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ON PRODUCT CONNECTION THEOREMS FOR MARKOV CHAINS

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Abstract: This paper addresses the problem of constructing multidimensional Markov chains with product form steady state distribution. Product connection theorems are established which guarantee that the product $\prod_\nu \pi^{(\nu)}_{n_\nu}$ of steady state probabilities $\pi^{(\nu)}_{n_\nu}$ related to ergodic Markov chains $\mathcal{X}^{(\nu)} = \{X^{(\nu)}(t) : t \in T\}$ represents the steady state probability $p(\mathfrak{n}) = p(n_1, n_2, \ldots)$ of an ergodic multidimensional Markov chain of random vectors $X^{(\nu)}(t)$, irrespectively of dependency relations. Such results are closely related to statements about product form queueing networks. In fact, it is shown that the theorems of Jackson and Gordon-Newell fit into this framework, and the same is true with respect to BCMP-type queueing networks, although not explicitely discussed here. General product connection theorems, in principle, may form the basis for discovering product form solutions for a wider class of queueing networks.

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1. Introduction

Networks of queues with Poisson input processes and phase type service time distributions often are characterized by the peculiarity that, in equilibrium, the detailed state processes of isolated queues behave as independent Markov chains. A manifestation of this feature is given in form of the well known statements of Jackson, Gordon and Newell, and Baskett et al. on product form queueing networks. They all address a phenomenon which may be described by the fact that, under certain circumstances, the joint distribution of *strongly depending* random variables $X^{(1)}(t), \ldots, X^{(M)}(t)$ may assume product form as $t \to \infty$.

The aim of this note is to present sufficient conditions for this to occur. In particular, we shall show that for an ergodic Markov chain $\mathcal{L}$ of random vectors to possess product form solution it is sufficient, that its transition kernel is formed by state dependent linear forms of kernel elements of the component processes. This property applies, as we shall prove, not only to product form queueing networks with Poisson input, but also to certain *adaptive* networks in random environment. For example, when service processes in a Jackson network with MMPP input[1] are able to adapt proportionally to exogenous rate variations, the network generator matrix is proportional to that of a classical Jackson network. A similar result has been established by Zhu [9] by means of reversibility properties.

Our approach is based on an appropriate construction of the generator L of an M-dimensional Markov process $\mathcal{L} = \{(X^{(1)}(t), \ldots, X^{(M)}(t)) : t \geq 0\}$, where the processes $\{X^{(\nu)}(t) : t \geq 0\}$ are ergodic Markov chains. Adressing the general problem of determining product form queueing networks with non-Poissonian input, one of the chains $X^{(\nu)}(t)$ can be identified as the phase process of some Markovian arrival process (MAP), whereas the other processes are taken as state processes of the respective network stations. An important particularity is the interpretation of $\mathcal{L}$ in view of constructive steps. If, for example, a generator L is constructed in a way such that the corresponding chain $\mathcal{L}$ has product form steady state distribution, the interpretation of $\mathcal{L}$ to be the state process of the network under MAP input, in general, is not possible. In fact, an elegant way to find a solution for queueing networks with fixed service time distributions and MAP input seems to be outside the limits of what is possible. This is mainly due to the fact that no closed form expression is known for the output process of queueing stations with MAP input [1]. Nevertheless, what we can present is a product form queueing network with input rates varying

[1] MMPP = Markov modulated Poisson process.

under *control* of a MAP, and with correspondingly varying service rates.

The paper is organized as follows. In Section 2 the basic product connection theorems are established together with some generalization which possibly may form a basis for discovering product form conditions for a wider class of queueing networks. Section 3 is devoted to examples, i.e. to the product form theorems of Jackson and Gordon-Newell, which are reformulated in terms of product connection theorems. In Section 4 it is demonstrated how to construct a product form queueing network with non-Poissonian input. Section 5 gives a short resumé.

2. Product Connection Properties

We consider a family of M ergodic homogeneous Markov chains $\mathcal{X}^{(\nu)} = \{X^{(\nu)}(t) : t \in T\}$ with state spaces $E^{(\nu)} \subset \mathbb{N}_0$, transition intensity matrices $A^{(\nu)} = ((\alpha^{(\nu)}_{n_\nu\, n^*_\nu}))_{n_\nu\, n^*_\nu \in E^{(\nu)}}$, and steady state distributions $\pi^{(\nu)} = (\pi^{(\nu)}_0, \pi^{(\nu)}_1, \ldots)$, $\nu = 1, \ldots, M$. The term *transition intensity matrix* has to be understood in the following way: If a chain $\mathcal{X}^{(\nu)}$ is a discrete time chain with transition probability matrix $\tilde{A}^{(\nu)}$, then its transition intensity matrix $A^{(\nu)}$ is defined as $A^{(\nu)} = \tilde{A}^{(\nu)} - \mathbf{I}$ ($\mathbf{I}$ the identity matrix). On the other hand, if $\mathcal{X}^{(\nu)}$ is a continuous time chain, then with the transition intensity matrix $A^{(\nu)}$ we mean its generator matrix. Consequently, $\pi^{(\nu)} A^{(\nu)} = \mathbf{o}$ for any ergodic chain $\mathcal{X}^{(\nu)}$ with steady state probability vector $\pi^{(\nu)}$ ($\mathbf{o}$ the zero vector). Notice that any chain $\mathcal{X}^{(\nu)}$, $\nu \in \{1, \ldots, M\}$, is irreducible iff for each pair $(n_\nu, n^*_\nu) \in E^{(\nu)} \times E^{(\nu)}$ with $n_\nu \neq n^*_\nu$ there are states $n_{\nu_0} = n_\nu,\ n_{\nu_1}, \ldots, n_{\nu_r},\ n_{\nu_{r+1}} = n^*_\nu$ with $r \geq 0$ and $n_{\nu_j} \neq n_{\nu_{j+1}}$ for each $j \in \{0, \ldots, r\}$, such that $\prod_{j=0}^{r} \alpha^{(\nu)}_{n_{\nu_j}\, n_{\nu_{j+1}}} > 0$. Throughout this note it is assumed that either all chains $\mathcal{X}^{(\nu)}$ are irreducible discrete time chains ($T \subset \mathbb{N}_0$) with finite state spaces, or irreducible continuous time chains ($T \subset \mathbb{R}$) with countable state spaces. Additionally, it is taken for granted that the regularity conditions

$$0 < |\alpha^{(\nu)}_{n_\nu\, n_\nu}| < \infty \quad \text{for } \ n_\nu < \infty, \quad \nu \in \{1, \ldots, M\} \tag{1}$$

are satisfied.

2.1. Notation

Let $\mathbb{R}_+$ denote the set of positive reals, $\mathbb{N}_0$ the set of non-negative integers. We set $E = E^{(1)} \times \ldots \times E^{(M)}$ and $\mathcal{M} = \{1, \ldots, M\}$, and use the shortcuts

$\bigotimes_{\nu=1}^{k} E^{(i_\nu)} = E^{(i_1)} \times \ldots \times E^{(i_k)}$, $\bigotimes_{j \notin \{i_1,\ldots,i_k\}} E^{(j)} = E^{\overline{(i_1 \ldots i_k)}}$, and $\mathfrak{n}^{\overline{(i_1 \ldots i_k)}}$ for an element of $E^{\overline{(i_1 \ldots i_k)}}$ with components n_j equal to those of $\mathfrak{n} = (n_1, \ldots, n_M)$ for $j \notin \{i_1, \ldots, i_\ell\}$. For any given subset $\{i_1, \ldots, i_k\}$ of $\mathcal{M}$ of cardinality k let

$$\psi^{(i_1 \ldots i_k)} : E^{\overline{(i_1 \ldots i_k)}} \longrightarrow \mathbb{R}_+ \cup \{0\}$$

denote a bounded non-negative function. For given $\psi^{(i_1 \ldots i_k)}$ we set

$$\xi^{i_1 \ldots i_k}(\mathfrak{n}) = (-1)^k \, \psi^{(i_1 \ldots i_k)}(\mathfrak{n}^{\overline{(i_1 \ldots i_k)}}) \prod_{\nu=1}^{k} \alpha^{(i_\nu)}_{n_{i_\nu} n_{i_\nu}}$$

for any $\mathfrak{n} = (n_1, \ldots, n_M) \in E$, and define generalized elementary symmetric functions by

$$\eta_k(\mathfrak{n}) = \sum_{\Omega_k} \xi^{i_1 \ldots i_k}(\mathfrak{n}) \ ,$$

where Ω_k denotes the set of all (different) subsets of $\mathcal{M}$ of cardinality k. For example, if $M = 5$, then (here we omit the argument $\mathfrak{n}$)

$$\begin{aligned}
\eta_2 &= \xi^{12} + \xi^{13} + \xi^{14} + \xi^{15} + \xi^{23} + \xi^{24} + \xi^{25} + \xi^{34} + \xi^{35} + \xi^{45} \ , \\
\eta_3 &= \xi^{123} + \xi^{124} + \xi^{125} + \xi^{134} + \xi^{135} + \xi^{145} + \xi^{234} \\
&\quad + \xi^{235} + \xi^{245} + \xi^{345} \ ,
\end{aligned}$$

2.2. Simple Product Theorems

Consider the set Ψ of mappings $\psi : \mathfrak{P}(\mathcal{M}) \longrightarrow E^*$ from the set of all subsets of $\mathcal{M}$ into the set of all non-negative and bounded real functions over E. We set $\psi(\omega_\ell) = \psi^{(i_1 \ldots i_\ell)}$ for $\omega_\ell = \{i_1, \ldots, i_\ell\} \in \Omega_\ell \subset \mathfrak{P}(\mathcal{M})$. Let $\mathcal{F} = \{\mathcal{X}^{(\nu)} : \nu \in \mathcal{M}\}$ be a family of ergodic irreducible regular Markov chains $\mathcal{X}^{(\nu)} = \{X^{(\nu)}(t) : t \in T\}$ with state spaces $E^{(\nu)} \subset \mathbb{N}_0$, transition intensity matrices $A^{(\nu)} = ((\alpha^{(\nu)}_{n_\nu n^*_\nu}))_{n_\nu n^*_\nu \in E^{(\nu)}}$, and steady state distributions $\pi^{(\nu)} = (\pi^{(\nu)}_0, \pi^{(\nu)}_1, \ldots)$.

Definition 1. The family $\mathcal{F}$ is called a *cadence family*, if there is a mapping $\psi = \psi(\mathcal{F}) \in \Psi$, which satisfies the following conditions:

(1) For any pair $(\mathfrak{n}, \mathfrak{n}^*) \in E \times E$ there is a sequence $\mathfrak{n}_0, \ldots, \mathfrak{n}_{s+1}$ of elements of E with $\mathfrak{n}_0 = \mathfrak{n}$, $\mathfrak{n}_{s+1} = \mathfrak{n}^*$, and $\mathfrak{n}_j \neq \mathfrak{n}_{j+1}$, $\mathfrak{n}_j$ and $\mathfrak{n}_{j+1}$ differing only in

components $n_{j,i_1},\dots,n_{j,i_{\ell(j)}}$ for each $j\in\{0,\dots,s\}$, such that

$$\prod_{j=0}^{s}\psi^{(i_1\dots i_{\ell(j)})}(\mathfrak{n}_j^{\overline{(i_1\dots i_{\ell(j)})}})\prod_{\nu=1}^{\ell(j)}\alpha^{(i_\nu)}_{n_{j,i_\nu}\,n_{j+1,i_\nu}}>0\ , \tag{2}$$

where $\mathfrak{n}_j^{\overline{(i_1\dots i_{\ell(j)})}}$ is built from $\mathfrak{n}_j=(n_{j1},\dots,n_{jM})$ by deleting the components n_{ji_ν} for $\nu=1,\dots,\ell(j)$.

(2) In case that $\mathcal{F}$ is a family of discrete time Markov chains it is assumed that all state spaces are finite and the sum over all functional values $\psi^{(i_1\dots i_k)}(\mathfrak{n}^{\overline{(i_1\dots i_k)}})$ is 1, i.e.

$$\sum_{\ell=1}^{M}\sum_{\Omega_\ell}\psi^{(i_1\dots i_k)}(\mathfrak{n}^{\overline{(i_1\dots i_k)}})=1\ . \tag{3}$$

Lemma 1. *Let $\mathcal{F}$ be a cadence family of discrete time Markov chains with associate mapping $\Psi(\mathcal{F})$. Define the matrix $L=((L_{\mathfrak{n}\,\mathfrak{n}^*}))_{\mathfrak{n}\,\mathfrak{n}^*\in E}$ through*

$$L_{\mathfrak{n}\,\mathfrak{n}^*}= \tag{4}$$

$$\begin{cases}\psi^{(i_1,\dots,i_\ell)}(\mathfrak{n}^{\overline{(i_1\dots i_\ell)}})\cdot\prod_{\nu=1}^{\ell}\alpha^{(i_\nu)}_{n_{i_\nu}\,n^*_{i_\nu}}, & \text{if } n^*_{i_\nu}\neq n_{i_\nu} \text{ for each } i_\nu\in\{i_1,\dots,i_\ell\},\\ & n^*_j=n_j \text{ for } j\notin\{i_1,\dots,i_\ell\},\ 1\le\ell\le M,\\ -\sum_{\ell=1}^{M}\eta_\ell(\mathfrak{n}), & \text{if } \mathfrak{n}^*=\mathfrak{n}.\end{cases}$$

Then the sum $\sum_{\mathfrak{n}^*}L_{\mathfrak{n}\,\mathfrak{n}^*}$ equals zero, and the matrix L is to be interpreted as the transition intensity matrix of an irreducible Markov chain $\mathcal{L}$ with state space E, and $0<|L_{\mathfrak{n}\,\mathfrak{n}}|<\infty$ for each $\mathfrak{n}$ with finite components.

