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GENERALIZED SELF ADJOINTNESS AND APPLICATIONS TO DIFFERENTIAL EQUATIONS DERIVABLE FROM A VARIATIONAL PRINCIPLE

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Abstract: Any system of second-order differential equations which has coefficients that do not depend explicitly on the time can be represented by a vector field on a tangent bundle. An approach to the problem of deciding whether the conditions for the equations to be derivable from a given Lagrangian by means of differential geometric ideas are formulated.

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1. Introduction

The question as to whether a given system of differential equations is variational or not has been of considerable interest. To state the problem more explicitly, suppose one has a given system of differential equations, then can one find out

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whether they are variational or not, and if so, can a Lagrange function for these equations be constructed [1]. The problem in this form is often referred to as the inverse problem of the calculus of variations [2]. In addition to this aspect of the problem, there have been recent advances with many applications to differential geometry [3], [4]. In quantum mechanics and quantum field theory, the question of the existence of a Lagrangian which is necessary for a quantization procedure such as path integrals has been of interest. In this short note, we would like to consider some mathematical aspects related to this problem.

Any system of second-order differential equations which has coefficients that do not depend explicitly on the time can be represented by a particular vector field on a tangent bundle [5], [6]. This vector field is usually referred to as a second order differential equation for the bundle. In the application just mentioned, one would like to interpret the conditions for the particular equations to be derivable from a Lagrangian in terms of differential geometric methods which are associated with the second order differential equation vector field.

The main interest here will be to formulate a generalized idea of self-adjointness and show how it is linked to the problem introduced above. This formulation is illustrated with respect to several model problems associated with a given contact two-form structure. It can be shown for example that the Euler or Lagrangian vector field, is self-adjoint under the definition used here with respect to the exterior derivative of the Cartan form. The resulting formalism leads to a natural formulation of symmetries, as well as canonical transformations.

2. Review and Formulation of the Problem

A vector field Γ on the tangent bundle $\pi : T(M) \to M$ of a differentiable manifold M of dimension n, is called a second-order differential equation if for each point (q, u) of $T(M)$, where $u \in T_q(u)$

$$\pi_* \Gamma_{(q,u)} = u. \tag{1}$$

Evolution space will then be thought of as the first jet bundle of smooth maps, which is called $J^1(R, M)$, with $R = \mathbb{R}$. The points of $J^1(R, M)$ are defined in terms of smooth maps $R \to M$, that is curves in M. Let us identify the parameter on any curve with the time, considered as an additional coordinate. Time may be incorporated by forming the manifold $R \times M$. Any curve σ in M defines another curve in $R \times M$, called its graph, given by $r \to (t, \sigma(t))$. In terms of coordinates, (t, q^a, u^a), a section $t \to (t, \sigma^a(t), u^a(t))$, will be the 1-jet of the curve σ if and only if

$$u^a(t) = \dot{\sigma}^a(t), \tag{2}$$

for all t in the domain of σ. This condition may be expressed in terms of the n local contact 1-forms θ^a, which have coordinate expression

$$\theta^a = dx^a - u^a\, dt. \tag{3}$$

Any vector field Γ on $J^1(R, M)$ whose integral curves are all 1-jets of curves in M must satisfy

$$< \Gamma, \theta^a >= 0, \qquad < \Gamma, dt >= 1. \tag{4}$$

The vector field Γ is the second-order differential equation field, and can be written in terms of coordinates as follows

$$\Gamma = \frac{\partial}{\partial t} + u^a \frac{\partial}{\partial q^a} + f^a \frac{\partial}{\partial u^a}\,. \tag{5}$$

