_{\mathrm{i=1}}^{\mathrm{i=n}} \alpha_{\mathrm{i},1}\chi_{\mathrm{E}_{\mathrm{i},1}} + \sum_{\mathrm{i=1}}^{\mathrm{i=n}} \alpha_{\mathrm{i},2}\chi_{\mathrm{E}_{\mathrm{i},2}},$$

where

$$E_{\mathrm{i}} = \pi^{-1}(\widetilde{E}_{\mathrm{i}}),\ E_{\mathrm{i}} = E_{\mathrm{i},1} \cup E_{\mathrm{i},2},\ E_{\mathrm{i},1} \cap E_{\mathrm{i},2} = \emptyset,\ \pi(E_{\mathrm{i},1}) = \pi(E_{\mathrm{i},2}) = \widetilde{E}_{\mathrm{i}}$$

and $\alpha_{\mathrm{i},1} = \alpha_{\mathrm{i},2} = \widetilde{\alpha}_{\mathrm{i}}$, i.e. $\pi^*(\widetilde{t}) = t$.

Then: $ess\ \sup_{\mathbf{S}} \left|\widetilde{t}(s)\right| = ess\ \sup_{\mathbf{R}} |t(x)|$ and

$$\left|\widetilde{t}\right|^{\mathrm{p}} = \sum_{\mathrm{i=1}}^{\mathrm{i=n}} |\widetilde{\alpha}_{\mathrm{i}}|^{\mathrm{p}}\ \chi_{\widetilde{\mathrm{E}}_{\mathrm{i}}} \ \text{ and }\ \pi^*(\left|\widetilde{t}\right|^{\mathrm{p}}) = |t|^{\mathrm{p}} = \sum_{\mathrm{i=1}}^{\mathrm{i=n}} |\alpha_{\mathrm{i,1}}|^{\mathrm{p}}\ \chi_{\mathrm{E_{i,1}}} + \sum_{\mathrm{i=1}}^{\mathrm{i=n}} |\alpha_{\mathrm{i,2}}|^{\mathrm{p}}\ \chi_{\mathrm{E_{i,2}}}$$

Consequently, $|t|^{\mathrm{p}} \in \mathcal{F}_{\mathrm{s}}(\mathbf{R})$, and since, due to (3),

$$\mu(E_{\mathrm{i,1}}) = \mu(E_{\mathrm{i,2}}) = \widetilde{\mu}(\widetilde{E}_{\mathrm{i}}),$$

we have

$$\sum_{\mathrm{i=1}}^{\mathrm{i=n}} |\alpha_{\mathrm{i,1}}|^{\mathrm{p}}\ \mu(E_{\mathrm{i,1}}) = \sum_{\mathrm{i=1}}^{\mathrm{i=n}} |\alpha_{\mathrm{i,2}}|^{\mathrm{p}}\ \mu(E_{\mathrm{i,2}}) = \sum_{\mathrm{i=1}}^{\mathrm{i=n}} |\widetilde{\alpha}_{\mathrm{i}}|^{\mathrm{p}}\ \widetilde{\mu}(\widetilde{E}_{\mathrm{i}}).$$

In other words,

$$\int_{\mathbf{S}} \left|\widetilde{t}\right|^{\mathrm{p}} d\widetilde{\mu} = \int_{\mathrm{A}} |t|^{\mathrm{p}}\, d\mu = \int_{\mathbf{h}\mathrm{A}} |t|^{\mathrm{p}}\, d\mu$$

and thus, for any p, $0 < p < \infty$,

$$\int_{\mathbf{S}} \left|\widetilde{t}\right|^{\mathrm{p}} d\widetilde{\mu} = \frac{1}{2}\left[\int_{\mathrm{A}} |t|^{\mathrm{p}}\, d\mu + \int_{\mathbf{h}\mathrm{A}} |t|^{\mathrm{p}}\, d\mu\right] = \frac{1}{2}\int_{\mathbf{R}} |t|^{\mathrm{p}}\, d\mu = \frac{1}{2}\int_{\mathbf{R}} \left|\pi^*(\widetilde{t})\right|^{\mathrm{p}} d\mu. \quad (8)$$