Proof. 1. We introduce the notation

$$\begin{aligned}S^{i_1\dots i_h}_{j_1\dots j_\ell} &:= \sum_{n^*_{i_1}\neq n_{i_1}}\dots\sum_{n^*_{i_h}\neq n_{i_h}}\sum_{n^*_{j_1}}\dots\sum_{n^*_{j_\ell}}L_{\mathfrak{n}\,\mathfrak{m}^*}\ ,\\ S_{j_1\dots j_\ell} &:= \sum_{n^*_{j_1}}\dots\sum_{n^*_{j_\ell}}L_{\mathfrak{n}\,\mathfrak{m}^*}\ ,\\ S^{i_1\dots i_h} &:= \sum_{n^*_{i_1}\neq n_{i_1}}\dots\sum_{n^*_{i_h}\neq n_{i_h}}L_{\mathfrak{n}\,\mathfrak{m}^*}\ ,\end{aligned}$$

where $\mathfrak{m}^*=(m^*_1,\dots,m^*_M)$ with $m^*_\nu=n^*_\nu$ for $\nu\in\{i_1,\dots,i_h\}$ or $\nu\in\{j_1,\dots,j_\ell\}$, and $m^*_\nu=n_\nu$ in all other cases. Clearly, if there is no such $h\ge1$ and no such

$\ell \geq 1$, then $\mathfrak{m}^* = \mathfrak{n}$, and the corresponding sum degenerates to $S = L_{\mathfrak{n}\,\mathfrak{n}}$. Notice that, according to (4), for any $h \leq M$ the sum $S^{i_1\ldots i_h}$ takes the form

$$S^{i_1\ldots i_h} = (-1)^h \, \psi^{(i_1\ldots i_h)}(\mathfrak{n}^{\overline{(i_1\ldots i_h)}}) \prod_{\nu=1}^{h} \alpha^{(i_\nu)}_{n_{i_\nu}\,n_{i_\nu}} = \xi^{i_1\ldots i_h}(\mathfrak{n}) \,,$$

since $\sum_{n_\nu{}^* \neq n_\nu} \alpha_{n_\nu\, n_\nu^*} = -\,\alpha_{n_\nu\, n_\nu}$ for all $\nu \in \{1, \ldots, M\}$. With this symbolism in mind we may write

$$\begin{aligned}
\sum_{\mathfrak{n}^*} L_{\mathfrak{n}\,\mathfrak{n}^*} &= S_{1\ldots M} = S^1_{2\ldots M} + S_{2\ldots M} \\
&= S^{12}_{3\ldots M} + S^1_{3\ldots M} + S^2_{3\ldots M} + S_{3\ldots M} \\
&= S^{123}_{4\ldots M} + S^{12}_{4\ldots M} + S^{13}_{4\ldots M} + S^1_{4\ldots M} \\
&\quad + S^{23}_{4\ldots M} + S^2_{4\ldots M} + S^3_{4\ldots M} + S_{4\ldots M} \\
&\vdots \\
&= \sum_{\ell=1}^{M} \eta_\ell(\mathfrak{n}) + L_{\mathfrak{n}\,\mathfrak{n}} = 0 \,,
\end{aligned}$$

and L can be interpreted as the transition intensity matrix of some M-dimensional Markov chain $\mathcal{L}$, where $0 < \sum_{\ell=1}^{M} |\eta_\ell(\mathfrak{n})| = |L_{\mathfrak{n}\,\mathfrak{n}}| < \infty$ for finite $\mathfrak{n}$.

2. We have to show that $\mathcal{L}$ is irreducible. This is a consequence of property (1) of a cadence family: For any pair $(\mathfrak{n}, \mathfrak{n}^*) \in E \times E$ there is a sequence $\mathfrak{n}_0, \ldots, \mathfrak{n}_{s+1}$ of elements of E, such that $\prod_{j=0}^{s} L_{\mathfrak{n}_j \mathfrak{n}_{j+1}} > 0$ according to (4) and (2). □

As mentioned in the proof above, the property (1) of a cadence family represents a sufficient condition for proving irreducibility of $\mathcal{L}$. This condition could be replaced by a weaker condition, if the set of Markov chains $\mathcal{X}^{(\nu)}$ has special properties: We say, that a finite set of irreducible Markov chains $\mathcal{X}^{(\nu)}$ with state spaces $E^{(\nu)}$ $(\nu = 1, \ldots, M)$ has the property of *synchronous reachability*, if for any pair $\mathfrak{n} = (n_1, \ldots, n_M)$, $\mathfrak{n}^* = (n_1^*, \ldots, n_M^*)$ there is a common integer $s = s(\mathfrak{n}, \mathfrak{n}^*) \geq 0$, such that one can find a sequence $\mathfrak{n}_0 = \mathfrak{n}, \mathfrak{n}_1, \ldots, \mathfrak{n}_{s+1} = \mathfrak{n}^*$ with $\mathfrak{n}_j \neq \mathfrak{n}_{j+1}$ $\forall\ j \in \{0, \ldots, s\}$, for which the product $\prod_{j=0}^{s} \alpha^{(\nu)}_{n_{\nu_j}\, n_{\nu_{j+1}}}$ is positive (here $\mathfrak{n}_j = (n_{1_j}, \ldots, n_{M_j})$ $\forall\ j \in \{0, \ldots, s\}$). It is not excluded that for some intermediate index ν_j, corresponding to chain $\mathcal{X}^{(\nu)}$, a previously reached state may be repeatedly acquired; in particular, $n_{\nu_j} = n_\nu$ or $n_{\nu_j} = n_\nu^*$ is possible for $j \in \{2, \ldots, s-1\}$ if $s \geq 1$. So, in many practical cases, synchronous reachability will be on hand.

Lemma 2. *Let $\mathcal{F}$ be a family of discrete time Markov chains with the property of synchronous reachability, and assume that all state spaces $E^{(\nu)}$,*

$\nu \in \mathcal{M}$, are of cardinality ≥ 3. If there is some fixed $k \in \mathcal{M}$, such that $\psi^{(i_1 \ldots i_k)}(\mathfrak{n}^{\overline{(i_1 \ldots i_k)}}) > 0$ for each $\mathfrak{n}^{\overline{(i_1 \ldots i_k)}} \in E^{\overline{(i_1 \ldots i_k)}}$ and each subset $\{i_1, \ldots, i_k\}$ of cardinality k, then the matrix $L = ((L_{\mathfrak{n}\,\mathfrak{n}^*}))_{\mathfrak{n}\,\mathfrak{n}^* \in E}$, defined through (4), is the transition intensity matrix of some irreducible regular Markov chain $\mathcal{L}$ with state space E.

Proof. Based on the statement of Lemma 1, only irreducibility of $\mathcal{L}$ has to be shown.

1. Let $\mathfrak{n} = (n_1, \ldots, n_M)$ and $\mathfrak{n}^* = (n_1^*, \ldots, n_M^*)$ be state vectors of the chain $\mathcal{L}$ with exactly k different components $n_{i_1} \neq n_{i_1}^*, \ldots, n_{i_k} \neq n_{i_k}^*$ and $M-k$ equal components, such that $\psi^{(i_1,\ldots,i_k)}(\mathfrak{n}^{\overline{(i_1 \ldots i_k)}}) > 0$, and $L_{\mathfrak{n}\,\mathfrak{n}^*} = \psi^{(i_1,\ldots,i_k)}(\mathfrak{n}^{\overline{(i_1 \ldots i_k)}}) \cdot \prod_{\nu=1}^{k} \alpha^{(i_\nu)}_{n_{i_\nu}\, n^*_{i_\nu}}$. For $\prod_{\nu=1}^{k} \alpha^{(i_\nu)}_{n_{i_\nu}\, n^*_{i_\nu}} > 0$ reachability of $\mathfrak{n}^*$ from $\mathfrak{n}$ is obvious, so let $\prod_{\nu=1}^{k} \alpha^{(i_\nu)}_{n_{i_\nu}\, n^*_{i_\nu}} = 0$; then, since each chain $\mathcal{X}^{(\nu)}$ is irreducible, there is, for any state $n_{i_\ell} \in E^{(i_\ell)}$, an integer $s^{(\ell)}$ and a sequence of states $n_0 = n_{i_\ell},\ n_1, \ldots, n_{s^{(\ell)}},\ n_{s^{(\ell)}+1} = n^*_{i_\ell}$ with $s^{(\ell)} \geq 0$ and $n_j \neq n_{j+1}\ \forall\ j \in \{0, \ldots, s^{(\ell)}\}$, such that we have $\alpha^{(i_\ell)}_{n_0\, n_1} \cdot \alpha^{(i_\ell)}_{n_1\, n_2} \cdot \ldots \cdot \alpha^{(i_\ell)}_{n_{s^{(\ell)}}\, n_{s^{(\ell)}+1}} > 0$. Due to the property of synchronous reachability we can assume, that $s^{(\ell)}$ takes the same value $s^{(\ell)} = s$ for all $\ell \in \{1, \ldots, k\}$. Consequently, there are corresponding state vectors $\mathfrak{n}_0, \ldots, \mathfrak{n}_{s+1}$ with $\mathfrak{n}_0 = \mathfrak{n}$, $\mathfrak{n}_{s+1} = \mathfrak{n}^*$, which differ pairwise from each other in exactly k components and define a positive product $L_{\mathfrak{n}_0\, \mathfrak{n}_1} \cdot \ldots \cdot L_{\mathfrak{n}_s\, \mathfrak{n}^*} > 0$, implying that $\mathfrak{n}^*$ is reachable from $\mathfrak{n}$.

2. Assume that two state vectors $\mathfrak{n}$ and $\mathfrak{n}^*$ of $\mathcal{L}$ differ from each other in $h \neq k$ components. If $h > k$, then we can consider a sequence of $r + 1 \geq 2$ state vectors $\mathfrak{n}_0^*, \mathfrak{n}_1^*, \ldots, \mathfrak{n}_r^*$ with $\mathfrak{n}_0^* = \mathfrak{n}$, which differ pairwise from each other in exactly k components, whereas the last vector $\mathfrak{n}_r^*$ differs from $\mathfrak{n}^*$ in less than k components. Due to the previous reasoning, $\mathfrak{n}_r^*$ is reachable from $\mathfrak{n}$, i.e. $\prod_{h=0}^{r-1} L_{\mathfrak{n}_h^*\, \mathfrak{n}_{h+1}^*} > 0$. Let $m_{i_1}^* \neq n_{i_1}^*, \ldots, m_{i_\ell}^* \neq n_{i_\ell}^*$ be the $\ell < k$ components of $\mathfrak{n}_r^*$, which are different from their "peer" components in $\mathfrak{n}^*$. Bearing in mind, that $k \leq M$, and that it was required that every state space $E^{(\nu)}$ has at least 3 elements ($\nu = 1, \ldots, M$), we may arbitrarily select $k-\ell$ components $\tilde{n}_{j_\kappa} \in E^{(j_\kappa)}$, $\tilde{n}_{j_\kappa} \neq n_{j_\kappa}$, $\tilde{n}_{j_\kappa} \neq n^*_{j_\kappa}$ for $j_\kappa \notin \{i_1, \ldots, i_\ell\}$ ($\kappa = 1, \ldots, k - \ell$), such that a state vector $\mathfrak{x} = (x_1, \ldots, x_M) \in E$ can be constructed with $x_{i_\kappa} = n_{i_\kappa} \neq n^*_{i_\kappa}$ for $\kappa \in \{1, \ldots, \ell\}$, $x_{j_\kappa} = \tilde{n}_{j_\kappa}$ for $\kappa \in \{1, \ldots k - \ell\}$, and $x_\nu = n_\nu$ for all other components. Then $\mathfrak{n}_r^*$ and $\mathfrak{x}$ differ in exactly k components, and the same is true for $\mathfrak{x}$ and $\mathfrak{n}^*$. Consequently, $L_{\mathfrak{n}_r^*\, \mathfrak{x}} L_{\mathfrak{x}\, \mathfrak{n}^*} > 0$, yielding $\prod_{h=0}^{r-1} L_{\mathfrak{n}_h^*\, \mathfrak{n}_{h+1}^*} L_{\mathfrak{n}_r^*\, \mathfrak{x}} L_{\mathfrak{x}\, \mathfrak{n}^*} > 0$, which implies that $\mathfrak{n}^*$ is reachable from $\mathfrak{n}$. □

Remark 1. Everything mentioned so far remains true also for continuous time Markov processes if we propose that in Lemma 1 all functions $\psi^{(i_1 \dots i_k)}(\mathfrak{n})$ are zero for $k \geq 2$. This setting corresponds to the fact that the simultaneous occurrence of more than one Markov event has probability zero.

