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HOLOMORPHIC VECTOR BUNDLES ON OPEN SUBSETS OF STEIN MANIFOLDS

E. Ballico
Dept. of Mathematics
University of Trento
38050 Povo (TN), ITALY
e-mail: ballico@science.unitn.it

Abstract: Let Z be a connected complex Stein manifold and X an open subset of Z such that every holomorphic function on X extends to a holomorphic function on Z. Assume $X \neq Z$. Here we prove the existence of non-trivial holomorphic vector bundles on X. If Z is affine and X is an algebraic Zariski open subset of Z we prove the existence of algebraic vector bundles on X which are not trivial as holomorphic vector bundles.

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1. The Statements

Let Z be a complex Stein space. A famous theorem of Grauert says that the holomorphic and the topological classification of complex vector bundles on Z are the same. Quite often (for instance if Z is contractible or if Z is an open Riemann surface ([2], Th. 30.4)) this implies that every holomorphic vector bundle on Z is holomorphically trivial. The aim of this paper is to show that for open subsets of Z, which are not Stein, this is never the case (at least in the smooth case). For any complex space X let $\mathcal{O}_X$ be the sheaf of germs of holomorphic functions on X and $\mathbf{O}(X) = H^0(X, \mathcal{O}_X)$ the $\mathbf{C}$-algebra of all holomorphic functions on X. We prove the following result.

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Theorem 1. *Let Z be a smooth connected n-dimensional Stein manifold and X an open subset of Z with $X \neq Z$ and such that the restriction map $\mathbf{O}(Z) \to \mathbf{O}(X)$ is bijective, i.e. such that Z is the envelope of holomorphy of X. Then for every integer $r \geq n$ there is a holomorphic vector bundle E on X with rank$(E) = r$ and E not trivial.*

We will also work in the category of algebraic varieties. Abusing notations, for any algebraic scheme Y let $\mathcal{O}_Y$ be the sheaf of regular functions on Y. Y is smooth if and only if the associated complex space Y_{hol} is smooth. For any complex algebraic scheme $(X, \mathcal{O}_X)$ let $\mathcal{O}_{X_{hol}}$ be the sheaf of germs of holomorphic functions on X_{hol}. If E is an algebraic coherent sheaf on X, let E_{hol} denote the associated coherent $\mathcal{O}_{X_{hol}}$-sheaf on X_{hol}. E is locally free if and only if E_{hol} is locally free. We prove the following results.

Theorem 2. *Let X be an open subset of the smooth n-dimensional affine variety Z defined over an algebraically closed field $\mathbf{K}$. Fix an integer $r \geq n$. Assume $X \neq Z$ and that the restriction map $\rho_{Z,X} : H^0(Z, \mathcal{O}_Z) \to H^0(X, \mathcal{O}_X)$ is bijective. Then there exists a non-trivial algebraic vector bundle E on X with rank$(E) = r$.*

Theorem 3. *Let X be a Zariski open subset of the smooth n-dimensional complex affine variety Z. Assume $X \neq Z$ and that the restriction map $\rho_{Z,X}$: $H^0(Z, \mathcal{O}_Z) \to H^0(X, \mathcal{O}_X)$ is bijective. For every integer $r \geq n$ there is an algebraic vector bundle E on X with rank$(E) = r$ and such that E_{hol} is not holomorphically trivial.*

J. Winkelmann proved the existence of non-trivial holomorphic vector bundles on any positive dimensional compact complex manifold ([7]).

2. The Proofs

Lemma 1. *Let Y be a complex space or an algebraic variety and $L \in Pic(Y)$ such that L is not trivial. Then for all integers $a > 0$ the vector bundle $L \oplus \mathcal{O}_Y^{\oplus a}$ is not trivial.*

Proof. Since $\det(L \oplus \mathcal{O}_Y^{\oplus a}) \cong L$, $\det(L \oplus \mathcal{O}_Y^{\oplus a})$ is not trivial. Hence $L \oplus \mathcal{O}_Y^{\oplus a}$ is not trivial. □

Remark 1. Lemma 1 means that a non-trivial line bundle is not stably trivial. Hence if a complex space Y has a non-trivial line bundle, then for all integers $r \geq 1$ there is a rank r holomorphic vector bundle on Y which is not stably trivial. We recall that on any complex manifold Y with $H^1(Y, \mathcal{O}_Y) \neq 0$ there is a non-trivial line bundle (e.g. use the exponential sequence or that $H^1(Y, \mathcal{O}_Y)$ is isomorphic to the tangent space at $\mathcal{O}_Y$ of the Picard functor of Y).