Obviously, the correspondence $\widetilde{t} \longleftrightarrow t$ between the measurable step functions on $\mathbf{S}$ and $\mathbf{h}$-invariant measurable step functions on $\mathbf{R}$ is a vector space isomorphism. It is known that the set of measurable step functions on $\mathbf{S}$ is dense in the corresponding vector space $\mathcal{L}^{\mathrm{p}}(\mathbf{S},\widetilde{\mu})$. Using a symmetrisation argument, we can easily show that the set of $\mathbf{h}$-invariant, respectively $\mathbf{h}$-antiinvariant measurable step functions on $\mathbf{R}$ is dense in $\mathcal{L}^{\mathrm{p}}_{\mathrm{s}}(\mathbf{R},\mu)$, respectively $\mathcal{L}^{\mathrm{p}}_{\mathrm{a}}(\mathbf{R},\mu)$. Consequently (8) is fulfielled when replacing $\widetilde{t}$ by a function $\widetilde{f} \in \mathcal{L}^{\mathrm{p}}(\mathbf{S},\widetilde{\mu})$, $0 < p < \infty$. Also, $ess\ \sup_{\mathbf{S}} \left|\widetilde{f}(s)\right| < \infty$ if and only if $ess \sup_{\mathbf{R}} |f(x)| < \infty$, where $f = \pi^*(\widetilde{f})$. Therefore (i) is completely proved.

The identity (ii) results directly from the similar decomposition of $\mathcal{F}(\mathbf{R})$ and the obvious fact that $\mathcal{F}_{\mathrm{s}}(\mathbf{R}) \bigcap \mathcal{L}^{\mathrm{p}}(\mathbf{R},\mu) = \mathcal{L}^{\mathrm{p}}_{\mathrm{s}}(\mathbf{R},\mu)$ and $\mathcal{F}_{\mathrm{a}}(\mathbf{R}) \bigcap \mathcal{L}^{\mathrm{p}}(\mathbf{R},\mu) = \mathcal{L}^{\mathrm{p}}_{\mathrm{a}}(\mathbf{R},\mu)$.

The relation (7) is an easy corollary of (8), and (6) can be obtained writing (8) for $p = 1$ and for $\widetilde{t}$ instead for $\left|\widetilde{t}\right|$.

To obtaine (iv) we need only to notice that for $\widetilde{f}$, $\widetilde{g} \in \mathcal{B}(\mathbf{S})$, $\widetilde{f} = \widetilde{g}$ $\widetilde{\mu}-$a.e. if and only if $\pi^{*}(\widetilde{f}) = \pi^{*}(\widetilde{g})$ $\mu-$a.e. Therefore, we can define the pull-back operator π^{*} for equivalence classes $\widetilde{\mathbf{f}}$ of functions from $\mathcal{B}(\mathbf{S})$ which are equal $\widetilde{\mu}-$a.e: $\pi^{*}(\widetilde{\mathbf{f}}) := \widetilde{\mathbf{f}} \circ \pi$. Moreover, $\widetilde{\mathbf{f}} \in L^{\mathrm{p}}(\mathbf{S},\widetilde{\mu})$ if and only if $\pi^{*}(\widetilde{\mathbf{f}}) \in L^{\mathrm{p}}(\mathbf{R},\mu)$. Then the identity in (iv) becomes obvious by noticing that.

$$\int\limits_{\mathbf{S}} \widetilde{\mathbf{f}}\ \overline{\widetilde{\mathbf{g}}}\ \mathrm{d}\widetilde{\mu} = \int\limits_{\mathbf{R}} \frac{1}{2}\pi^{*}(\widetilde{\mathbf{f}})\ \overline{\pi^{*}(\widetilde{\mathbf{g}})}\ d\mu. \qquad \square$$

7. Final Remarks

We notice that, although very simple, the formula (6) is far from trivial. Indeed, it contains the quintessence of the method we adopted to **canonically** reduce the analysis on nonorientable surfaces to the analysis on their orientable double covers. As an illusration, let us deal with the following two cases:

1. If $\mathbf{S}$ is the Möbius strip M_{r} and $\widetilde{\lambda}$ is the Lebesgue measure on M_{r}, then the formula (6) becomes

$$\int\limits_{\mathrm{M_r}} \widetilde{f} d\widetilde{\lambda} = \frac{1}{4}\int\limits_{\mathrm{A_r}} (\widetilde{f} \circ \pi)(z) \left[1 + \frac{1}{|z|^{4}}\right]\ dxdy = \frac{1}{2}\int\limits_{\mathrm{A_r}} (\widetilde{f} \circ \pi)(z)\ dxdy\,, \tag{9}$$

where A_{r} is the annulus serving as the orientable double cover of M_{r}.