Remark 2. As has been shown in [8], the requirement of synchronous reachability for the family $\mathcal{F}$ of Markov chains is not at all necessary if all functions $\psi^{(i_1 \dots i_k)}(\mathfrak{n})$ are zero for $k \geq 2$. Also, it it not necessary in this case to require, that state spaces $E^{(\nu)}$ are of cardinality greater or equal to 3. Consequently, in this case, Lemma 2 holds for a set of non-negative functions $\psi^{(i)} : E^{(\bar{i})} \longrightarrow \mathbb{R}_+ \cup \{0\}$ with $\sum_{i=1}^{M} \psi^{(i)}(\mathfrak{n}^{(\bar{i})}) = c > 0, \;\; \forall\, \mathfrak{n} \in E$, and for an arbitrary family of regular irreducible ergodic Markov chains, which either are all discrete time chains (in which case $c = 1$), or are all continuous time chains. In general, when all functions $\psi^{(i_1 \dots i_\ell)}$ are the null-function $\psi^{(i_1 \dots i_\ell)} \equiv 0$ for $\ell > k_0$ for some $k_0 < M$ ($k_0 \geq 1$), then all generalized elementary symmetric functions η_ℓ are zero for $\ell > k_0$. In particular, if $\psi^{(i_1 \dots i_\ell)} \not\equiv 0$ only for $\ell = k_0$, then $L_{\mathfrak{n}\mathfrak{n}} = -\eta_{k_0}(\mathfrak{n})$, and $\sum_{\mathfrak{n}^* \neq \mathfrak{n}} L_{\mathfrak{n}^*\mathfrak{n}} = \eta_{k_0}(\mathfrak{n})$.

Theorem 1. *The Markov chain $\mathcal{L}$, defined through its transition intensity matrix (4), is ergodic with stationary distribution $\mathcal{P} = \{p(\mathfrak{n}) : \mathfrak{n} \in E\}$ given by*

$$p(\mathfrak{n}) = \prod_{i=1}^{M} \pi_{n_i}^{(i)} \tag{5}$$

for $\mathfrak{n} = (n_1, \dots, n_M) \in E$, $n_i \in E^{(i)}$.

Proof. Obviously, in case of a cadence family of discrete time Markov chains, the chain $\mathcal{L}$ is ergodic due to the finiteness of the state space. Further, for both, the discrete time and the continuous time version, the chain $\mathcal{L}$ is irreducible and regular according to Lemma 1. It remains to show that $p(\mathfrak{n})$ is a positive stationary distribution of $\mathcal{L}$ (implying ergodicity), i.e. that

$$\sum_{\mathfrak{n}} \prod_{i=1}^{M} \pi_{n_i}^{(i)} L_{\mathfrak{n}\,\mathfrak{n}^*} = 0 = \prod_{i=1}^{M} \pi_{n_i^*}^{(i)} \sum_{\mathfrak{n}} L_{\mathfrak{n}^*\,\mathfrak{n}}\,.$$

Let us first assume that all Markov processes are discrete time chains. By formally repeating the steps in the proof of Lemma 1 we introduce the following corresponding notation:

$$\begin{aligned}
\sigma^{i_1\ldots i_h}_{j_1\ldots j_\ell} &:= \sum_{n_{i_1}\neq n^*_{i_1}} \cdots \sum_{n_{i_h}\neq n^*_{i_h}} \sum_{n_{j_1}} \cdots \sum_{n_{j_\ell}} \prod_{\nu=1}^{h} \pi^{(i_\nu)}_{n_{i_\nu}} \prod_{\mu=1}^{\ell} \pi^{(j_\mu)}_{n_{j_\mu}} \\
&\quad \times \prod_{\substack{\kappa\notin\{i_1,\ldots,i_h\}\\ \kappa\notin\{j_1,\ldots,j_\ell\}}} \pi^{(\kappa)}_{n^*_\kappa} \cdot L_{\mathfrak{m}\,\mathfrak{n}^*} \ , \\
\sigma_{j_1\ldots j_\ell} &:= \sum_{n_{j_1}} \cdots \sum_{n_{j_\ell}} \prod_{\mu=1}^{\ell} \pi^{(j_\mu)}_{n_{j_\mu}} \prod_{\kappa\notin\{j_1,\ldots,j_\ell\}} \pi^{(\kappa)}_{n^*_\kappa} \cdot L_{\mathfrak{m}\,\mathfrak{n}^*} \ , \\
\sigma^{i_1\ldots i_h} &:= \sum_{n_{i_1}\neq n^*_{i_1}} \cdots \sum_{n_{i_h}\neq n^*_{i_h}} \prod_{\nu=1}^{h} \pi^{(i_\nu)}_{n_{i_\nu}} \prod_{\kappa\notin\{i_1,\ldots,i_h\}} \pi^{(\kappa)}_{n^*_\kappa} \cdot L_{\mathfrak{m}\,\mathfrak{n}^*} \ ,
\end{aligned}$$

where this time $\mathfrak{m} = (m_1, \ldots, m_M)$ is a state vector with $m_\kappa = n_\kappa$ for $\kappa \in \{i_1, \ldots, i_h\}$ or $\kappa \in \{j_1, \ldots, j_\ell\}$, and $m_\kappa = n^*_\kappa$ in all other cases. If there is no such $h \geq 1$ and no such $\ell \geq 1$, then $\mathfrak{m} = \mathfrak{n}^*$, and the corresponding sum degenerates to $\sigma = \prod_{\kappa=1}^{M} \pi^{(\kappa)}_{n^*_\kappa} \cdot L_{\mathfrak{n}^*\,\mathfrak{n}^*}$. For constant $\mathfrak{n}^* \in E$, in order to shorten notation let us write

$$\psi^{(i_1,\ldots,i_k)}(\mathfrak{n}^{*\overline{(i_1\ldots i_k)}}) =: c^*_{i_1\ldots i_k} \ .$$

Then any sum $\sigma^{i_1\ldots i_h}$ takes the form

$$\begin{aligned}
\sigma^{i_1\ldots i_h} &= (-1)^h \, c^*_{i_1\ldots i_h} \prod_{\nu=1}^{h} \pi^{(i_\nu)}_{n^*_{i_\nu}} \prod_{\kappa\notin\{i_1,\ldots,i_h\}} \pi^{(\kappa)}_{n^*_\kappa} \cdot \alpha^{(i_\nu)}_{n^*_{i_\nu}\, n^*_{i_\nu}} \\
&= \prod_{\kappa=1}^{M} \pi^{(\kappa)}_{n^*_\kappa} \cdot (-1)^h \, c^*_{i_1\ldots i_h} \prod_{\nu=1}^{h} \alpha^{(i_\nu)}_{n^*_{i_\nu}\, n^*_{i_\nu}} = \prod_{\kappa=1}^{M} \pi^{(\kappa)}_{n^*_\kappa} \cdot \xi^{i_1\ldots i_h} \\
&=: \zeta^{i_1\ldots i_h} \ ,
\end{aligned}$$

since, instead of $\sum_{n_\nu{}^*\neq n_\nu} \alpha_{n_\nu\, n^*_\nu} = -\alpha_{n_\nu\, n_\nu}$ as in Lemma 1, here we have $\sum_{n_\nu\neq n^*_\nu} \pi^{(\nu)}_{n_\nu} \alpha^{(\nu)}_{n_\nu\, n^*_\nu} = -\pi^{(\nu)}_{n^*_\nu} \alpha^{(\nu)}_{n^*_\nu\, n^*_\nu}$ for $\nu = 1, \ldots, M$.

Now, the expression $\sum_{\mathfrak{n}} \prod_{i=1}^{M} \pi_{n_i}^{(i)} L_{\mathfrak{n}\,\mathfrak{n}^*}$ can be written as

$$\begin{aligned}
\sum_{\mathfrak{n}} \prod_{i=1}^{M} \pi_{n_i}^{(i)} L_{\mathfrak{n}\,\mathfrak{n}^*} &= \sigma_{1\ldots M}, \\
\sigma_{1\ldots M} &= \sigma^{1}_{2\ldots M} + \sigma_{2\ldots M} \\
&= \sigma^{12}_{3\ldots M} + \sigma^{1}_{3\ldots M} + \sigma^{2}_{3\ldots M} + \sigma_{3\ldots M} \\
&= \sigma^{123}_{4\ldots M} + \sigma^{12}_{4\ldots M} + \sigma^{13}_{4\ldots M} + \sigma^{1}_{4\ldots M} + \sigma^{23}_{4\ldots M} \\
&\qquad + \sigma^{2}_{4\ldots M} + \sigma^{3}_{4\ldots M} + \sigma_{4\ldots M} \\
&\vdots
\end{aligned}$$

such that, based on the same reasoning as in the proof of Lemma 1, we are finally lead to

$$\sum_{\mathfrak{n}} \prod_{i=1}^{M} \pi_{n_i}^{(i)} L_{\mathfrak{n}\,\mathfrak{n}^*} = \sum_{\ell=1}^{M} \tilde{\eta}_\ell(\mathfrak{n}) + \prod_{\kappa=1}^{M} \pi_{n^*_\kappa}^{(\kappa)} \cdot L_{\mathfrak{n}^*\,\mathfrak{n}^*} ,$$

where $\tilde{\eta}_\ell(\mathfrak{n})$ is the ℓ^{th} generalized elementary symmetric function of the expressions $\zeta^{i_1\ldots i_\ell}(\mathfrak{n})$, viz.

$$\begin{aligned}
\tilde{\eta}_\ell(\mathfrak{n}) &= \sum_{\Omega_\ell} \zeta^{i_1\ldots i_\ell}(\mathfrak{n}) = \prod_{\kappa=1}^{M} \pi_{n^*_\kappa}^{(\kappa)} \cdot \sum_{\Omega_\ell} \xi^{i_1\ldots i_\ell}(\mathfrak{n}) \\
&= \prod_{\kappa=1}^{M} \pi_{n^*_\kappa}^{(\kappa)} \cdot \sum_{\Omega_\ell} (-1)^k \, c^*_{i_1\ldots i_\ell} \prod_{\nu=1}^{\ell} \alpha^{(i_\nu)}_{n^*_{i_\nu}\, n^*_{i_\nu}} = \prod_{\kappa=1}^{M} \pi_{n^*_\kappa}^{(\kappa)} \cdot \eta_\ell(\mathfrak{n}) \,.
\end{aligned}$$

From (4) we have $\sum_{\ell=1}^{M} \eta_\ell(\mathfrak{n}) = -L_{\mathfrak{n}^*\,\mathfrak{n}^*}$, which proves the assertion. In the case that the participating Markov processes are continuous time parameter processes everything remains true with the additional simplification that only functions of the form $\psi^{(i)}$ occur, and all generalized elementary symmetric function η_ℓ are zero for $\ell > 1$. □

In the defining expression (4) for the transition intensity matrix it was required that exactly the ℓ components $n^*_{i_1}, \ldots, n^*_{i_\ell}$ (corresponding to the mark of the function $\psi^{(i_1\ldots i_\ell)}$), and only these, are different from their "peer" components $n_{i_1}, \ldots, n_{i_\ell}$. One may be tempted to weaken this condition, requiring $n^*_j = n_j$ for $j \notin \{i_1, \ldots, i_\ell\}$, if $\mathfrak{n}^* \neq \mathfrak{n}$, *and* admitting equal components $n^*_{i_\nu} = n_{i_\nu}$ $(1 \le \nu \le \ell,\ 1 \le \ell \le M)$. In this case, if $\psi^{(i_1,\ldots,i_\ell)}$ is chosen as a positive function independently of the number of components $n^*_{i_\nu}$ with $n^*_{i_\nu} \neq n_{i_\nu}$, then, as is easily seen, the following consistency condition has to be fulfilled for $1 < \ell < M$:

$$\psi^{(i_1,\ldots,i_\ell)}(\mathfrak{n}^{\overline{(i_1\ldots i_k)}}) \cdot \prod_{\nu=1}^{\ell} \alpha^{(i_\nu)}_{n_{i_\nu}\, n^*_{i_\nu}} = \text{const.}, \quad \forall \{i_1,\ldots,i_\ell\} \subset \{1,\ldots,M\}\ .$$

This case is of marginal interest. Rather, we may state the following generalization, where *all* transition intensity matrices $A^{(\nu)}$ participate in each component of the transition intensity matrix L. To this end set

$$\vartheta_\ell(\mathfrak{n}) = \sum_{\Omega_\ell} (-1)^\ell \psi^{(i_1\ldots i_\ell)}(\mathfrak{n}^{\overline{(i_1\ldots i_k)}}) \prod_{\nu=1}^{M} \alpha^{(\nu)}_{n_\nu,n_\nu}\ .$$