Proposition 1. *Let Z be a reduced and irreducible complex Stein space with dim$(Z) = 2$ and Y an open subset of Z such that the restriction map $\mathbf{O}(Z) \to \mathbf{O}(Y)$ is bijective. Assume $Y \neq Z$. Then for every integer $r \geq 1$ there is a a holomorphic vector bundle E on Y with rank$(E) = r$ and E not stably trivial.*

Proof. Since the restriction map $\mathbf{O}(Z) \to \mathbf{O}(Y)$ is bijective and $Y \neq Z$, Y is not Stein. We have $H^1(Y, \mathcal{O}_Y) \neq 0$ (see [3], p. 159, for domains in $\mathbf{C}^n$, and [1] for a more general result and the vanishing of $H^i(Y, \mathcal{O}_Y)$ for every $i \geq 2$ ([7])). Apply Remark 1. □

Proof of Theorem 2. First we assume $n = 2$. Since X is not affine, we have $H^1(X, \mathcal{O}_X) \neq 0$ ([6], Th. 4.1, and the vanishing of $H^i(X, \mathcal{O}_X)$ for every $i \geq 2$ ([5])). Hence, the connected component of Pic(X) containing $\mathcal{O}_X$ has positive dimension and we conclude by Lemma 1 and Remark 1. Now assume $n \geq 3$. Let W be the union of all hypersurfaces of Z contained in $Z \backslash X$. Since Z is smooth, W is an effective Cartier divisor (perhaps empty). By [6], Prop. 2.3, $Z \backslash W$ is affine. Since every regular function on $Z \backslash W$ is regular at each point of X and the restriction map $\rho_{Z,X}$ is bijective, $W = \emptyset$. Fix $P \in Z \backslash X$. Since $r \geq n$, it is easy to prove the existence of $r+1$ regular functions $u_1, \dots, u_{r+1}$ on Z such that $\{P\} = \{u_1 = \cdots = u_{r+1} = 0\}$. The functions $u_1, \dots, u_{r+1}$ induce an injective map $u : \mathcal{O}_Z \to \mathcal{O}_Z^{\oplus(r+1)}$ of coherent sheaves. Set $F = \text{Coker}(u)$ and $E = F|X$. The coherent sheaf F is locally free on $Z \backslash \{P\}$, and hence E is locally free. By construction F has a length one resolution by locally free sheaves, i.e. it has homological dimension at most one. Since Z is smooth and $\dim(Z) \geq 3$, F is reflexive ([4], 1.3). F is not locally free near P because each u_i vanishes at P, and hence Im(u) is not a direct factor of $\mathcal{O}_Z^{\oplus(r+1)}$ in any neighborhood of P. Assume that E is trivial and take r regular sections $s_1, \dots, s_r$ of E which induce a trivialization of E, i.e. an isomorphism $E \cong \mathcal{O}_X^{\oplus r}$. Since F is reflexive, Z is normal and $Z \backslash X$ has codimension at least two, each s_i, $1 \leq i \leq r$, is the restriction of a unique $f_i \in H^0(Z, F)$. Let A be the set of all $Q \in Z \backslash \{P\}$ such that $f_1(Q), \ldots, f_r(Q)$ are not a basis of the fiber $F|\{Q\}$. Since the determinant of an $r \times r$ matrix of regular functions is a regular function and $F|Z\backslash\{P\}$ is locally free, A is an effective Cartier divisor of $Z\backslash\{P\}$. Since $A \cap X = \emptyset$ and $Z\backslash X$ has codimension at least two in Z, $A = \emptyset$. Thus $F|Z\backslash\{P\}$ is trivial. Since F is reflexive, F is uniquely determined by its restriction to $Z\backslash\{P\}$ ([4], 1.6). Thus F is trivial. Hence F is locally free at P, contradicting our construction of the sheaf F and concluding the proof of Theorem 2.