The inner product on $L^{2}(M_{\mathrm{r}};\widetilde{\lambda})$ is given by:

$$\begin{aligned}\langle \widetilde{f}, \widetilde{g}\rangle_{L^2(M_{\mathrm{r}};\widetilde{\lambda})} &= \int\limits_{\mathrm{M_r}} \widetilde{f}\ \overline{\widetilde{g}}\ d\widetilde{\lambda} = \frac{1}{4}\int\limits_{\mathrm{A_r}} (\widetilde{f} \circ \pi)(z)\overline{(\widetilde{g} \circ \pi)}(z) \left[1 + \frac{1}{|z|^{4}}\right]\ dx\ dy \\ &= \frac{1}{2}\int\limits_{\mathrm{A_r}} (\widetilde{f} \circ \pi)(z)\overline{(\widetilde{g} \circ \pi)}(z)\ dx\ dy = \langle \frac{1}{\sqrt{2}}\pi^{*}(\widetilde{f}), \frac{1}{\sqrt{2}}\pi^{*}(\widetilde{g})\rangle\end{aligned}$$

for every $\widetilde{f},\widetilde{g} \in L^2(M_r; \widetilde{\lambda})$, where $\overline{\widetilde{g}}$ is the complex conjugate of $\widetilde{g}$. Here $\widetilde{f}$ and $\widetilde{g}$ are class representatives of $\widetilde{\mathbf{f}}$ and $\widetilde{\mathbf{g}}$.

2. Suppose that a signed Borel measure μ is given on a symmetric Riemann surface (**R**,**h**). Let $\mu_{\rm s}$ be the **h**-invariant component of μ and $\widetilde{\mu} = (\pi^*)^{-1}(\mu_{\rm s})$. Let us consider the following two measure spaces:

$$(\mathbf{R}, \mathcal{B}(\mathbf{R}), \mu_{\rm s}) \quad \text{and} \quad (\mathbf{R}/\langle \mathbf{h} \rangle, \mathcal{B}(\mathbf{R}\langle \mathbf{h} \rangle), \widetilde{\mu})$$

and keep in mind that $\mathbf{R}/\ \langle \mathbf{h} \rangle$ is a nonorientable surface. The Theorem 5 assures us that no difficulty can appear when integrating functions of $\mathcal{F}(\mathbf{R}/\langle \mathbf{h} \rangle)$ on this second measure space. Indeed, it is enough to integrate uniquely **h**-invariant functions on the measure space (**R**, $\mathcal{B}$(**R**), μ_s). We should however point out that identifying classes of symmetric Riemann surfaces is a laborious task in Teichmüller spaces theory. It has been shown in [4], [5] that the symmetric compact Riemann surfaces of genus one are precisely the tori **C**/**Z**, where **Z** is the lattice

$$\{n + m\ i\tau \mid\ n, m \in \mathbf{Z},\ \tau \geq 1\}.$$

For such a surface, $\mathbf{R}/\ \langle \mathbf{h}\ \rangle$ is a Klein bottle. Integrating (with respect to $\widetilde{\mu}$) on a Klein bottle might appear a rather descouraging task. In reality, due to Theorem 5, such a task reduces to integrating (with respect to $\mu_s = \pi^*(\widetilde{\mu})$) **h**–invariant functions on a torus **C**/**Z**.

References

[1] C. Andreian Cazacu and V. Stanciu, BMO-QC mappings between klein surfaces, *Libertas Mathematica*, **XX** (2000), 7-13.

[2] I. Barza, Integration on nonorientable Riemann surfaces, In: *Almost Complex Structures* (Ed-s: K. Sekigawa, S. Dimiev), World Scientific (1995), 63-97.

[3] I. Barza, Calculus on Nonorientable Riemann Surfaces, *Libertas Mathematica*, **XV** (1995), 1-45.

[4] I. Barza, The dianalytic morphisms of the Klein bottles, *Complex Analysis, Juensuu 1987* (Ed-s: I. Laine, S. Rikman, T. Sorvali), Springer Verlag, Lecture Notes in Mathematics, **1351**, 38-51.

[5] I. Barza, On a problem of S. Stoilow, *Ann. Univ. Bucharest*, **36** (1987), 6-8.

[6] I. Barza and D. Ghisa, Boundary value problems on nonorientable surfaces, *Rev. Roum. Math. Pure et Appl.*, **XLIII**, No. 1-2 (1998), 1-13.

[7] I. Barza, D. Ghisa and S. Ianus, Some remarks on nonorientable surfaces, *Publications de L'Institut Mathé Matique, Nouvelle Série*, **63**, No. 77 (1998), 47-54.

[8] C. Blatter, Zur Riemannschen Geometry im Grossen auf dem Möbiusband, *Composito Math.*, **15** (1961), 88-107.

[9] C. Constantinescu and A. Cornea, *Ideale Ränder Riemannscher Flächen*, Springer Verlag (1963).

[10] D. Ghisa, Potentials on nonorientable surfaces, *Libertas Mathematica*, **XX** (2000), 23-31.

[11] P.M. Pu, Some inequalities in certain nonorientable manifolds, *Pacific J. Math.*, **2** (1952), 55-71.

[12] G. Springer, *Introduction to Riemann Surfaces*, Addison-Wesley Publishing Company Inc. (1957).