Theorem 2. *Define, under the presumptions of Theorem 1, the matrix* $\tilde{L} = ((\tilde{L}_{\mathfrak{n}\,\mathfrak{n}^*}))_{\mathfrak{n}\,\mathfrak{n}^*\in E}$ *through*

$$\tilde{L}_{\mathfrak{n}\,\mathfrak{n}^*} = \tag{6}$$

$$\begin{cases} \psi^{(i_1,\ldots,i_\ell)}(\mathfrak{n}^{\overline{(i_1\ldots i_\ell)}}) \cdot \prod_{\nu=1}^{M} \alpha^{(\nu)}_{n_\nu,n_\nu{}^*}\,, & \text{if } n^*_{i_\nu} \neq n_{i_\nu} \text{ for each} \\ & i_\nu \in \{i_1,\ldots,i_\ell\}, \\ & n^*_j = n_j \text{ for } j \notin \{i_1,\ldots,i_\ell\}, \\ & 1 \le \ell \le M, \\ & \\ -\sum_{\ell=1}^{M} \vartheta_\ell(\mathfrak{n})\,, & \text{if } \mathfrak{n}^* = \mathfrak{n}\ . \end{cases}$$

Then $\tilde{L}$ is the transition intensity matrix of some homogeneous ergodic Markov chain $\tilde{\mathcal{L}}$ with steady state distribution

$$p_M(\mathfrak{n}) = \prod_{i=1}^{M} \pi^{(i)}_{n_i}\ . \tag{7}$$

Proof. We use the notational shortcuts

$$\begin{aligned} \psi^{(i_1,\ldots,i_k)}(\mathfrak{n}^{\overline{(i_1\ldots i_k)}}) &=: c_{i_1\ldots i_k}\,, \quad \psi^{(i_1,\ldots,i_k)}(\mathfrak{n}^{*\overline{(i_1\ldots i_k)}}) = c^*_{i_1\ldots i_k}\,, \\ \prod_{\nu=1}^{M} \alpha^{(\nu)}_{n_\nu\, n_\nu} &=: A_{1\ldots M}\,, \quad \prod_{\nu=1}^{M} \alpha^{(\nu)}_{n^*_\nu\, n^*_\nu} =: A^*_{1\ldots M}\ . \end{aligned}$$

Then $\sum_{\mathfrak{n}^*} \tilde{L}_{\mathfrak{n}\,\mathfrak{n}^*}$ can be rewritten as

$$\sum_{\mathfrak{n}^*} \tilde{L}_{\mathfrak{n}\,\mathfrak{n}^*} = \sum_{\ell=1}^{M} \sum_{\Omega_\ell} \tilde{\xi}^{i_1\ldots i_\ell}(\mathfrak{n}) + \tilde{L}_{\mathfrak{n}\,\mathfrak{n}}\,,$$

where now

$$\tilde{\xi}^{i_1\dots i_\ell}(\mathfrak{n}) = (-1)^\ell \cdot c_{i_1\dots i_\ell} \cdot A_{1\dots M}$$

for each subset of cardinality ℓ, such that $\sum_{\mathfrak{n}^*} \tilde{L}_{\mathfrak{n}\,\mathfrak{n}^*} = 0$ according to (6). Thus, $\tilde{L}_{\mathfrak{n}\,\mathfrak{n}^*}$ is the transition intensity matrix of some regular and irreducible Markov chain $\tilde{\mathcal{L}}$, as can be shown in litterally the same way as before. Further, $\mathcal{P}_M = \{p_M(\mathfrak{n}) : \mathfrak{n} \in E\}$ is an equilibrium distribution of $\tilde{\mathcal{L}}$ with $p_M(\mathfrak{n})$ as in (7) because of

$$\begin{aligned} \sum_{\mathfrak{n}} \prod_{i=1}^{M} \pi_{n_i}^{(i)} \tilde{L}_{\mathfrak{n}\,\mathfrak{n}^*} &= \prod_{\kappa=1}^{M} \pi_{n_\kappa^*}^{(\kappa)} \cdot \sum_{\ell=1}^{M} \sum_{\Omega_\ell} (-1)^\ell \cdot c^*_{i_1\dots i_\ell} \cdot A^*_{1\dots M} \\ &\quad + \prod_{\kappa=1}^{M} \pi_{n_\kappa^*}^{(\kappa)} \cdot L_{\mathfrak{n}^*\mathfrak{n}^*} \\ &= p_M(\mathfrak{n}^*) \cdot \sum_{\Omega_\ell} \vartheta_\ell(\mathfrak{n}^*) - p_M(\mathfrak{n}^*) \cdot \sum_{\Omega_\ell} \vartheta_\ell(\mathfrak{n}^*) = 0 \ , \quad \square \end{aligned}$$

As a special case we have the following corollary.

Corollary 1. *Let $\mathcal{F}$ be a cadence family as in Theorem 2, and let, for some constant $c > 0$, the matrix $\tilde{L} = ((\tilde{L}_{\mathfrak{n}\,\mathfrak{n}^*}))_{\mathfrak{n}\,\mathfrak{n}^* \in E}$ be defined through*

$$\tilde{L}_{\mathfrak{n}\,\mathfrak{n}^*} = \begin{cases} c \cdot \prod_{\nu=1}^{M} \alpha^{(\nu)}_{n_\nu\, n_\nu^*}, & \text{if } \mathfrak{n}^* \neq \mathfrak{n}, \\ -c \cdot \prod_{\nu=1}^{M} \alpha^{(\nu)}_{n_\nu\, n_\nu} \cdot \sum_{\ell=1}^{M} (-1)^\ell \binom{M}{\ell}, & \text{if } \mathfrak{n}^* = \mathfrak{n}\ . \end{cases} \tag{8}$$

Then $\tilde{L}$ is the transition intensity matrix of some homogeneous ergodic Markov chain $\tilde{\mathcal{L}}$ with steady state distribution $p_M(\mathfrak{n}) = \prod_{i=1}^{M} \pi^{(i)}_{n_i}$.

Proof. Here we obtain $\sum_{\mathfrak{n}^*} \tilde{L}_{\mathfrak{n}\,\mathfrak{n}^*} = \sum_{\ell=1}^{M} \sum_{\Omega_\ell} \tilde{\xi}^{i_1\dots i_\ell}(\mathfrak{n}) + \tilde{L}_{\mathfrak{n}\,\mathfrak{n}}$, with $\tilde{\xi}^{i_1\dots i_\ell}(\mathfrak{n}) = (-1)^\ell \cdot c \cdot \prod_{\nu=1}^{M} \alpha^{(\nu)}_{n_\nu\, n_\nu}$, such that the corresponding generalized elementary symmetric functions are

$$\vartheta_\ell(\mathfrak{n}) := \sum_{\Omega_\ell} \tilde{\xi}^{i_1\dots i_\ell}(\mathfrak{n}) = c \cdot (-1)^\ell \cdot \binom{M}{\ell} \cdot \prod_{\nu=1}^{M} \alpha^{(\nu)}_{n_\nu\, n_\nu} \ ,$$

resulting in $\sum_{\mathfrak{n}^*} \tilde{L}_{\mathfrak{n}\,\mathfrak{n}^*} = 0$ according to (8). The remaining part of the proof is analogous to that of Theorem 2, where the last equation has to be replaced by

$$\sum_{\mathfrak{n}} \prod_{i=1}^{M} \pi_{n_i}^{(i)} \tilde{L}_{\mathfrak{n}\,\mathfrak{n}^*} = \prod_{\kappa=1}^{M} \pi_{n_\kappa^*}^{(\kappa)} \cdot \sum_{\ell=1}^{M} \vartheta_\ell(\mathfrak{n}) + \prod_{\kappa=1}^{M} \pi_{n_\kappa^*}^{(\kappa)} \cdot \tilde{L}_{\mathfrak{n}^*\mathfrak{n}^*} = 0\ . \qquad \square$$

2.3. General Product Connection Conditions

Applications of Theorems 1 and 2 have been presented in [8], where product form queueing networks with Poisson input process and input depending service rates have been considered. BCMP-type networks, especially Jackson and Gordon-Newell networks, are characterized by (possibly state depending) service rates which do not vary according to input variations. In Section 3 we show that these networks can be handled, too, by the theory of product connection of Markov chains. Another important issue addresses the question for conditions, which imply product form for queueing networks with non-Poissonian input.
As we shall see, the inclusion of queueing network problems requires a somewhat more general formulation of product connection criteria. Here the construction of the "connecting" kernel $L_{\mathfrak{n}\mathfrak{n}^*}$ has to take into account coefficient functions $\varphi^{i_1\dots i_\ell}(\mathfrak{n},\mathfrak{n}^*)$, which depend upon both arguments, $\mathfrak{n}$ and $\mathfrak{n}^*$, rather than only on $\mathfrak{n}^{\overline{(i_1\dots i_k)}} \in E^{\overline{(i_1\dots i_k)}}$.

We use the following slightly generalized notation: For the set $\mathfrak{P}(\mathcal{M})$ of all subsets of $\mathcal{M}=\{1,\dots,M\}$ let $f:\mathfrak{P}(\mathcal{M})\longrightarrow (E\times E)^*$ describe a mapping into the set of non-negative and uniformly bounded functions over $E\times E$. Denote $f(\omega_\ell)=\varphi^{i_1\dots i_\ell}: E\times E\longrightarrow \mathbb{R}_+\cup\{0\}$ for $\omega_\ell=\{i_1,\dots,i_\ell\}\in\Omega_\ell\subset\mathfrak{P}(\mathcal{M})$.

We assume, that f satisfies the following *product conditions*:

(i) For any pair $(\mathfrak{n},\mathfrak{n}^*)\in E\times E$ there is a sequence $\mathfrak{n}_0,\dots,\mathfrak{n}_{s+1}$ of elements of E with $\mathfrak{n}_0=\mathfrak{n}$, $\mathfrak{n}_{s+1}=\mathfrak{n}^*$, and $\mathfrak{n}_j\neq\mathfrak{n}_{j+1}$, $\mathfrak{n}_j$ and $\mathfrak{n}_{j+1}$ differing in components $n_{j,i_1},\dots,n_{j,i_{\ell(j)}}$ for each $j\in\{0,\dots,s\}$, such that

$$\prod_{j=0}^{s}\varphi^{(i_1\dots i_{\ell(j)})}(\mathfrak{n}_j,\mathfrak{n}_{j+1})\prod_{\nu=1}^{\ell(j)}\alpha^{(i_\nu)}_{n_{j,i_\nu}\,n_{j+1,i_\nu}}>0\ .$$

If $\mathcal{F}$ is a family of discrete time Markov chains, then

$$\sum_{\ell=1}^{M}\sum_{\{i_1,\dots,i_\ell\}\in\Omega_\ell}\varphi^{(i_1\dots i_\ell)}(\mathfrak{n},\mathfrak{n}^*)=1,\quad \forall\,(\mathfrak{n},\mathfrak{n}^*)\in E\times E\ .$$

$$(ii)\quad \sum_{\ell=1}^{M}\sum_{\{i_1,\dots,i_\ell\}\in\Omega_\ell}\sum_{n^*_{i_1}\neq n_{i_1}}\cdots\sum_{n^*_{i_\ell}\neq n_{i_\ell}}\varphi^{(i_1\dots i_\ell)}(\mathfrak{n},\mathfrak{m}^*)\cdot\prod_{\nu=1}^{\ell}\alpha^{(i_\nu)}_{n_{i_\nu}n^*_{i_\nu}}=$$

$$\sum_{\ell=1}^{M}\sum_{\{i_1,\dots,i_\ell\}\in\Omega_\ell}\sum_{n^*_{i_1}\neq n_{i_1}}\cdots\sum_{n^*_{i_\ell}\neq n_{i_\ell}}\varphi^{(i_1\dots i_\ell)}(\mathfrak{m}^*,\mathfrak{n})\cdot\prod_{\nu=1}^{\ell}\frac{\pi^{(i_\nu)}_{n^*_{i_\nu}}}{\pi^{(i_\nu)}_{n_{i_\nu}}}\alpha^{(i_\nu)}_{n^*_{i_\nu}n_{i_\nu}}$$

for $(\mathfrak{n}, \mathfrak{m}^*) \in E \times E$, $\mathfrak{m}^* = (m_1^*, \dots, m_M^*)$ with $m_{i_\nu}^* = n_{i_\nu}^* \neq n_{i_\nu}$, $\forall \nu \in \{1, \dots, \ell\}$ and $m_j^* = n_j$ for $j \notin \{i_1, \dots, i_\ell\}$.

Remark 3. The use of functions $\varphi^{(i_1 \dots i_\ell)}(\mathfrak{n}, \mathfrak{m}^*)$ enables to formally include products of more than one generator element in the definition of the generator matrix L even in case of continuous time Markov processes. For example, if

$$\mathfrak{m}^* = \mathfrak{n}_{(n_i^*, n_j^*)} = (n_1, \dots, n_{i-1}, n_i^*, n_{i+1}, \dots, n_{j-1}, n_j^*, n_{j+1}, \dots, n_M),$$

then $\varphi^{(ij)}(\mathfrak{n}, \mathfrak{m}^*)$ may be defined as $\varphi^{(ij)}(\mathfrak{n}, \mathfrak{m}^*) = \frac{c}{\alpha^{(j)}_{n_j\, n_j^*}}$, such that $\varphi^{(ij)}(\mathfrak{n}, \mathfrak{m}^*) \cdot \alpha^{(i)}_{n_i\, n_i^*} \alpha^{(j)}_{n_j\, n_j^*}$ reduces to an expression describing the arrival rate of only one event.