The integral curves with initial t coordinate 0 have t for parameter and satisfy

$$\dot{q}^a(t) = u^a(t), \qquad \dot{u}^a(t) = f^a(t, q(t), u(t)). \tag{6}$$

They are the 1-jets of the solution curves of the second-order differential equations

$$\ddot{q}^a = f^a(t, q, \dot{q}). \tag{7}$$

Thus, $J^1(R, M)$ is a vector bundle over $R \times M$, and is equipped with a system of 1-forms, the contact forms, with local basis $\{\theta^a\}$. Let X be a vector field on $R \times M$. There is a unique vector field $X^{(1)}$ on $J^1(R, M)$, called its first prolongation, such that $\pi_* X^{(1)} = X$ and $L_{X^{(1)}}\theta^a$ is a linear combination of the basic contact 1-forms. In coordinates, if X takes the form

$$X = \tau \frac{\partial}{\partial t} + \xi^a \frac{\partial}{\partial q^a}, \tag{8}$$

with τ, ξ^a local functions, then

$$X^{(1)} = \tau \frac{\partial}{\partial t} + \xi^a \frac{\partial}{\partial q^a} + \eta^a \frac{\partial}{\partial u^a}, \tag{9}$$

where η^a is defined by

$$\eta^a = \dot{\xi}^a - u^a \dot{\tau} = \frac{\partial \xi^a}{\partial t} + u^b \frac{\partial \xi^a}{\partial q^b} - u^a(\frac{\partial \tau}{\partial t} + u^b \frac{\partial \tau}{\partial q^b}). \tag{10}$$

It follows from (10) that

$$(\frac{\partial}{\partial t})^{(1)} = \frac{\partial}{\partial t},$$

and for any vector fields X, Y on $R \times M$, $[X^{(1)}, Y^{(1)}] = [X, Y]^{(1)}$. It is a straightforward consequence of this definition that for any vector field X on $R \times M$, and any second-order differential equation field Γ, the vector field

$$V = L_{X^{(1)}}\Gamma - \dot{\tau}\Gamma,$$

is vertical, where $\tau = i_X\, dt$. The addition of a vertical vector field to a second-order differential equation field leads to a new second-order differential equation field [5], [6]. The effect of the action of the flow of $X^{(1)}$ on Γ is to transform it into a new second order differential equation field, but with a change of parametrization if $\dot{\tau} \neq 0$.

One can also say that a vector at a point of $J^1(R, M)$ is vertical if it is tangent to the fibre of $\pi : J^1(R, M) \to R \times M$. For example, the vector fields $V_a = \partial/\partial u^a$ form a local basis of vertical vector fields. If Γ is a given second order vector field given in (5), then a set of local vector fields H_a, in terms of this second-order differential equation field Γ are given by

$$H_a = \frac{\partial}{\partial q^a} - Q_a^b \frac{\partial}{\partial u^b}, \qquad Q_a^b = -\frac{1}{2}\frac{\partial f^b}{\partial u^a}. \tag{11}$$

Elements of the vector field system spanned by the H_a are called horizontal. The vector fields $\{H_a, V_a, \Gamma\}$ form a local vector field basis on $J^1(R, M)$. The dual basis of 1-forms is given by $\{\theta^a, \psi^a, dt\}$, where using (3)

$$\psi^a = du^a - f^a dt + Q_b^a \theta^b = Q_b^a\, dx^b + du^a - (f^a + Q_b^a u^b)\, dt. \tag{12}$$

A type $(1, 1)$ tensor field S can be defined on $J^1(R, M)$ by

$$S = V_a \otimes \theta^a.$$

Then S vanishes on vertical vectors, and on second order differential equation fields. For any vector field Z on $J^1(R, M)$, it is clear that $S(Z)$ is vertical. One can define the vertical lift X^v to $J^1(R, M)$ by using S

$$X^v = S(X^{(1)}).$$

A Lagrangian, for example, determines a second-order differential equation field Γ. The integral curves of Γ are the 1-jets of the solution curves of the Euler-Lagrange equations. Suppose that the Lagrangian is regular, so that

$$V_a(V_b(L)) = \frac{\partial^2 L}{\partial u^a \partial u^b}$$

is nonsingular. Thus, there exists g^{bk} such that

$$\frac{\partial^2 L}{\partial u^a \partial u^b} g^{bk} = \delta^k_a.$$

Then Γ, which is called the Euler-Lagrange field, is given by [7],

$$\Gamma = \frac{\partial}{\partial t} + u^a \frac{\partial}{\partial q^a} + \Lambda^a \frac{\partial}{\partial u^a}, \tag{13}$$

where Λ^a are defined in terms of the Lagrangian as follows

$$\Lambda^i = g^{ij}(-\frac{\partial^2 L}{\partial u^j \partial q^k}\dot{q}^k - \frac{\partial^2 L}{\partial u^j \partial t} + \frac{\partial L}{\partial q^j}).$$

