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A NOTE ON ROTATIONS AND INTERVAL EXCHANGE TRANSFORMATIONS ON 3-INTERVALS

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Abstract: We show that no interval exchange transformation can be a conjugacy between an irrational rotation and an interval exchange transformation on 3-intervals with corresponding permutation $(1, 2, 3) \to (3, 2, 1)$, and rationally independent discontinuity points.

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1. Introduction

Interval exchange transformations were first studied by Katok and Stepin [8], and Keane [5], and are defined as follows. Let $I = [0, 1)$, $n \geq 2$ and $\alpha = (\alpha_1, \cdots, \alpha_n)$ a probability vector with $\alpha_i > 0$. Define $\beta_0 = 0$ and $\beta_i = \sum_{k=1}^{i} \alpha_k$, and set $I_i = [\beta_{i-1}, \beta_i)$. Let τ be a permutation of $\{1, 2, \cdots, n\}$,

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and consider the probability vector $\alpha^\tau = (\alpha_{\tau^{-1}(1)}, \cdots, \alpha_{\tau^{-1}(n)})$. Note that $\alpha_{\tau^{-1}(i)} > 0$ for all i. Let $\beta_0^\tau = 0$ and $\beta_i^\tau = \sum_{k=1}^{i} \alpha_{\tau^{-1}(k)}$, and set $I_i^\tau = [\beta_{i-1}^\tau, \beta_i^\tau)$.

Define $T : I \to I$ by

$$Tx = x - \beta_{i-1} + \beta_{\tau(i)-1}^\tau ,$$

if $x \in I_i$. T is called an (α, τ) interval exchange transformation on n intervals. It is clear that T is invertible, $T\beta_{i-1} = \beta_{\tau(i)-1}^\tau$ and T maps I_i isometrically onto $I_{\tau(i)}^\tau$. Further, T is continuous except possibly at $\{\beta_1, \cdots, \beta_{n-1}\}$. At these points T is right continuous. Note that T is continuous at β_i if and only if $\tau(i+1) = \tau(i) + 1$. In other words, T is discontinuous at β_i if and only if $T\beta_{i-1}, T\beta_i$ do not appear in this order as consecutive terms in the ordered set $\{\beta_0^{\tau_n}, \cdots, \beta_{2n}^{\tau_n}\}$. We say T is in *standard form* if T is discontinuous at β_i for all $i = 1, 2, \cdots, n-1$ or equivalently, if $\tau(i+1) \neq \tau(i)+1$ for all $i = 1, 2, \cdots, n-1$. Notice that any interval exchange transformation on n intervals can be written in standard form as an interval exchange transformation on m intervals with $m \leq n$. Since if T is not in standard form, then T is continuous at β_i for some i, then $\tau(i+1) = \tau(i)+1$, and so T maps the interval $[\beta_{i-1}, \beta_{i+1})$ isometrically onto $[\beta_{\tau(i)-1}^\tau, \beta_{\tau(i)+1}^\tau)$. Thus, we can redefine T on intervals with end points

$$\{\beta_0, \cdots, \beta_{i-1}, \beta_{i+1}, \cdots, \beta_n\}.$$

We repeat this process until all the remaining β's are discontinuity points of T.

The permutation τ corresponding to T is said to be irreducible if

$$\tau(\{1, 2, \cdots, k\}) \neq \{1, 2, \cdots, k\}, \text{ for all } k = 0, 1, \cdots, n-1.$$

Note that if τ is reducible, then T can be decomposed into two interval exchange transformations, one on $[0, \beta_k)$ and the other on $[\beta_k, 1)$. We assume throughout this paper that T is irreducible.

Interval exchange transformations have been studied by several authors. Here we mention few of the known results. In [5], Keane studied the minimality of such transformations, and in [6] questions concerning unique ergodicity were investigated. It is easy to see that if $n = 2$, T corresponds to a rotation and if $n = 3$, then T can be seen as an induced transformation of a rotation. Thus, if the β's are rationally independent, then in both cases T is uniquely ergodic. Keynes and Newton [7], and also Keane [6] gave examples of interval exchange transformations that are not uniquely ergodic. Masur [10], and independently