For the following we introduce generalized elementary symmetric functions in a similar form as before: Let for $\mathfrak{n} \in E$

$$\begin{aligned} \vartheta_\ell(\mathfrak{n}) &= \sum_{\{i_1, \dots, i_\ell\} \in \Omega_\ell} \sum_{n_{i_1}^* \neq n_{i_1}} \cdots \sum_{n_{i_\ell}^* \neq n_{i_\ell}} \varphi^{(i_1 \dots i_\ell)}(\mathfrak{n}, \mathfrak{m}^*) \cdot \prod_{\nu=1}^{\ell} \alpha^{(i_\nu)}_{n_{i_\nu} n_{i_\nu}^*} \\ &=: \sum_{\{i_1, \dots, i_\ell\} \in \Omega} \tau^{i_1 \dots i_\ell}(\mathfrak{n}) \end{aligned} \tag{9}$$

denote the ℓ^{th} generalized elementary symmetric function of all expressions of type $\tau^{i_1 \dots i_\ell}(\mathfrak{n})$.

Lemma 3. *Assume, that $f(\omega_\ell) = \varphi^{i_1 \dots i_\ell} : E \times E \longrightarrow \mathbb{R}_+ \cup \{0\}$ for any $\omega_\ell = \{i_1, \dots, i_\ell\} \in \Omega_\ell \subset \Omega$ satisfies the first product condition (i). Define the matrix $L = ((L_{\mathfrak{n}\, \mathfrak{n}^*}))_{\mathfrak{n}\, \mathfrak{n}^* \in E}$ through*

$$L_{\mathfrak{n}\,\mathfrak{n}^*} = \tag{10}$$

$$\begin{cases} \varphi^{(i_1, \dots, i_\ell)}(\mathfrak{n}, \mathfrak{n}^*) \cdot \prod_{\nu=1}^{\ell} \alpha^{(i_\nu)}_{n_{i_\nu}\, n_{i_\nu}^*}, & \text{if } n_{i_\nu}^* \neq n_{i_\nu} \text{ for each} \\ & i_\nu \in \{i_1, \dots, i_\ell\}, \\ & n_j^* = n_j \quad \text{for} \\ & j \notin \{i_1, \dots, i_\ell\}, \;\; 1 \le \ell \le M \\ -\sum_{\ell=1}^{M} \vartheta_\ell(\mathfrak{n}), & \text{if } \mathfrak{n}^* = \mathfrak{n}. \end{cases}$$

Then the sum $\sum_{\mathfrak{n}^*} L_{\mathfrak{n}\,\mathfrak{n}^*}$ equals zero, and the matrix L is to be interpreted as the transition intensity matrix of some irreducible Markov chain $\mathcal{L}$ with state space E.

Proof. Due to the definition of the functional values $\vartheta_\ell(\mathfrak{n})$ in (9) the sum $\sum_{\mathfrak{n}^*} L_{\mathfrak{n}\mathfrak{n}^*}$ equals zero, such that L has the required properties of a generator matrix defining a Markov chain $\mathcal{L}$. The regularity of $\mathcal{L}$ is an immediate consequence from the regularity of all participating chains $\mathcal{X}^{(\nu)}$ ($\nu = 1, \dots, M$) together with the requirement of boundedness of all functions $\varphi^{(i_1 \dots i_\ell)}$ ($\{i_1, \dots, i_\ell\} \in \Omega_\ell$). Finally, as is easily seen, condition (i) is the necessary and sufficient condition for irreducibility of $\mathcal{L}$. □

Let us call a family $\mathcal{F}$ of ergodic Markov chains $\mathcal{X}^{(\nu)}$ ($\nu = 1, \dots, M$) a *general cadence family*, if the product conditions (i) and (ii) are satisfied. As another generalization it is allowed in the following, that the state space E is a proper subset of the Cartesian product $\bigotimes_{\nu=1}^{M} E^{(\nu)} = E^{(1)} \times \ldots \times E^{(M)}$.

Theorem 3. *Let $\mathcal{F} = \{\mathcal{X}^{(\nu)} : \nu \in \mathcal{M}\}$ be a general cadence family of Markov chains $\mathcal{X}^{(\nu)} = \{X^{(\nu)}(t) : t \in T\}$ with state spaces $E^{(\nu)} \subset \mathbb{N}_0$, transition intensity matrices $A^{(\nu)} = ((\alpha^{(\nu)}_{n_\nu\, n^*_\nu}))_{n_\nu\, n^*_\nu \in E^{(\nu)}}$, and steady state distributions $\pi^{(\nu)} = (\pi^{(\nu)}_0, \pi^{(\nu)}_1, \dots)$. Let $E \subset \bigotimes_{\nu=1}^{M} E^{(\nu)}$. Define the transition intensity matrix L of some irreducible Markov chain $\mathcal{L}$ through (10). Then $\mathcal{L}$ is ergodic with steady state distribution $\mathcal{P} = \{p(\mathfrak{n}) : \mathfrak{n} = (n_1, \dots, n_M) \in E\}$, where*

$$p(\mathfrak{n}) = \frac{1}{G} \cdot \prod_{i=1}^{M} \pi^{(i)}_{n_i}\,, \qquad G = \sum_{\mathfrak{n} \in E} \prod_{i=1}^{M} \pi^{(i)}_{n_i}.$$

Proof. Applying Lemma 3, we only have to show that $\sum_{\mathfrak{n}^*} p(\mathfrak{n}^*) \cdot L_{\mathfrak{n}^*\mathfrak{n}} = 0$ for any $\mathfrak{n} \in E$. Now, $\sum_{\mathfrak{n}^*} p(\mathfrak{n}^*) \cdot L_{\mathfrak{n}^*\mathfrak{n}} = \sum_{\mathfrak{n}^* \neq \mathfrak{n}} p(\mathfrak{n}^*) \cdot L_{\mathfrak{n}^*\mathfrak{n}} + p(\mathfrak{n}) \cdot L_{\mathfrak{n}\mathfrak{n}}$, and

$$\begin{aligned}
&\sum_{\mathfrak{n}^* \neq \mathfrak{n}} p(\mathfrak{n}^*) L_{\mathfrak{n}^*\mathfrak{n}} \\
&= \sum_{\ell=1}^{M} \sum_{\{i_1,\dots,i_\ell\} \in \Omega_\ell} \sum_{n^*_{i_1} \neq n_{i_1}} \cdots \sum_{n^*_{i_\ell} \neq n_{i_\ell}} \prod_{\nu=1}^{\ell} \pi^{(i_\nu)}_{n^*_{i_\nu}} \prod_{\kappa \notin \{i_1,\dots,i_\ell\}}^{\ell} \pi^{(\kappa)}_{n_\kappa} \cdot L_{\mathfrak{m}^*\,\mathfrak{n}} \\
&= p(\mathfrak{n}) \sum_{\ell=1}^{M} \sum_{\{i_1,\dots,i_\ell\} \in \Omega_\ell} \sum_{n^*_{i_1} \neq n_{i_1}} \cdots \sum_{n^*_{i_\ell} \neq n_{i_\ell}} \prod_{\nu=1}^{\ell} \frac{\pi^{(i_\nu)}_{n^*_{i_\nu}}}{\pi^{(i_\nu)}_{n_{i_\nu}}} \cdot L_{\mathfrak{m}^*\,\mathfrak{n}}\,,
\end{aligned}$$

where the right hand side — due to condition (ii) — equals $p(\mathfrak{n}) \cdot \sum_{\ell=1}^{M} \vartheta_\ell(\mathfrak{n}) = -p(\mathfrak{n}) \cdot L_{\mathfrak{n}\mathfrak{n}}$. Consequently $\sum_{\mathfrak{n}^*} p(\mathfrak{n}^*) \cdot L_{\mathfrak{n}^*\mathfrak{n}} = 0$. □

3. Simple Product Form Queueing Networks

In this section we give some application examples, which address the well known Jackson and Gordon-Newell networks. We use for any $\mathfrak{n} = (n_1, \ldots, n_M) \in \mathbb{N}_0^M$ the shortcuts:

$$\begin{aligned}
\mathfrak{n}_{(n_i \pm 1)} &= (n_1, \ldots, n_{i-1}, n_i \pm 1, n_{i+1}, \ldots, n_M)\ , \\
\mathfrak{n}_{(n_i - 1, n_j + 1)} &= (n_1, \ldots, n_{i-1}, n_i - 1, n_{i+1}, \ldots, \\
&\quad\ n_{j-1}, n_j + 1, n_{j+1}, \ldots, n_M)\ , \\
\mathcal{M} &= \{1, \ldots, M\}\ .
\end{aligned}$$

3.1. Jackson Networks

Consider a family $\mathcal{F} = \{\mathcal{X}^{(i)} : i \in \mathcal{M}\}$ of M regular continuous-time birth-death-processes (BD's) $\mathcal{X}^{(i)} = \{X^{(i)}(t) : t \in T\}$ with state spaces $E^{(i)} = \mathbb{N}_0$, steady state distributions $\pi^{(i)} = (\pi_0^{(i)}, \pi_1^{(i)}, \ldots)$ and transition intensity matrices $A^{(i)} = ((\alpha^{(i)}_{n_i\, n_i^*}))_{n_i\, n_i^* \in E^{(i)}}$. Then

$$\alpha^{(i)}_{n_i\, n_i+1} =: \lambda^{(i)}_{n_i} > 0, \quad \alpha^{(i)}_{n_i+1\, n_i} =: \mu^{(i)}(n_i + 1) > 0, \quad \forall\, n_i \in \mathbb{N}_0,\ i \in \mathcal{M}$$

and

$$\pi^{(i)}_{n_i} = c \cdot \frac{\lambda_0^{(i)} \cdot \ldots \cdot \lambda^{(i)}_{n_i - 1}}{\mu^{(i)}(1) \cdot \ldots \cdot \mu^{(i)}(n_i)} \quad \text{for } n_i \geq 1\ ,$$

where $c = \pi_0^{(i)} = \left(1 + \sum_{n_i=1}^{\infty} \frac{\lambda_0^{(i)} \cdot \ldots \cdot \lambda^{(i)}_{n_i-1}}{\mu^{(i)}(1) \cdot \ldots \cdot \mu^{(i)}(n_i)}\right)^{-1}$ $(i = 1, \ldots, M)$. In particular,

$$\pi^{(i)}_{n_i} \cdot \lambda^{(i)}_{n_i} = \pi^{(i)}_{n_i+1} \cdot \mu^{(i)}(n_i + 1), \quad \forall\, n_i \geq 0\ . \tag{11}$$

Let $Q = ((q_{ij}))_{i,j \in \{0,1,\ldots,M\}}$ denote the transition probability matrix of some irreducible discrete-time $(M+1)$-state Markov chain, such that

$$\sum_{j=1}^{M} q_{ij} = 1 - q_{i0}, \quad \sum_{i=1}^{M} q_{0i} = 1\ . \tag{12}$$

We show next, that the well known result of Jackson can be subsumed in the framework of general product connection theorems.

In addition to $\mathcal{F}$ consider a two-state continuous time Markov process $\mathcal{X}^{(0)}$ with state space $\{s,\bar{s}\}$, generator elements $\alpha^{(0)}_{s\bar{s}}=\alpha^{(0)}_{\bar{s}s}=\gamma$ and steady state probabilities $\pi^{(0)}_{s}$, $\pi^{(0)}_{\bar{s}}$, where $\pi^{(0)}_{s}=\pi^{(0)}_{\bar{s}}$.

Referring to Remark 3 with respect to products of generator elements let, for $E=\mathbb{N}_0^M$ and $\mathfrak{n},\mathfrak{n}^*\in E$, the functions $\varphi^{(\ell)}:E\times E\longrightarrow\mathbb{R}_+$ and $\varphi^{(k\ell)}:E\times E\longrightarrow\mathbb{R}_+$ be defined through

$$\varphi^{(\ell)}(\mathfrak{n},\mathfrak{n}^*)=\begin{cases} q_{\ell 0} & \text{for } \mathfrak{n}^*=\mathfrak{n}_{(n_\ell-1)},\ \ell\in\mathcal{M}\,, \\ 0 & \text{in all other cases}\,, \end{cases}$$

$$\varphi^{(k\ell)}(\mathfrak{n},\mathfrak{n}^*)=\begin{cases} \dfrac{q_{0\ell}}{\alpha^{(\ell)}_{n_\ell\,n_\ell+1}} & \text{for } k=0,\ \mathfrak{n}^*=\mathfrak{n}_{(n_\ell+1)},\ \ell\in\mathcal{M}\,, \\ \dfrac{q_{k\ell}}{\alpha^{(\ell)}_{n_\ell\,n_\ell+1}} & \text{for } \mathfrak{n}^*=\mathfrak{n}_{(n_k-1,n_\ell+1)},\ k,\ell\in\mathcal{M}\,, \\ 0 & \text{in all other cases}\,, \end{cases}$$

and set $\varphi^{i_1\dots i_k}(\mathfrak{n},\mathfrak{n}^*)\equiv 0$ for any subset $\{i_1,\dots,i_k\}$ of $\mathcal{M}\cup\{0\}$ of cardinality $k>2$.