Since $\{H_a, V_a, \Gamma\}$ and $\{\theta^a, \psi^a, dt\}$ are dual local bases, the vector field Γ is uniquely determined by the equations

$$< \Gamma, \theta^a >= 0, \quad < \Gamma, \psi^a >= 0, \quad < \Gamma, dt >= 1. \tag{14}$$

3. Generalized Self-adjointness

Recall that a subspace of a tangent space to a differential manifold is called a Lagrangian subspace for a two-form if the two-form vanishes on each pair of vectors from the subspace. The following theorem has been proved in [6] and it will be referred to in the following.

Theorem. *Let Γ be a second-order differential equation on the tangent bundle of a differentiable manifold N. Necessary and sufficient conditions for Γ to be derivable from a regular Lagrangian are that there exists on $T(N)$ a two-form ω, of maximal rank, for which $L_\Gamma \omega = 0$, and such that all vertical subspaces are Lagrangian both for ω and for $i_H\, d\omega$, where H is any horizontal vector.*

Recall that f is a constant of motion for the dynamical vector field Y if $L_Y f = 0$. Suppose one considers the trivial bundle $R \times N$ over the real line $R = \mathbb{R}$, where N is a $2n$-dimensional real C^∞-differentiable manifold. Here N could be TM in which case, N is the evolution space of the system. Suppose

$\varphi \in \Omega^1(R \times N)$ is a given one-form for which $d\varphi$ is of constant rank $2n$. The two-form $\omega = d\varphi$ defines an exact contact structure on $R \times N$.

Definition 1. A vector field X on $R \times N$ is called self-adjoint with respect to the given two-form $\omega\, d\varphi$

$$i_X\, \omega = 0, \qquad i_X\, dt = 1. \tag{15}$$

Similarly, X is self-adjoint with respect to the given contact structure if $i_X\, d\varphi = 0$ holds.

The first of these conditions states that X must be a characteristic vector field of $d\varphi$. Since $\mathcal{C}(d\varphi)$ is one-dimensional, the two conditions define a unique vector field on $R \times N$. It is easy to verify that the original one-form φ may be replaced in the above definition by any one-form φ' for which $d\varphi = d\varphi'$, that is, which locally differs from φ by at most a total differential.

Proposition 1. *Suppose that $\omega = d\varphi$ and $i_H \omega$ are Lagrangian. A necessary condition for the two-form $\omega = d\varphi$ to satisfy $L_\Gamma \omega = 0$ is that the vector field Γ on $R \times N$ be self-adjoint is that*

$$i_\Gamma d\varphi = 0.$$

Proof. From the homotopy identity

$$L_\Gamma \omega = i_\Gamma\, d\omega + d i_\Gamma \omega = 0.$$

Since $d\omega = 0$ follows from ω, the requirement the $i_\Gamma\, d\varphi = 0$ ensures that $L_\Gamma \omega = 0$.

Let N be the tangent bundle TM of the n-dimensional manifold M, and consider a regular time-dependent Lagrange function $L \in C^\infty(R \times TM)$.

Proposition 2. *The vector field Γ defined by (13) is self-adjoint with respect to the derivative of the Cartan form, $\omega = d\theta$, where θ is defined by,*

$$\theta = (L - \dot{q}^i \frac{\partial L}{\partial \dot{q}^i})\, dt + \frac{\partial L}{\partial \dot{q}^i}\, dq^i. \tag{16}$$

Proof. It is clear that the equation $i_\Gamma dt = 1$ holds. It has been shown in [9] that $i_\Gamma \, d\theta = 0$ holds as well, which means that all the conditions for generalized self-adjointness are satisfied.