Veech [13], [14], [15], [16], [17] showed that almost every minimal interval exchange transformation is uniquely ergodic. Later Boshernitzan [2] gave another proof of this result by more elementary means. Some of the spectral properties were studied by Veech in a series of papers [15], [16], [17]. Oseledets [11] and Goodson [4] constructed ergodic interval exchange transformations with simple spectrum. Recently, Berthé, Chekhova and Ferenczi [1] proved that every ergodic interval exchange transformation on three intervals has simple spectrum. The first interval exchange transformation with continuous spectrum was given by Katok and Stepin [8], their example is also an exchange on three intervals. In [1], the authors gave other examples of exchanges on three intervals with continuous spectrum. They asked whether it is possible to construct an example of an exchange on three intervals with discrete spectrum, and hence conjugate to an irrational rotation. This question was answered in the affirmative by Ferenczi, Holton and Zamboni in [3], where they gave a large family of examples. In this paper we show that any such conjugacy with an irrational rotation has infinitely many discontinuities. This is done by using a recent result of Simin Li [9], where he gave necessary and sufficient conditions for an interval exchange transformation to be conjugate to an irrational rotation via a conjugation that is itself an interval exchange.

2. Non-trivial Exchanges on 3-intervals

Let $0 < l < m < 1$ with $1, l, m$ rationally independent. Consider the interval exchange transformation T given by

$$Tx = \begin{cases} x + 1 - l\,, & x \in [0, l), \\ x + 1 - l - m\,, & x \in [l, m), \\ x - m\,, & x \in [m, 1). \end{cases}$$

T corresponds to the permutation $(1,2,3) \to (3,2,1)$. Notice that T is the only interval exchange transformation on 3-intervals which is irreducible and in standard form. Moreover, by a result of Keane [5], T is minimal. We call T a non-trivial exchange transformation on 3-intervals. It is well known that T is an induced transformation of the interval exchange transformation S defined on $[0, 1-l+m)$ by

$$Sx = \begin{cases} x+1-l\,, & x \in [0,m), \\ x-m\,, & x \in [m, 1-l+m). \end{cases}$$

Since after normalization S is isomorphic to an irrational rotation, S is minimal and uniquely ergodic, and hence so is T.

Simin Li [9] gave recently necessary and sufficient conditions for an interval exchange transformation to be conjugate via an interval exchange transformation to an irrational rotation.

Theorem 1. (see [9]) *Let T be an interval exchange transformation, and let $d(T^n)$ be the number of discontinuities of T^n. Then, T is conjugate via an interval exchange to an irrational rotation if and only if:*

(i) T^n is minimal for all $n \geq 1$,

(ii) $\{d(T^n)\}$ is bounded by some integer $N > 0$ and

(iii) there exist $k > 0$ and $M \geq 2^{N^3+3N^2}$ such that $d(T^k) = d(T^{2k}) = \cdots = d(T^{Mk})$.

Theorem 2. *Let T be a non-trivial interval exchange transformation on 3-intervals with rationally independent discontinuity points. Let $D(T^n)$ be the set of discontinuity points of T^n, and let $d(T^n)$ denote the cardinality of $D(T^n)$. Then*

$$D(T^n) = \{T^{-i}l, T^{-j}m : 0 \leq i,j \leq n-1\},$$

and hence, $d(T^n) = 2n$.

Proof. The proof is done by induction on n. The result is true for $n = 1$. Suppose

$$D(T^k) = \{T^{-i}l, T^{-j}m : 0 \leq i,j \leq k-1\},$$

for $k = 1, 2, \cdots, n$. We prove the result for $k = n+1$. Let

$$0 < \beta_1 < \beta_2 < \cdots < \beta_{2n} < 1$$

be the discontinuities of T^n written in increasing order. By the induction hypothesis

$$D(T^n) = \{\beta_i : 1 \leq i \leq 2n\} = \{T^{-i}l, T^{-j}m : 0 \leq i,j \leq n-1\}.$$

Let $\beta_0 = 0$ and $\beta_{2n+1} = 1$. The underlying partition of T^n is given by

$$\mathcal{P}(T^n) = \{[\beta_i, \beta_{i+1}) : i = 0, 1, \cdots 2n\}.$$

Let τ_n be the permutation corresponding to T^n (notice that T^n is an interval exchange transformation). Then

$$T^n\{\beta_0, \beta_1, \cdots, \beta_{2n}\} = \{\beta_0^{\tau_n}, \beta_1^{\tau_n}, \cdots, \beta_{2n}^{\tau_n}\}$$

with $\beta_0 = \beta_0^{\tau_n} = 0$, and $T^n\beta_i = \beta^{\tau_n}_{\tau_n(i+1)-1}$ for $i = 0, 1, \cdots, 2n$. Furthermore, since $1, l$ and m are rationally independent, and each $\beta_i^{\tau_n}$ is a linear combination of $1, l$ and m with integer coefficients, it follows that $l, m \notin \{\beta_0^{\tau_n}, \beta_1^{\tau_n}, \cdots, \beta_{2n}^{\tau_n}\}$. Now invertibility of T implies that $T\beta_0^{\tau_n}, \cdots, T\beta_{2n}^{\tau_n}, Tm, Tl$ are all distinct.