Then the expression (10) defines the generator matrix L of some Markov process $\mathcal{L}$ as

$$L_{\mathfrak{n},\mathfrak{n}^*}=\begin{cases} \alpha^{(0)}_{n_0,n_0^*}\cdot q_{0i} & \text{for } n_0,n_0^*\in\{s,\bar{s}\},\ \mathfrak{n}^*=\mathfrak{n}_{(n_i+1)},\ i\in\mathcal{M}\,, \\ \alpha^{(i)}_{n_i,n_i-1}\cdot q_{ij} & \text{for } \mathfrak{n}^*=\mathfrak{n}_{(n_i-1,n_j+1)},\ i,j\in\mathcal{M}\,, \\ \alpha^{(i)}_{n_i,n_i-1}\cdot q_{i0} & \text{for } \mathfrak{n}^*=\mathfrak{n}_{(n_i-1)},\ i\in\mathcal{M}\,, \\ -(\vartheta_1+\vartheta_2) & \text{for } \mathfrak{n}^*=\mathfrak{n} \\ 0 & \text{in all other cases}\,. \end{cases}$$

We set $\alpha^{(0)}_{s\bar{s}}\cdot q_{0i}=\alpha^{(0)}_{\bar{s}s}\cdot q_{0i}=\gamma\cdot q_{0i}=\gamma^{(i)}$ for $i\in\mathcal{M}$.

Definition 2. $\mathcal{L}$ represents the state process of an *open queueing network* $\mathcal{N}_J$ *in equilibrium*, if the arrival rates $\alpha^{(i)}_{n_i\,n_1+1}=\lambda^{(i)}_{n_i}$ for $i\in\mathcal{M}$ are given as solutions of the following linear system of equations:

$$\gamma^{(i)}+\sum_{j=1}^{M}\lambda^{(j)}q_{ji}=\lambda^{(i)}\,,\quad \forall\, i\in\{1,\dots,M\}\,. \tag{13}$$

Notice, that these solutions necessarily are state-independent: $\lambda^{(i)}_{n_i} = \lambda^{(i)}$, $\forall i$. Equations (13) are called the *traffic equations* of the open queueing network. According to (12) these equations imply the validity of the following *global flow balance* equation:

$$\gamma = \sum_{i=1}^{M} \gamma^{(i)} = \gamma \cdot \sum_{i=1}^{M} q_{0i} = \sum_{i=1}^{M} \lambda^{(i)} q_{i0} \ . \tag{14}$$

Lemma 4. *The open queueing network with underlying process $\mathcal{L}$ is a product form (PF) queueing network with steady state distribution $\mathcal{P} = \{p(\mathfrak{n}) : \mathfrak{n} \in \mathbb{N}_0^M\}$, where*

$$p(\mathfrak{n}) = \prod_{i=1}^{M} \pi^{(i)}_{n_i}$$

is the probability of observing state set

$$\{(s, n_1, \dots, n_M)\} \cup \{(\bar{s}, n_1, \dots, n_M)\}$$

in equilibrium.

Proof. We have to show, that the product conditions (i) and (ii) are fulfilled.

Condition (i) is an immediate consequence from the fact, that the functions $\varphi^{\ell)}(\mathfrak{n}, \mathfrak{n}^*)$ and $\varphi^{(k\ell)}(\mathfrak{n}, \mathfrak{n}^*)$ ($\ell \in \{0, \dots, M\}$, $k \in \mathcal{M}$) are proportional to $q_{\ell 0}$ and $q_{k\ell}$, respectively, $Q = ((q_{\ell k}))$ representing the transition probability matrix of an irreducible Markov chain. The second product condition (ii) takes the following form:

$$\begin{aligned}
&\sum_{i=1}^{M} \Big(\varphi^{(0i)}(\mathfrak{n}, \mathfrak{n}_{(n_i+1)}) \alpha^{(0)}_{s\bar{s}} \alpha^{(i)}_{n_i n_i+1} + \varphi^{(i)}(\mathfrak{n}, \mathfrak{n}_{(n_i-1)}) \alpha^{(i)}_{n_i\, n_i-1} \Big) + \\
&\qquad + \sum_{i=1}^{M} \sum_{j=1}^{M} \varphi^{(ij)}(\mathfrak{n}, \mathfrak{n}_{(n_i-1, n_j+1)}) \alpha^{(i)}_{n_i,\, n_i-1} \\
&\quad = \sum_{i=1}^{M} \Big(\varphi^{(0i)}(\mathfrak{n}_{(n_i-1)}, \mathfrak{n}) \alpha^{(0)}_{\bar{s}s} \alpha^{(i)}_{n_i n_i+1} \frac{\pi^{(0)}_{\bar{s}}}{\pi^{(0)}_{s}} \frac{\pi^{(i)}_{n_i-1}}{\pi^{(i)}_{n_i}} \\
&\qquad + \varphi^{(i)}(\mathfrak{n}_{(n_i+1)}, \mathfrak{n}) \alpha^{(i)}_{n_i+1\, n_i} \frac{\pi^{(i)}_{n_i+1}}{\pi^{(i)}_{n_i}} \Big) + \\
&\qquad + \sum_{i=1}^{M} \sum_{j=1}^{M} \varphi^{(ij)}(\mathfrak{n}_{(n_i+1, n_j-1)}, \mathfrak{n}) \alpha^{(i)}_{n_i+1,\, n_i} \alpha^{(j)}_{n_j-1,\, n_j} \frac{\pi^{(i)}_{n_i+1}}{\pi^{(i)}_{n_i}} \frac{\pi^{(j)}_{n_j-1}}{\pi^{(j)}_{n_j}} \ ,
\end{aligned}$$

or explicitely

$$\sum_{i=1}^{M} \Big(\gamma^{(i)} + q_{i0} \cdot \mu^{(i)}(n_i)\Big) \; + \; \sum_{i=1}^{M}\sum_{j=1}^{M} q_{ij} \cdot \mu^{(i)}(n_i)$$

$$= \sum_{i=1}^{M} \Big(\gamma^{(i)} \cdot \frac{\pi^{(i)}_{n_i-1}}{\pi^{(i)}_{n_i}} + q_{i0} \cdot \mu^{(i)}(n_i+1) \cdot \frac{\pi^{(i)}_{n_i+1}}{\pi^{(i)}_{n_i}}\Big)$$

$$+ \; \sum_{i=1}^{M}\sum_{j=1}^{M} q_{ij} \cdot \mu^{(i)}(n_i+1) \cdot \frac{\pi^{(i)}_{n_i+1}}{\pi^{(i)}_{n_i}} \frac{\pi^{(j)}_{n_j-1}}{\pi^{(j)}_{n_j}} \; .$$

By exployting (12) one obtains for the left hand side

$$\sum_{i=1}^{M} \Big(\gamma^{(i)} + q_{i0} \cdot \mu^{(i)}(n_i)\Big) + \sum_{i=1}^{M}\sum_{j=1}^{M} q_{ij} \cdot \mu^{(i)}(n_i)$$

$$= \sum_{i=1}^{M} \Big(\gamma^{(i)} + \mu^{(i)}(n_i)\Big) \; .$$

Similarly, with (11), the right hand side can be written as

$$\sum_{i=1}^{M} \Big(\frac{\gamma^{(i)}}{\lambda^{(i)}} \cdot \mu^{(i)}(n_i) + q_{i0} \cdot \lambda^{(i)}\Big) + \sum_{i=1}^{M}\sum_{j=1}^{M} q_{ij} \cdot \frac{\lambda^{(i)}}{\lambda^{(j)}} \mu^{(j)}(n_j) \; ,$$

which, according to (13), is the same as

$$\sum_{i=1}^{M} \Big(\frac{\gamma^{(i)}}{\lambda^{(i)}} \cdot \mu^{(i)}(n_i) + q_{i0} \cdot \lambda^{(i)}\Big) + \sum_{j=1}^{M} \frac{\mu^{(j)}(n_j)}{\lambda^{(j)}} \cdot (\lambda^{(j)} - \gamma^{(j)})$$

$$= \sum_{i=1}^{M} \Big(q_{i0} \cdot \lambda^{(i)} + \mu^{(i)}(n_i)\Big) \; .$$

Substitution of the global flow balance equation proves the assertion due to $\pi^{(0)}_{s} + \pi^{(0)}_{\bar{s}} = 1$. □

3.2. Gordon-Newell Networks

Let again $\mathcal{F} = \{\mathcal{X}^{(i)} : i \in \mathcal{M}\}$ be a family of M regular continuous-time birth-death-processes (BD's) $\mathcal{X}^{(i)} = \{X^{(i)}(t) : t \in T\}$ with state spaces $E^{(i)} = \mathbb{N}_0$, as described in Example 1. Assume, that the matrix $Q = ((q_{ij}))_{i,j\in\{1,\ldots,M\}}$ is the transition probability matrix of some irreducible discrete-time M-state Markov chain, i.e.

$$\sum_{j=1}^{M} q_{ij} = 1 \ . \tag{15}$$

For some fixed positive integer $K \geq 1$ let

$$E_K = \left\{ \mathfrak{n} = (n_1, \ldots, n_M) \in \mathbb{N}_0^M : \sum_{\nu=1}^{M} n_\nu = K \right\} .$$

We then define for $\mathfrak{n}, \mathfrak{n}^* \in E_K$ a matrix $L^{(K)}$ by

$$L^{(K)}_{\mathfrak{n}\,\mathfrak{n}^*} = \begin{cases} \varphi^{(ij)}(\mathfrak{n}, \mathfrak{n}_{(n_i-1,n_j+1)}) \cdot \alpha^{(i)}_{n_i\,n_i-1} \cdot \alpha^{(j)}_{n_j\,n_j+1}, & \text{if } n^*_k = n_k \,, \\ & \quad \forall k \neq i, j \\ -\vartheta_2(\mathfrak{n}), & \text{if } \mathfrak{n}^* = \mathfrak{n} \\ 0, & \text{otherwise} \,, \end{cases} \tag{16}$$

where $\varphi^{(ij)} : E_K \times E_K \longrightarrow \mathbb{R}_+$ is given by

$$\varphi^{(ij)}(\mathfrak{n}, \mathfrak{n}^*) = \begin{cases} \frac{q_{ij}}{\alpha^{(j)}_{n_j\,n_j+1}} & \text{for } \mathfrak{n}^* = \mathfrak{n}_{(n_i-1,n_j+1)} \\ 0 & \text{in all other cases} \,. \end{cases}$$

Obviously, the product condition (i) is fulfilled for the set of all functions $\varphi^{(ij)}$ (compare proof of Lemma 4), such that $L^{(K)}$ represents the generator matrix of some irreducible regular Markov chain $\mathcal{L}^{(K)}$.

Definition 3. $\mathcal{L}^{(K)}$ represents the state process of a *closed queueing network $\mathcal{N}_{GN}$ in equilibrium* with K customers, if the arrival rates $\alpha^{(i)}_{n_i\,n_1+1} = \lambda^{(i)}_{n_i} = \lambda^{(i)}$ constitute a vector $\Lambda = (\lambda^{(1)}, \ldots, \lambda^{(M)})$ with positive state - independent components, which is proportional to any positive solution of the following homogeneous linear system of *traffic equations:*

$$\lambda^{(i)} = \sum_{j=1}^{M} \lambda^{(j)} q_{ji} \quad \forall\, i \in \{1, \ldots, M\} \ . \tag{17}$$

Lemma 5. *The closed queueing network $\mathcal{N}_{GN}$ with underlying process $\mathcal{L}^{(K)}$ is a product form queueing network with steady state distribution $\mathcal{P}_K = \{p_K(\mathfrak{n}) : \mathfrak{n} \in E_K\}$, where*

$$p_K(\mathfrak{n}) = \frac{1}{G_K} \cdot \prod_{i=1}^{M} \pi^{(i)}_{n_i} .$$

Proof. It has to be shown, that product condition (ii) is fulfilled, such that Theorem 3 applies. Condition (ii) takes the form

$$\sum_{i=1}^{M}\sum_{j=1}^{M} \varphi^{(i)}(\mathfrak{n}, \mathfrak{n}_{(n_i-1,n_j+1)}) \cdot \alpha^{(i)}_{n_i,\, n_i-1} =$$
$$\sum_{i=1}^{M}\sum_{j=1}^{M} \varphi^{(i)}(\mathfrak{n}_{(n_i+1,n_j-1)}, \mathfrak{n}) \cdot \alpha^{(i)}_{n_i+1,\, n_i} \cdot \frac{\pi^{(i)}_{n_i+1}}{\pi^{(i)}_{n_i}} \frac{\pi^{(j)}_{n_j-1}}{\pi^{(j)}_{n_j}} ,$$

and this equation indeed holds according to relations (11) and (17):

$$\sum_{i=1}^{M}\sum_{j=1}^{M} q_{ij} \cdot \mu^{(i)}(n_i) = \sum_{i=1}^{M} \mu^{(i)}(n_i) \stackrel{!}{=} \sum_{i=1}^{M}\sum_{j=1}^{M} q_{ij} \cdot \lambda^{(i)} \cdot \frac{\mu^{(j)}(n_j)}{\lambda^{(j)}}. \qquad \square$$