Let us introduce some examples. First, this definition of self-adjointness leads to the idea of a self-adjoint system of differential equations. In any local coordinate system $(t, x_1, \cdots, x_n)$ on the manifold $R \times N$, where N is a $2n$-dimensional real C^∞-differentiable manifold, consider a one-form with the structure

$$\varphi = H \, dt + G_j \, dx^j.$$

Here, H and G_j depend only on t and the x^j. This form can be differentiated in a straightforward way to give

$$d\varphi = (\frac{\partial G_i}{\partial t} - \frac{\partial H}{\partial x^i}) \, dt \wedge dx^i + \frac{\partial G_i}{\partial x^j} \, dx^j \wedge dx^i.$$

Let the vector field which is to be self-adjoint with respect to $d\varphi$ be represented by

$$X = \eta \frac{\partial}{\partial t} + \lambda^i \frac{\partial}{\partial x^i}.$$

Then, $i_X dt = 1$ implies that $\eta = 1$, and the condition $i_X d\varphi = 0$ implies that

$$[(\frac{\partial G_i}{\partial t} - \frac{\partial H}{\partial x^i}) + \lambda^j (\frac{\partial G_i}{\partial x^j} - \frac{\partial G_j}{\partial x^i})] \, dx^i - (\frac{\partial G_i}{\partial t} - \frac{\partial H}{\partial x^i}) \lambda^i \, dt = 0.$$

Since dt and dx^i are independent, this results in the following pair of equations

$$(\frac{\partial H}{\partial x^i} - \frac{\partial G_i}{\partial t}) = \lambda^j (\frac{\partial G_i}{\partial x^j} - \frac{\partial G_j}{\partial x^i}), \qquad (\frac{\partial G_i}{\partial t} - \frac{\partial H}{\partial x^i}) \lambda^i = 0.$$

As another example, it has been shown in [9] that, for a first order Lagrange function, the derivative of the Cartan form (16) can be written as $\alpha = d\theta = E + F$. Here, E and F are defined as the differential two-forms given by

$$E = E_i \, dq^i \wedge dt,$$
$$F = F_{ab}(dq^a - \dot{q}^a \, dt) \wedge (dq^b - \dot{q}^b \, dt) + G_{ab}(d\dot{q}^a - \ddot{q}^a \, dt) \wedge (dq^b - \dot{q}^b \, dt),$$

where the functions F_{ab} and G_{ab} are defined in the following way

$$F_{ab} = \frac{1}{2}(\frac{\partial^2 L}{\partial q^a \partial \dot{q}^b} - \frac{\partial^2 L}{\partial q^b \partial \dot{q}^a}), \qquad G_{ab} = \frac{\partial^2 L}{\partial \dot{q}^a \partial \dot{q}^b}.$$

Now $d\theta$ satisfies $i_\Gamma \, d\theta = 0$, where Γ is given in (13). If we require that α be closed, that is, $d\alpha = 0$, by direct calculation one can obtain the standard form of the Helmholtz conditions [9].

Finally, consider a different example. With the horizontal distribution spanned by H_i, one can consider $\{H_i, \partial/\partial u^i\}$ a basis of local vector fields for $T(M)$. The dual basis of one-forms is $\{dq^i, \theta^j\}$, where $\theta^j = du^j - \frac{1}{2}\partial_{q^s} f^j dq^s$. The Lie derivatives of these one-forms with respect to the second-order differential equation field $\Gamma = u^i \partial/\partial q^i + +f^i \partial/\partial u^i$ are written

$$L_\Gamma \, dq^i = du^i,$$

$$L_\Gamma \theta^i = L_\Gamma \, du^i - \frac{1}{2} L_\Gamma (\frac{\partial f^i}{\partial q^j} \, dq^j - \frac{1}{2}\frac{\partial f^i}{\partial q^j} L_\Gamma (dq^j)$$
$$= -\frac{1}{2}\Gamma(\frac{\partial f^i}{\partial q^j}) \, dq^j - \frac{1}{2}\frac{\partial f^i}{\partial q^j} \, du^j + df^i.$$

Let ω be the two-form defined by

$$\omega = a_{ij} \, dq^i \wedge dq^j + g_{ij} \, dq^i \wedge \theta^j, \qquad a_{ij} + a_{ji} = 0.$$