Suppose $l \in (\beta_{r-1}^{\tau_n}, \beta_r^{\tau_n})$, and $m \in (\beta_{s-1}^{\tau_n}, \beta_s^{\tau_n})$. We consider three cases.

Case 1. If $r = s$, then $T^{-n}l, T^{-n}m \in (\beta_{p-1}, \beta_p)$ where $p = {\tau_n}^{-1}(r)$. Since T is an order preserving isometry on $[\beta_{p-1}, \beta_p)$, it follows that $T^{-n}l < T^{-n}m$. The underlying partition of T^{n+1} is then given by

$$\begin{aligned}\mathcal{P}_1(T^{n+1}) = \{&[\beta_0, \beta_1), \cdots, [\beta_{p-2}, \beta_{p-1}), [\beta_{p-1}, T^{-n}l), \\ &[T^{-n}l, T^{-n}m), [T^{-n}m, \beta_p), [\beta_p, \beta_{p+1}), \cdots, [\beta_{2n}, \beta_{2n+1})\}.\end{aligned}$$

To prove the result, we need to show that

$$\{\beta_1, \cdots, \beta_{p-1}, T^{-n}l, T^{-n}m, \beta_p, \cdots, \beta_{2n}\}$$

is the set of discontinuity points of T^{n+1}. Let

$$D_1 = \{\beta_0, \cdots, \beta_{p-1}, T^{-n}l, T^{-n}m, \beta_p, \cdots, \beta_{2n}\}$$

and

$$E_1 = \{\beta_0^{\tau_n}, \cdots, \beta_{r-1}^{\tau_n}, l, m, \beta_r^{\tau_n}, \cdots, \beta_{2n}^{\tau_n}\},$$

both considered as ordered sets. Then $TD_1 = E_1$, and by discontinuity of T^n at β_p we have $T^n\beta_p \neq \beta_r^{\tau_n}$. Further, $\beta_i^{\tau_n} \in (0, l)$ for $1 \leq i \leq r-1$, and $\beta_i^{\tau_n} \in (m, l)$ for $r \leq i \leq 2n,$. Hence,

$$\begin{aligned} T^{n+1}D_1 &= TE_1 \\ &= \{Tm = 0, T\beta_r^{\tau_n}, \cdots, T\beta_{2n}^{\tau_n}, Tl = 1 - m, \\ &\quad T\beta_0^{\tau_n} = 1 - l, T\beta_1^{\tau_n}, \cdots, T\beta_{r-1}^{\tau_n}\}. \end{aligned}$$

Here the elements of TE_1 are listed in increasing order.

We first show that T^{n+1} is discontinuous at β_i for $i \neq p$. To do this, we need to prove that $T^{n+1}\beta_{i-1}$ and $T^{n+1}\beta_i$ do not appear in this order as consecutive terms in TE_1. By assumption, T^n is discontinuous at β_i, hence $T^n\beta_{i-1}$ and $T^n\beta_i$ do not appear as consecutive terms of the form $\beta_j^{\tau_n}, \beta_{j+1}^{\tau_n}$ in E_1. Let $I_0 = [0, l)$, $I_1 = [l, m)$ and $I_2 = [m, 1)$. If $T^n\beta_{i-1}, T^n\beta_i \in I_j$ for some $j = 0, 2$, then since T maps I_j isometrically onto TI_j, it follows that $T^{n+1}\beta_{i-1}$ and $T^{n+1}\beta_i$ cannot appear as consecutive terms in TE_1. If $T^n\beta_{i-1} \in I_j$ and $T^n\beta_i \in I_k$ for $j \neq k$, then either $T^n\beta_{i-1} \in I_0$ and $T^n\beta_i \in I_2$, or $T^n\beta_{i-1} \in I_2$ and $T^n\beta_i \in I_0$. In the first case we get $T^{n+1}\beta_i < 1 - m < T^n\beta_{i-1}$, and in the second case, we get $T^{n+1}\beta_{i-1} < 1 - m < T^{n+1}\beta_i$. Hence, $T^n\beta_{i-1}$ and $T^n\beta_i$ do not appear as consecutive terms of the form $\beta_j^{\tau_n}, \beta_{j+1}^{\tau_n}$ in E_1, and so T^{n+1} is discontinuous at β_i.