Proof of Theorem 3. As in the proof of Theorem 2 we obtain that $Z\backslash X$ contains no hypersurface of Z. Fix $P \in Z\backslash X$ and let F be the coherent sheaf on Z with $E = F|X$ locally free constructed in the proof of Theorem 2. The proof of Theorem 2 works for holomorphic sections $s_1, \ldots, s_r$ of E_{hol} and shows that E_{hol} is not holomorphically trivial.

Proof of Theorem 1. First assume $n = 2$. Since X is not Stein, we have $H^1(X, \mathcal{O}_X) \neq 0$ (see [3], p. 159, for domains in $\mathbf{C}^n$, and [1] for a more general result). Hence the connected component of $\mathrm{Pic}(X)$ containing $\mathcal{O}_X$ has positive dimension and we conclude by Lemma 1 and Remark 1. Now we assume $n \geq 3$. By Lemma 1 and Remark 1 we may (and will) assume $H^1(X, \mathcal{O}_X) = 0$. Let W be the union of all hypersurfaces of Z contained in $Z\backslash X$. Since Z is smooth, W is an effective Cartier divisor (perhaps empty). Hence $Z\backslash W$ is Stein. Since every holomorphic function on $Z\backslash W$ is holomorphic at each point of X and the restriction map $\mathbf{O}(Z) \to \mathbf{O}(X)$ is bijective, $W = \emptyset$. Fix $P \in Z\backslash X$. Since $r \geq n$, it is easy to prove the existence of $r+1$ holomorphic functions $u_1, \ldots, u_{r+1}$ on Z such that $\{P\} = \{u_1 = \cdots = u_{r+1} = 0\}$. The functions $u_1, \ldots, u_{r+1}$ induce an injective map $u : \mathcal{O}_Z \to \mathcal{O}_Z^{\oplus(r+1)}$ of coherent sheaves. Set $F = \mathrm{Coker}(u)$ and $E = F|X$. The coherent sheaf F is locally free on $Z\backslash\{P\}$, and hence E is locally free. By construction F has a length one resolution by locally free sheaves, i.e. it has homological dimension at most one. Since Z is smooth and $\dim(Z) \geq 3$, F is reflexive ([4], 1.3). F is not locally free near P because each u_i vanishes at P, and hence $\mathrm{Im}(u)$ is not a direct factor of $\mathcal{O}_Z^{\oplus(r+1)}$ in any neighborhood of P. Assume that E is trivial and take r regular sections $s_1, \ldots, s_r$ of E which induce a trivialization $E \cong \mathcal{O}_X^{\oplus r}$ of E. From the exact sequence on X

$$0 \to \mathcal{O}_X \to \mathcal{O}_X^{\oplus(r+1)} \to E \to 0 \tag{1}$$

and the assumption $H^1(X, \mathcal{O}_X) = 0$ we obtain that each section s_i, $1 \leq i \leq r$, lifts to $m_i \in H^0(X, \mathcal{O}_X^{\oplus(r+1)})$. Since the restriction map $\mathbf{O}(Z) \to \mathbf{O}(X)$ is bijective, each m_i gives a section $n_i \in H^0(Z, \mathcal{O}_Z^{\oplus(r+1)})$. The existence of the quotient map $\mathcal{O}_Z^{\oplus(r+1)} \to F$ shows that each n_i gives $a_i \in H^0(Z, F)$, $1 \le i \le r$. By construction $a_i|X = s_i$. Since Z is smooth and F is reflexive, there is $f_i \in H^0(Z, F)$ with $f_i|Z\backslash\{P\} = a_i$ ([4], 1.6). Now repeat verbatim the proof of Theorem 2 to conclude the proof of Theorem 1.

Remark 2. In the set-up of Theorems 1, 2 and 3 assume that Z is an open subset of $\mathbf{C}^n$ (or of $\mathbf{K}^n$ for Theorem 2). Then the same result is true taking any $r \ge n-1$. Indeed, look at the proofs of Theorems 2, 3 and 1 and take $P = (c_1, \dots, c_n) \in Z\backslash X$. Among the functions $u_1, \dots, u_{r+1}$ it is sufficient to take the functions $z_i - c_i$, $1 \le i \le n$, and $r+1-n$ arbitrary regular functions vanishing at P.

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