Terms of the form $\theta^i \wedge \theta^j$ are excluded from the expression for ω, so that the condition that vertical subspaces are Lagrangian for ω is fulfilled. Then

$$L_\Gamma \omega = \Gamma(a_{ij}) \, dq^i \wedge dq^j + a_{ij} (\frac{1}{2}\frac{\partial f^i}{\partial q^s} dq^s$$
$$+ \theta^i) \wedge dq^j + a_{ij} dq^i \wedge (\frac{1}{2}\frac{\partial f^j}{\partial q^s} dq^s + \theta^j)$$
$$+\Gamma(g_{ij}) \, dq^i \wedge \theta^j + g_{ij} (\frac{1}{2}\frac{\partial f^i}{\partial q^s} dq^s + \theta^i) \wedge \theta^j + g_{ij} \, dq^i \wedge (-\frac{1}{2} A^j_s \, dq^s + \frac{1}{2}\frac{\partial f^j}{\partial q^s}\theta^s)$$
$$= (\Gamma(a_{ij}) + a_{is}\frac{\partial f^s}{\partial q^j} - \frac{1}{2} g_{is} A^s_j) \, dq^i \wedge dq^j$$
$$+(2a_{ij} + \Gamma(g_{ij}) + \frac{1}{2} g_{is}\frac{\partial f^s}{\partial q^j} + \frac{1}{2} g_{sj}\frac{\partial f^s}{\partial q^i}) \, dq^i \wedge \theta^j + g_{ij}\theta^i \wedge \theta^j. \qquad (17)$$

Here, A^i_j is given by

$$A^i_j = \Gamma(\frac{\partial f^i}{\partial u^j}) - 2\frac{\partial f^i}{\partial q^j} - \frac{1}{2}\frac{\partial f^i}{\partial u^k}\frac{\partial f^k}{\partial u^j}.$$

Equating to zero the coefficients of linearly independent terms to obtain the conditions imposed by setting $L_\Gamma \omega$ equal to zero, one has to have

$$g_{ij} = g_{ji}.$$

Therefore, it follows from (17) that

$$a_{ij} = 0, \qquad \Gamma(g_{ij}) + \frac{1}{2} g_{ik} \frac{\partial f^k}{\partial q^j} + \frac{1}{2} g_{kj} \frac{\partial f^k}{\partial q^i} = 0.$$

These are, respectively, the skew symmetric and symmetric parts of the coefficient of $dq^i \wedge \theta^j$. Finally, there is the term

$$-\frac{1}{2} g_{ik} A^k_j \, dq^i \wedge dq^j = 0.$$

This implies that $g_{ik} A^k_j = g_{jk} A^k_i$. Therefore,

$$\omega = g_{ij} \, dq^i \wedge \theta^j.$$

This means that ω is of maximal rank if and only if $\det(g_{ij}) \neq 0$.

Proposition 3. *Let $\alpha \in \Omega^1(R \times M)$ be given. Then, there exists a vector field Y on $R \times M$ such that*

$$i_Y \, d\varphi = \alpha, \tag{18}$$

if and only if $i_X \alpha = 0$, where X is the self-adjoint vector field corresponding to $d\varphi$.

Proof. Suppose that $\alpha = i_Y \, d\varphi$, then $i_X i_Y \, d\varphi = -i_Y i_X \, d\varphi = 0$ by (14). Consider the map $Z \to i_Z \, d\varphi$. This is a linear map whose kernel is a one-dimensional subspace of the set of vector fields on $\mathbb{R} \times M$ consisting of multiples of a characteristic. If α lies in this image space, then there is some Y such that $i_Y \, d\varphi = \alpha$. The image space of these vector fields is then contained in the subspace of the image consisting of those one forms which are linear combinations of $dq^i - \dot{q}^i \, dt$ and $d\dot{q}^i - \Lambda^i \, dt$, which is just the subspace of those α such that $i_X \alpha = 0$. Since both spaces are $2n$-dimensional, they must coincide.

It follows immediately from this using (14) and (18), that the following holds.

Corollary. *Whenever Y is a solution of (18), the Lie bracket of X and Y belongs to the set of characteristics of $d\varphi$,*

$$[Y, X] = gX,$$

for some function $g \in C^\infty(R \times M)$.