Now, the discontinuity of T^n at β_p implies that $T^{n+1}\beta_p \neq T\beta_r^{\tau_n}$, and

$$Tm = 0 < T\beta_r^{\tau_n} < T^{n+1}\beta_p.$$

Hence $T^{n+1}(T^{-n}m) = Tm = 0$ and $T^{n+1}\beta_p$ do not appear as consecutive terms in TE_1. So T^{n+1} is discontinuous at β_p.

The discontinuity of T^{n+1} at $T^{-n}l$ follows from the fact that $T^{n+1}\beta_{p-1}$ is an interior point of TI_2, while $T^{n+1}(T^{-n}l) = 1 - m$ is the left end-point of TI_1. Finally, $T^{n+1}(T^{-n}m) = 0 < T\beta_r^{\tau_n} < 1 - m = T^{n+1}(T^{-n}l)$ implies that T^{n+1} is discontinuous at $T^{-n}m$. Therefore, $D_1 = D(T^{n+1})$.

Case 2. If $r < s$ and $p = \tau_n^{-1}r < \tau_n^{-1}s = q$, then $T^{-n}l \in (\beta_{p-1}, \beta_p)$ and $T^{-n}m \in (\beta_{q-1}, \beta_q)$. The discontinuity of T^n at β_p and β_q implies $T^n\beta_p \neq \beta_r^{\tau_n}$ and $T^n\beta_q \neq \beta_s^{\tau_n}$. The underlying partition of T^{n+1} is easily seen to be

$$\mathcal{P}_2(T^{n+1}) = \{[\beta_0, \beta_1), \cdots, [\beta_{p-2}, \beta_{p-1}), [\beta_{p-1}, T^{-n}l), [T^{-n}l, \beta_p),$$

$$[\beta_p, \beta_{p+1}), \cdots [\beta_{q-1}, T^{-n}m), [T^{-n}m, \beta_q), [\beta_q, \beta_{q+1}), \cdots, [\beta_{2n}, 1)\}.$$

To show the discontinuity of T^{n+1} at $\beta_1, \cdots, \beta_{2n}, T^{-n}l, T^{-n}m$, we consider the ordered sets

$$D_2 = \{\beta_0, \cdots, \beta_{p-1}, T^{-n}l, \beta_p, \cdots, \beta_{q-1}, T^{-n}m, \beta_q, \cdots, \beta_{2n}\}$$

and

$$E_2 = \{\beta_0^{\tau_n}, \cdots, \beta_{r-1}^{\tau_n}, l, \beta_r^{\tau_n}, \cdots, \beta_{s-1}^{\tau_n}, m, \beta_s^{\tau_n}, \cdots, \beta_{2n}^{\tau_n}\}.$$

Then $T^nD_2 = E_2$. Notice that $\beta_1^{\tau_n}, \cdots, \beta_{r-1}^{\tau_n}$ are interior points of I_0, $\beta_r^{\tau_n}, \cdots,$ $\beta_{s-1}^{\tau_n}$ are interior points of I_1 and $\beta_s^{\tau_n}, \cdots, \beta_{2n}^{\tau_n}$ are interior points of I_2. Thus

$$\begin{aligned} T^{n+1}D_2 &= TE_2 \\ &= \{Tm = 0, T\beta_s^{\tau_n}, \cdots, T\beta_{2n}^{\tau_n}, Tl = 1-m, \\ &\quad T\beta_r^{\tau_n}, \cdots, T\beta_{s-1}^{\tau_n}, T\beta_0^{\tau_n} = 1-l, \cdots, T\beta_{r-1}^{\tau_n}\}. \end{aligned}$$

Here the elements of TE_2 are listed in increasing order. We first prove that T^{n+1} is discontinuous at β_i for $i \neq p, q$. If $T^n\beta_{i-1}, T^n\beta_i \in I_j$, then since $T^n\beta_{i-1}, T^n\beta_i$ do not appear as consecutive terms in E_1 and since T is an isometry on I_j, we have that $T^{n+1}\beta_{i-1}$ and $T^{n+1}\beta_i$ are not consecutive terms of E_2, and thus T^{n+1} is discontinuous at β_i. If $T^n\beta_i \in I_j$ and $T^n\beta_{i-1} \in I_k$ for $k \neq j$, then we consider several cases.