4. On Markov Processes Related to Product-Form Queueing Networks with Non-Poissonian Input

Consider some finite state irreducible Markov process $\mathcal{X}^{(0)} = \{J_t : t \geq 0\}$ which represents the phase process of a Markovian arrival process (MAP) [3], [4]. Let $E^{(0)} = \{1, \ldots, m\}$ be the state space of $\mathcal{X}^{(0)}$, $\pi = (\pi_1, \ldots, \pi_m)$ its steady state distribution, and $D = D_0 + D_1$ its generator with $D_{0;jj^*} = \theta_j\, p_j(0, j^*)$ for $j \neq j^*$, $D_{1;jj^*} = \theta_j\, p_j(1, j^*)$, $D_{0;jj} = -\theta_j$. Here $p_j(n, j^*)$ gives the probability of a phase transition from j to j^* together with $n \in \{0, 1\}$ arrivals, and θ_j represents the overall rate of state changes of the MAP (phase changes and / or arrivals). The average rate of arrivals from phase j is $\gamma_j := \sum_{j^* \in E^{(0)}} D_{1;jj^*}$. Consider further an open product-form (PF) queueing network $\mathcal{N}$ with Poisson input of intensity $\gamma = \max_{j\in E^{(0)}} \sum_{j^*\in E^{(0)}} D_{1;jj^*} = \max_{j\in E^{(0)}} \gamma_j$. $\mathcal{N}$ still acquires equilibrium if γ is reduced to $\bar{\gamma}$ with

$$\bar{\gamma} = \sum_{j\in E^{(0)}} \sum_{j^*\in E^{(0)}} \pi_j D_{1;jj^*}. \tag{18}$$

A state of $\mathcal{N}$ at time $t \geq 0$ is described by $\mathfrak{x}_t = (x_1(t), \ldots, x_N(t)) \in \mathbb{N}_0^N$, where $x_\ell(t)$ is the number of customers in station ℓ ($\ell = 1, \ldots, N$). $\{\mathfrak{x}_t : t \geq 0\}$ defines a continuous time regular Markov chain; let $g = g(\bar{\gamma}) = ((g_{\mathfrak{x}\,\mathfrak{x}^*}))_{\mathfrak{x},\mathfrak{x}^* \in \mathbb{N}_0^N}$ be its generator. We assume that the steady state distribution of $\mathcal{N}$ takes the form

$$p(\bar{\gamma}, \mathfrak{x}) = \prod_{\ell=1}^{N} \beta_{x_\ell}^{(\ell)} \,, \tag{19}$$

where for each $\ell \in \{1, \ldots, N\}$, $\beta_{x_\ell}^{(\ell)}$ represents the steady state probability for finding x_ℓ customers in station ℓ in equilibrium. In order to describe internal state changes we use the shortcuts:

$$\begin{aligned}
\mathfrak{x}_{(x_\ell \pm 1)} &= (x_1, \ldots, x_{\ell-1}, x_\ell \pm 1, x_{\ell+1}, \ldots, x_N)\,, \\
\mathfrak{x}_{(x_\ell - 1, x_k + 1)} &= (x_1, \ldots, x_{\ell-1}, x_\ell - 1, x_{\ell+1}, \ldots, \\
&\quad\; x_{k-1}, x_k + 1, x_{k+1}, \ldots, x_N)\,.
\end{aligned}$$

Let $Q = ((q_{\ell\,k}))_{\ell \in \{1,\ldots,N\}, k \in \{0,1,\ldots,N\}}$ be the routing matrix of the queueing network; $q_{\ell\,k}$ symbolizes the probability for entering station k after leaving station ℓ, and $q_{\ell\,0}$ is the probability for leaving the network after service completion in station ℓ. Similarly, we set $q_{0\ell}$ for the probability, that a newly arriving customer from outside enters station ℓ, $1 \leq \ell \leq N$. This leads to $\sum_{k=1}^{N} q_{\ell\,k} = 1 - q_{\ell\,0}$ for $1 \leq \ell \leq N$, $\quad \sum_{\ell=1}^{N} q_{0\,\ell} = 1$, such that the matrix $((g_{\mathfrak{x}\,\mathfrak{x}^*}))_{\mathfrak{x},\mathfrak{x}^* \in \mathbb{N}_0^N}$ can be written as

$$g_{\mathfrak{x}\,\mathfrak{x}^*}(\bar{\gamma}) = \begin{cases} q_{0\ell} \cdot \bar{\gamma}\,, & \text{if } \mathfrak{x}^* = \mathfrak{x}_{(x_\ell+1)}\,, \\ q_{\ell\,k} \cdot \mu^{(\ell)}(x_\ell)\,, & \text{if } \mathfrak{x}^* = \mathfrak{x}_{(x_\ell-1, x_k+1)}\,, \\ q_{\ell\,0} \cdot \mu^{(\ell)}(x_\ell)\,, & \text{if } \mathfrak{x}^* = \mathfrak{x}_{(x_\ell-1)}\,, \\ 0 & \text{in any other case}\,. \end{cases} \tag{20}$$

$\mu^{(\ell)}(x_\ell)$ represents the service completion rate at station ℓ when there are x_ℓ customers resident. We assume, that each station ℓ in isolation, when fed by a Poisson process, is characterized by a birth-death process with average arrival rate $\lambda^{(\ell)}$. Then the following local balance equations hold:

$$\lambda^{((\ell)} = \frac{\beta_{x_\ell+1}^{(\ell)}}{\beta_{x_\ell}^{(\ell)}} \cdot \mu^{(\ell)}(x_\ell + 1)\,, \qquad x_\ell \in \mathbb{N}_0, \;\; \ell = 1, \ldots, N\,. \tag{21}$$

Since stability of the network implies stability of each station, we have $\lambda^{(\ell)} < \mu^{(\ell)}(x_\ell)$ for all $x_\ell \in \mathbb{N}_0$, $\ell \in \{1, \ldots, N\}$, and the $\lambda^{(\ell)}$ are obtained as solutions of the system of traffic equations

$$\bar{\gamma} \cdot q_{0\,\ell} + \sum_{k=1}^{N} \lambda^{(k)} \cdot q_{k\ell} = \lambda^{(\ell)} \qquad (\ell = 1, \dots, N)\ . \tag{22}$$

Let a matrix $L = ((L_{\mathfrak{n}\,\mathfrak{n}^*}))_{\mathfrak{n},\mathfrak{n}^* \in E}$ be defined in accordance to expression (10). With $M = 2$, this matrix reduces to

$$L_{\mathfrak{n}\,\mathfrak{n}^*} = \begin{cases} \varphi^{(\nu)}(\mathfrak{n}, \mathfrak{n}^*) \cdot \alpha^{(\nu)}_{n_\nu\, n^*_\nu}\,, & \text{if } \ n^*_\nu \neq n_\nu \ \text{ and } n^*_\iota = n_\iota\,, \\ & \text{for } \ \nu, \iota \in \{1,2\},\ \nu \neq \iota\,, \\ \varphi^{(12)}(\mathfrak{n}, \mathfrak{n}^*) \cdot \alpha^{(1)}_{n_1\, n^*_1}\, \alpha^{(2)}_{n_2\, n^*_2}\,, & \text{if } \ n^*_1 \neq n_1 \ \text{ and } n^*_2 \neq n_2\,, \\ -\Big(\vartheta_1(\mathfrak{n}) + \vartheta_2(\mathfrak{n})\Big)\,, & \text{if } \ \mathfrak{n}^* = \mathfrak{n}\,, \\ 0 & \text{in any other case}\ , \end{cases} \tag{23}$$

where $\vartheta_1(\mathfrak{n})$ and $\vartheta_2(\mathfrak{n})$ are defined in (9). We write $\mathfrak{n} = (j, \mathfrak{x})$ for the elements of $E = E^{(1)} \times \mathbb{N}_0^N$, j the phase identifier and $\mathfrak{x}$ representing a state vector of the queueing network. Since a change of phase in $\mathcal{X}^{(1)}$ may or may not be accompanied by an external arrival of the MAP, we have to distinguish two possibilities for both, the case that the network state vector $\mathfrak{x}$ remains unchanged and the case, that $\mathfrak{x}$ changes together with the phase transition. This can be reflected in the definition of the functions $\varphi^{(\nu)}$ $(\nu = 1, 2)$ and $\varphi^{(12)}$.

Formally, according to (20) and (23), the matrix $L = ((L_{\mathfrak{n}\,\mathfrak{n}^*}))_{\mathfrak{n},\mathfrak{n}^* \in E}$ takes the form

$$L_{\mathfrak{n}\,\mathfrak{n}^*} = \tag{24}$$

$$\begin{cases} \varphi^{(1)}((j,\mathfrak{x}),(j^*,\mathfrak{x})) \cdot D_{jj^*}\,, & \text{if } \ j^* \neq j\ ,\ \ \mathfrak{x}^* = \mathfrak{x}\,, \\ \varphi^{(2)}((j,\mathfrak{x}),(j,\mathfrak{x}_{(x_\ell+1)})) \cdot q_{0\ell} \cdot \bar{\gamma}, & \text{if } \ j^* = j\ ,\ \ \mathfrak{x}^* = \mathfrak{x}_{(x_\ell+1)}\,, \\ \varphi^{(12)}((j,\mathfrak{x}),(j^*,\mathfrak{x}_{(x_\ell+1)})) \cdot D_{jj^*} \cdot q_{0\ell} \cdot \bar{\gamma}, & \text{if } \ j^* \neq j\ ,\ \ \mathfrak{x}^* = \mathfrak{x}_{(x_\ell+1)}\,, \\ \varphi^{(2)}((j,\mathfrak{x}),(j,\mathfrak{x}_{(x_\ell-1,x_k+1)})) \cdot q_{\ell k} \cdot \mu^{(\ell)}(x_\ell), & \text{if } \ j^* = j\ ,\ \ \mathfrak{x}^* = \mathfrak{x}_{(x_\ell-1,x_k+1)}\,, \\ \varphi^{(12)}((j,\mathfrak{x}),(j^*,\mathfrak{x}_{(x_\ell-1,x_k+1)})) \cdot D_{jj^*} \cdot q_{\ell k} \cdot \mu^{(\ell)}(x_\ell), & \text{if } \ j^* \neq j\ ,\ \ \mathfrak{x}^* = \mathfrak{x}_{(x_\ell-1,x_k+1)}\,, \\ \varphi^{(2)}((j,\mathfrak{x}),(j,\mathfrak{x}_{(x_\ell-1)}) \cdot q_{\ell 0} \cdot \mu^{(\ell)}(x_\ell), & \text{if } \ j^* = j\ ,\ \ \mathfrak{x}^* = \mathfrak{x}_{(x_\ell-1)}\,, \\ \varphi^{(12)}((j,\mathfrak{x}),(j^*,\mathfrak{x}_{(x_\ell-1)}) \cdot D_{jj^*} \cdot q_{\ell 0} \cdot \mu^{(\ell)}(x_\ell), & \text{if } \ j^* \neq j\ ,\ \ \mathfrak{x}^* = \mathfrak{x}_{(x_\ell-1)}\,, \\ -\Big(\vartheta_1(\mathfrak{n}) + \vartheta_2(\mathfrak{n})\Big), & \text{if } \ \mathfrak{n}^* = \mathfrak{n}0, \text{in any other case}\ . \end{cases}$$

As a simplifying set-up, the corresponding functional values on the left hand side can be assumed to be independent of the specific stations where changes take place. This means, that station specific identifiers may be ignored, and the functional values of $\varphi^{(\nu)}$ ($\nu = 1$ or $\nu = 2$) and $\varphi^{(12)}$ can be written as matrix entries. More precisely, for any pair of station indices $\ell, k \in \{1, \ldots, N\}$, set

$$\begin{aligned}
\varphi^{(1)}((j,\mathfrak{z}),(j^*,\mathfrak{z})) &= \varphi^{(1)}((j^*,\mathfrak{z}),(j,\mathfrak{z})) =: U_{jj^*} \ , \\
\varphi^{(2)}((j,\mathfrak{z}),(j,\mathfrak{x}^{(j)}_{(x_\ell+1)})) &= \varphi^{(2)}((j,\mathfrak{x}^{(j)}_{(x_\ell-1)}),(j,\mathfrak{z})) =: A_{jj} \ , \\
\varphi^{(12)}((j,\mathfrak{z}),(j^*,\mathfrak{x}^{(j)}_{(x_\ell+1)})) &= \varphi^{(12)}((j^*,\mathfrak{x}^{(j)}_{(x_\ell-1)}),(j,\mathfrak{z})) =: A_{jj^*} \\
&\quad \text{for } j^* \neq j \ ,
\end{aligned}$$

$$
\begin{aligned}
\varphi^{(2)}((j,\mathfrak{z}),(j,\mathfrak{x}^{(j)}_{(x_\ell-1,x_k+1)})) &= \varphi^{(2)}((j,\mathfrak{x}^{(j)}_{(x_\ell+1,x_k-1)}),(j,\mathfrak{z})) =: B_{jj}\ , \\
\varphi^{(12)}((j,\mathfrak{z}),(j^*,\mathfrak{x}^{(j)}_{(x_\ell-1,x_k+1)})) &= \varphi^{(12)}((j^*,\mathfrak{x}^{(j)}_{(x_\ell+1,x_k-1)}),(j,\mathfrak{z})) =: B_{jj^*} \\
& \text{for } j^* \neq j\ , \\
\varphi^{(2)}((j,\mathfrak{z}),(j,\mathfrak{x}^{(j)}_{(x_\ell-1)})) &= \varphi^{(2)}((j,\mathfrak{x}^{(j)}_{(x_\ell+1)}),(j,\mathfrak{z})) =: C_{jj}\ , \\
\varphi^{(12)}((j,\mathfrak{z}),(j^*,\mathfrak{x}^{(j)}_{(x_\ell-1)})) &= \varphi^{(12)}((j^*,\mathfrak{x}^{(j)}_{(x_\ell+1)}),(j,\mathfrak{z})) =: C_{jj^*} \\
& \text{for } j^* \neq j\ .
\end{aligned}
$$