Proof. Using the homotopy identity, $L_X = i_X d + d \, i_X$,

$$i_{[Y,X]} \, d\varphi = i_Y L_X \, d\varphi - L_X i_Y \, d\varphi = i_Y (i_X d + d i_X) \, d\varphi - (i_X d + d i_X) i_Y \, d\varphi = 0.$$

This will be applied to the case with symmetries in the next section.

4. Symmetries and Canonical Transformations

One says that a vector field Y is a symmetry of a certain tensor field if that tensor field is invariant under the flow of Y. Thus, in this sense, Y is a symmetry of another vector field X if and only if

$$L_Y X = [Y, X] = 0, \tag{19}$$

where L_Y represents the Lie derivative [8], [10]. Thus Y is a symmetry of, for example, the 2-form $d\varphi$ if and only if

$$L_Y \, d\varphi = 0. \tag{20}$$

On the other hand, for differential equations, what one is really interested in is that the flow of Y maps integral curves of Γ into integral curves. For this to hold, the system need not be strictly invariant since one can change the parametrization along integral curves. This is reflected in the requirement that the Lie derivative satisfy

$$L_Y \Gamma = [Y, \Gamma] = h\Gamma. \tag{21}$$

A vector field satisfying (21) will be called a dynamical symmetry of Γ. A special class of dynamical symmetries for Lagrangian systems is provided by the symmetries of the derivative of the Cartan form $d\theta$.

Proposition 4. *A $d\theta$-symmetry is a dynamical symmetry of the Lagrangian vector field Γ.*

Proof. Since $i_{[X,Y]}\beta = i_X L_Y \beta - L_Y i_X \beta$, then applying this to $d\theta$ with X replaced by Γ, we obtain

$$i_{[\Gamma,Y]} \, d\theta = i_\Gamma L_Y \, d\theta - L_Y i_\Gamma \, d\theta = 0,$$

since $L_Y \, d\theta = 0$ and $i_\Gamma \, d\theta = 0$.

Since the set of characteristic vector fields of $d\theta$ is one-dimensional, it follows that $[\Gamma, Y]$ must be proportional to Γ. Let Y be a vector field that satisfies (21) and is given by

$$Y = \tau(t, q, \dot{q}) \frac{\partial}{\partial t} + \xi^i(t, q, \dot{q}) \frac{\partial}{\partial q^i} + \eta^i(t, q, \dot{q}) \frac{\partial}{\partial \dot{q}^i}. \tag{22}$$

Given this and (10), a long calculation shows that their Lie bracket is given as follows [9],

$$[Y, \Gamma] = (\eta^i - \Gamma(\xi^i)) \frac{\partial}{\partial q^i} + (Y(\Lambda^i) - \Gamma(\eta^i)) \frac{\partial}{\partial \dot{q}^i} - \Gamma(\tau) \frac{\partial}{\partial t}.$$

In order for Y to be a dynamical symmetry, this has to be proportional to Γ, and so matching coefficients, one has

$$Y(\Lambda^i) - \Gamma(\eta^i) = -\Lambda^i \Gamma(\tau), \quad \eta^i - \Gamma(\xi^i) = -\dot{q}^i \Gamma(\tau).$$

By identifying this with $h\Gamma$, one obtains h explicitly,

$$h = -\Gamma(\tau). \tag{23}$$

The way in which the concept of symmetry enters a certain theory often depends on the specific nature of the systems to which it is applied, as well as the type of framework in which the analysis is done, which could be either analytical or geometrical. In the case of a self-adjoint system, it seems natural to introduce the idea of symmetry in terms of the contact form $d\varphi$, which completely determines the structure of the corresponding vector field.

Definition 2. A mapping $F \in Diff(R \times N)$ is called a symmetry of the contact form $d\varphi$ iff

$$F_* \, d\varphi = d\varphi. \tag{24}$$

A mapping F is called a canonical symmetry iff F is a symmetry of $d\varphi$ and moreover satisfies $F_* \, dt = dt$.