- If $T^n\beta_i \in I_2$ and $T^n\beta_{i-1} \in I_0$ or I_1, then since $T^n\beta_{i-1} \neq l$ we have $T^{n+1}\beta_i < 1 - m < T^{n+1}\beta_{i-1}$.
- If $T^n\beta_i \in I_1$ and $T^n\beta_{i-1} \in I_2$, then since $T^n\beta_i \neq l$ it follows that $T^{n+1}\beta_{i-1} < 1 - m < T^{n+1}\beta_i$.
- If $T^n\beta_i \in I_1$ and $T^n\beta_{i-1} \in I_0$, then $T^{n+1}\beta_i < T^{n+1}\beta_{i-1}$.
- If $T^n\beta_i \in I_0$ and $T^n\beta_{i-1} \in I_1$, then since $i \neq q$ we have $T^{n+1}\beta_{i-1} < T\beta_{s-1}^{\tau_n} < T\beta_0^{\tau_n} \leq T^{n+1}\beta_i$.
- If $T^n\beta_i \in I_0$ and $T^n\beta_{i-1} \in I_2$, then $T^{n+1}\beta_{i-1} < 1 - m < T^{n+1}\beta_i$.

In all the above cases we see that T^{n+1} is not continuous at β_i.

The discontinuity of T^{n+1} at β_p and β_q follows from the fact that $T^{n+1}\beta_p \neq T\beta_r^{\tau_n}$ and $T^{n+1}\beta_q \neq T\beta_s^{\tau_n}$, so that neither $T^{n+1}\beta_p$ and $T^{n+1}(T^{-n}l)$ nor $T^{n+1}\beta_q$ and $T^{n+1}(T^{-n}m)$ appear as consecutive terms in TE_2. Finally, from $T^{n+1}(T^{-n}l) = 1 - m < 1 - l < T\beta_{r-1}^{\tau_n}$ and $T^{n+1}(T^{-n}m) = 0 < 1 - m < T\beta_{s-1}^{\tau_n}$ we have that T^{n+1} is discontinuous at $T^{n+1}(T^{-n}l)$ and $T^{n+1}(T^{-n}m)$. Thus, $D_2 = D(T^{n+1})$.

Case 3. If $r < s$ and $p = \tau_n^{-1}r > \tau_n^{-1}s = q$, then the underlying partition of T^{n+1} is given by

$$\mathcal{P}_3(T^{n+1}) = \{[\beta_0, \beta_1), \cdots, [\beta_{q-2}, \beta_{q-1}), [\beta_{q-1}, T^{-n}m), [T^{-n}m, \beta_q), [\beta_q, \beta_{q+1}, \cdots, [\beta_{p-2}, \beta_{p-1}), [\beta_{p-1}, T^{-n}l), [T^{-n}l\beta_p), [\beta_p, \beta_{p+1}), \cdots, [\beta_{2n}, 1)\}.$$

Let

$$D_3 = \{\beta_0, \cdots, \beta_{q-1}, T^{-n}m, \beta_q, \cdots, \beta_{p-1}, T^{-n}l, \beta_p, \cdots, \beta_{2n}\}$$

and

$$E_3 = \{\beta_0^{\tau_n}, \cdots, \beta_{r-1}^{\tau_n}, l, \beta_r^{\tau_n}, \cdots, \beta_{s-1}^{\tau_n}, m, \beta_s^{\tau_n}, \cdots, \beta_{2n}^{\tau_n}\}.$$

Then

$$\begin{aligned} T^{n+1}D_3 &= TE_3 \\ &= \{Tm = 0, T\beta_s^{\tau_n}, \cdots, T\beta_{2n}^{\tau_n}, Tl = 1 - m, \\ &\quad T\beta_r^{\tau_n}, \cdots, T\beta_{s-1}^{\tau_n}, T\beta_0^{\tau_n} = 1 - l, \cdots, T\beta_{r-1}^{\tau_n}\}. \end{aligned}$$

The elements of D_3, E_3 and TE_3 are listed in increasing order. A similar argument as in the above two cases shows that $D_3 = D(T^{n+1})$. Thus, the theorem is proved.

Theorem 3. *No interval exchange transformation can be a conjugacy between an irrational rotation and a non-trivial interval exchange transformation on 3-intervals with rationally independent discontinuity points.*

Proof. By Theorem 2 and Li's theorem, the result follows.

Remark. By unique ergodicity, the above theorem holds if we replace conjugacy by measure preserving isomorphism.

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