U is associated with pure phase changes of the MAP, A corresponds to external arrivals, B to network-internal movements, and C to departures from the network. All other pairs $(\mathfrak{n},\mathfrak{n}^*)$ shall be mapped to 0. We have $g_{\mathfrak{x}\mathfrak{x}_{(x_\ell+1)}} = q_{0\ell}\cdot\bar{\gamma}$, $g_{\mathfrak{x}\mathfrak{x}_{(x_\ell-1,x_k+1)}} = q_{\ell k}\cdot\mu^{(\ell)}(x_\ell)$, and $g_{\mathfrak{x}\mathfrak{x}_{(x_\ell-1)}} = q_{\ell 0}\cdot\mu^{(\ell)}(x_\ell)$. Therefore, the matrix L can be written as

$$
L_{\mathfrak{n}\mathfrak{n}^*} = \begin{cases}
U_{jj^*}\cdot D_{jj^*}, & \text{if } j^*\neq j\ ,\ \mathfrak{x}^* = \mathfrak{x}\,, \\
A_{jj}\cdot q_{0\ell}\cdot\bar{\gamma}, & \text{if } j^* = j\ ,\ \mathfrak{x}^* = \mathfrak{x}_{(x_\ell+1)}\,, \\
A_{jj^*}\cdot D_{jj^*}\cdot q_{0\ell}\cdot\bar{\gamma}, & \text{if } j^*\neq j\ ,\ \mathfrak{x}^* = \mathfrak{x}_{(x_\ell+1)}\,, \\
B_{jj}\cdot q_{\ell k}\cdot\mu^{(\ell)}(x_\ell), & \text{if } j^* = j\ ,\ \mathfrak{x}^* = \mathfrak{x}_{(x_\ell-1,x_k+1)}\,, \\
B_{jj^*}\cdot D_{jj^*}\cdot q_{\ell k}\cdot\mu^{(\ell)}(x_\ell), & \text{if } j^*\neq j\ ,\ \mathfrak{x}^* = \mathfrak{x}_{(x_\ell-1,x_k+1)}\,, \\
C_{jj}\cdot q_{\ell 0}\cdot\mu^{(\ell)}(x_\ell), & \text{if } j^* = j\ ,\ \mathfrak{x}^* = \mathfrak{x}_{(x_\ell-1)}\,, \\
C_{jj^*}\cdot D_{jj^*}\cdot q_{\ell 0}\cdot\mu^{(\ell)}(x_\ell), & \text{if } j^*\neq j\ ,\ \mathfrak{x}^* = \mathfrak{x}_{(x_\ell-1)}\,, \\
-\Big(\vartheta_1(\mathfrak{n})+\vartheta_2(\mathfrak{n})\Big), & \text{if } \mathfrak{n}^* = \mathfrak{n}\,, \\
0, & \text{in any other case}\ .
\end{cases} \tag{25}
$$

Notice that (25), in general, is *not* the generator matrix of queueing network $\mathcal{N}$ for the case that $\mathcal{N}$ has MAP-input! The next statement gives the definition of a Markov process describing the state process of some product form queueing network with non-Poissonian input and varying service rates. We set:

$$
\max_{j\in\{1,\ldots,m\}} |D_{jj}| = \delta\ ,
$$

and choose $U_{jj} = U_{jj^*} = 1$, $A_{jj} = B_{jj} = C_{jj} = 1 - |D_{jj}| \cdot \frac{\varepsilon}{\delta}$, and $A_{jj^*} = B_{jj^*} = C_{jj^*} = \frac{\varepsilon}{\delta}$ for all $j, j^* \in \{1, \ldots, m\}$, $j^* \neq j$, and some ε with $0 \leq \varepsilon < 1$. These settings guarantee that all matrix entries are positive.

Theorem 4. *Let the Markov chain $\mathcal{L}$ be determined by the following generator matrix L:*

$$L_{\mathfrak{n}\,\mathfrak{n}^*} = \begin{cases} D_{jj^*}, & \text{if } j^* \neq j\ ,\ \ \mathfrak{x}^* = \mathfrak{x}\,, \\ (1 - \frac{\varepsilon}{\delta}|D_{jj}|) \cdot q_{0\ell} \cdot \bar{\gamma}, & \text{if } j^* = j\ ,\ \ \mathfrak{x}^* = \mathfrak{x}_{(x_\ell+1)}\,, \\ \frac{\varepsilon}{\delta} \cdot D_{jj^*} \cdot q_{0\ell} \cdot \bar{\gamma}, & \text{if } j^* \neq j\ ,\ \ \mathfrak{x}^* = \mathfrak{x}_{(x_\ell+1)}\,, \\ (1 - \frac{\varepsilon}{\delta}|D_{jj}|) \cdot q_{\ell k} \cdot \mu^{(\ell)}(x_\ell), & \text{if } j^* = j\ ,\ \ \mathfrak{x}^* = \mathfrak{x}_{(x_\ell-1,x_k+1)}\,, \\ \frac{\varepsilon}{\delta} \cdot D_{jj^*} \cdot q_{\ell k} \cdot \mu^{(\ell)}(x_\ell), & \text{if } j^* \neq j\ ,\ \ \mathfrak{x}^* = \mathfrak{x}_{(x_\ell-1,x_k+1)}\,, \\ (1 - \frac{\varepsilon}{\delta}|D_{jj}|) \cdot q_{\ell 0} \cdot \mu^{(\ell)}(x_\ell), & \text{if } j^* = j\ ,\ \ \mathfrak{x}^* = \mathfrak{x}_{(x_\ell-1)}\,, \\ \frac{\varepsilon}{\delta} \cdot D_{jj^*} \cdot q_{\ell 0} \cdot \mu^{(\ell)}(x_\ell), & \text{if } j^* \neq j\ ,\ \ \mathfrak{x}^* = \mathfrak{x}_{(x_\ell-1)}\,, \\ -\Big(\vartheta_1(\mathfrak{n}) + \vartheta_2(\mathfrak{n})\Big), & \text{if } \mathfrak{n}^* = \mathfrak{n}\,, \\ 0, & \text{in any other case}\,. \end{cases}$$

Then $\mathcal{L}$ is irreducible ergodic with steady state distribution

$$P(j, \mathfrak{x}) = P(j, (x_1, \ldots, x_N)) = \pi_j \cdot \prod_{\ell=1}^{N} \beta_{x_\ell}^{(\ell)}\ .$$

Proof. The irreducibility of $\mathcal{L}$ is obvious, so only the second condition for a product connection has to be proved.

Referring to the form of condition (ii) in Section 2.3, the left hand side is easily seen to be identical with the expression

$$\sum_{j^* \neq j} U_{jj^*} \cdot D_{jj^*} + \sum_{\ell=1}^{N} q_{0\ell} \cdot \bar{\gamma} \cdot T(A) + \sum_{\ell=1}^{N} \sum_{k=1}^{N} q_{\ell k} \cdot \mu^{(\ell)}(x_\ell) \cdot T(B) + \sum_{\ell=1}^{N} q_{\ell 0} \cdot \mu^{(\ell)}(x_\ell) \cdot T(C)\ ,$$

where, for any $(m \times m)$-matrix V, the operator T is defined through

$$T(V) = V_{jj} + \sum_{\substack{j^* \in E^{(1)} \\ j^* \neq j}} V_{jj^*} D_{jj^*} \; .$$

Similarly, the right hand side of the expression in condition (ii) takes the form

$$\begin{aligned}\sum_{j^* \neq j} \frac{\pi_{j^*}}{\pi_j} \cdot U_{j^*j} \cdot D_{j^*j} + \sum_{\ell=1}^{N} q_{0\ell} \cdot \bar{\gamma} \cdot \frac{\beta^{(\ell)}_{x_\ell - 1}}{\beta^{(\ell)}_{x_\ell}} \cdot H(A) & \\ + \sum_{\ell=1}^{N} \sum_{k=1}^{N} q_{\ell k} \cdot \mu^{(\ell)}(x_\ell + 1) \cdot \frac{\beta^{(\ell)}_{x_\ell + 1}\, \beta^{(k)}_{x_k - 1}}{\beta^{(\ell)}_{x_\ell}\, \beta^{(k)}_{x_k}} \cdot H(B) & \\ + \sum_{\ell=1}^{N} q_{\ell 0} \cdot \mu^{(\ell)}(x_\ell + 1) \cdot \frac{\beta^{(\ell)}_{x_\ell + 1}}{\beta^{(\ell)}_{x_\ell}} \cdot H(C) \; , &\end{aligned}$$

where now the operator H is defined through

$$H(V) = V_{jj} + \sum_{\substack{j^* \in E^{(1)} \\ j^* \neq j}} V_{j^*j} \frac{\pi_{j^*}}{\pi_j} D_{j^*j}$$

for any $(m \times m)$-matrix V. The specific settings for matrices U, A, B, C imply

$$\sum_{j^* \neq j} U_{jj^*} \cdot D_{jj^*} - \sum_{j^* \neq j} \frac{\pi_{j^*}}{\pi_j} \cdot U_{j^*j} \cdot D_{j^*j} = \sum_{j^* \neq j} D_{jj^*} - \sum_{j^* \neq j} \frac{\pi_{j^*}}{\pi_j} \cdot D_{j^*j} = 0$$

as well as $T(A) = T(B) = T(C) = H(A) = H(B) = H(C) = 1$, such that the second product condition in this case reduces to the corresponding expression for a product form queueing network with Poisson input of intensity $\bar{\gamma}$ (compare with Section 3). □

For each ε, $0 \leq \varepsilon < 1$, $\mathcal{L}$ may be interpreted as the state process of some product form queueing network with variable input and service rates changing according to the variation of phases of a MAP. Note, that for $\varepsilon = 0$ the network has Poisson input with intensity $\bar{\gamma}$ and behaves independently with respect to the MAP.

5. Summary

We consider a family $\mathcal{X}^{(\nu)} = \{X^{(\nu)}(t) : t \in T\}$ of M ergodic homogeneous Markov chains $\mathcal{X}^{(\nu)} = \{X^{(\nu)}(t) : t \in T\}$ with steady state distributions $\pi^{(\nu)} = (\pi_0^{(\nu)}, \pi_1^{(\nu)} \ldots)$. For some ergodic M-dimensional Markov chain $\mathcal{L} = \{(X^{(1)}(t), \ldots, X^{(M)}(t)) : t \in T\}$ with transition intensity matrix L and steady state distribution $p(\mathfrak{n}) = p(n_1, \ldots, n_M) = \prod_\nu \pi_{n_\nu}^{(\nu)}$, the relation

$$\prod_\nu \pi_{n_\nu}^{(\nu)} \sum_{\mathfrak{n}^* \neq \mathfrak{n}} L_{\mathfrak{n}\mathfrak{n}^*} = \sum_{\mathfrak{n}^* \neq \mathfrak{n}} \prod_\nu \pi_{n_\nu^*}^{(\nu)} L_{\mathfrak{n}^*\mathfrak{n}}$$

represents the "product connection condition", saying that $\prod_\nu \pi_{n_\nu}^{(\nu)}$ is the equilibrium distribution of $\mathcal{L}$. The problem of finding product form steady state distributions for multidimensional Markov chains (e.g. for the state processes of Markovian queueing networks) can so be traced back to the task to construct the matrix L in a way such that the above relation is met. In this note we have presented several constructions of this type, defining the entries of L as products of generator elements associated with the chains $\mathcal{X}^{(\nu)}$. In Section 2 we proved that it is sufficient to define the components of L as product expressions of generator elements of the participating chains with factors being independent of the destination state. In Section 3 it is shown that the well known theorems of Jackson and Gordon-Newell (and, although not stated explicitely, the BCMP theorem) can be deduced by this method. Finally, in Section 4, a more general setting gives first hints how to investigate queueing networks with non-Markovian input.

Future research should deal with representations of queueing networks as quasi birth-death processes (QBDs) [6], [5], [2] and their integration into this framework.

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