A mapping $F \in Diff(R \times M)$ is called a symmetry of a vector field Y iff

$$F_* Y = Y. \tag{25}$$

To put this in other words, a symmetry of a vector field transforms the set of integral curves of that vector field onto itself, without altering the parametrization of these curves. Consider now a symmetry F of $d\varphi$. If F acts on (14), one obtains using (24)

$$i_{F_* X} \, d\varphi = 0.$$

Thus, $F_* X$ belongs to $\mathcal{C}(d\varphi)$, and since this is one-dimensional and contains X, there must exist a function $h \in C^\infty(R \times N)$ such that

$$F_* X = hX. \tag{26}$$

Therefore, a symmetry of $d\varphi$ will in general not be a symmetry of the self-adjoint vector field X. By (26), it follows that F permutes integral curves of X among themselves, but allows for a change of parametrization along the curves.

Finally, an extension of the classical notion of the canonical transformation into this type of formalism will be introduced. It should be recalled that, roughly speaking, a transformation is canonical if every Hamiltonian system is transformed into a Hamiltonian system.

It is useful to state formally a fact which has been used repeatedly here. Each p-form β on $R \times M$ with $1 \leq p \leq 2n+1$ admits a splitting into a form $\beta^{(1)}$ and a form $\beta^{(2)}$ such that

$$\beta = \beta^{(1)} + \beta^{(2)} \wedge dt, \tag{27}$$

with $\beta^{(2)} \in \Omega^{p-1}(\mathbb{R} \times M)$ and $i_{\partial/\partial t}\beta^{(1)} = 0$. This means that $\beta^{(1)}$ contains no terms in dt.

One can then introduce the equivalence class $[\beta]$ of p-forms β defined as follows,

$$[\beta] = \{\gamma \in \Omega^p(\mathbb{R} \times M) | \beta \wedge dt = \gamma \wedge dt\}\,. \tag{28}$$

Now the exterior derivative can be extended to (28) by putting

$$d[\beta] = \{d\beta' | \beta' \in [\beta]\}.$$

Here, φ will represent a one form on $R \times M$ such that $d\varphi$ is a contact form which determines a self-adjoint vector field according to the given definition. Of course, on account of the decomposition (27), φ can be written in the form

$$\varphi = \varphi^{(1)} + H\, dt,$$

for some $H \in C^\infty(R \times M)$. Thus, $\varphi^{(1)}$ can be thought of as the analogue of the canonical one form $p_i\, dq^i$ which occurs in the phase space description of Hamiltonian systems. If canonical transformations are to preserve the structure of self-adjoint vector fields, some condition must be imposed on the transformation of time.

Definition 3. A map $F \in Diff(R \times M)$ is called a canonical transformation with respect to the equivalence class $[\beta]$ iff

$$F_* dt = dt, \qquad F_*\, d[\beta] = d[\beta].$$

From this, it follows that F is canonical with respect to $[\beta]$ iff $F_*\, dt = dt$ and

$$F_*\, d\varphi = d(\varphi - \sigma\, dt), \tag{29}$$

for some function $\sigma \in C^\infty(R \times M)$.

Proposition 5. *A canonical transformation F is a symmetry of $d\varphi$ if and only if it is a symmetry of the corresponding self-adjoint vector field.*

Proof. Let F be a symmetry of $d\varphi$. If F acts on the first self-adjoint property in (14), we obtain $i_{F_*X}\, d\varphi = 0$, hence there must exist a function $f \in C^\infty(R \times M)$ such that $F_*X = fX$,

$$f =< fX, dt >=< F_*X, dt >=< X, F_*dt >=< X, dt >$$

and when F is canonical, implies $f = 1$. Hence, F is a symmetry of X.

Conversely, if $F_*X = X$, it follows that

$$0 = i_X\, d\varphi = i_{F_*X}\, d\varphi = i_X F_*\, d\varphi.$$

Since F is a canonical transformation, using (29) we have that $F_*\, d\varphi = d\varphi - d\,\sigma \wedge dt$, for some function σ. Since $i_X\, dt = 1$, we have

$$0 = i_X(d\varphi - d\sigma \wedge dt) = -i_X(d\sigma \wedge dt) = -(i_X d\sigma)\, dt + d\sigma.$$

Solving this relation, we have $d\sigma = (i_X\, d\sigma)\, dt$, it then follows that $d\sigma \wedge dt = 0$, thus $F_*d\varphi = d\varphi$ is a symmetry of the contact form $d\varphi$.

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