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THE RUDIN-CARLESON THEOREM FOR NON-HOMOGENEOUS DIFFERENTIAL FORMS

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Abstract: Let Ω be a bounded domain of $\mathbb{R}^n$ such that its boundary is a Lyapunov hypersuface Σ and $\mathbb{R}^n - \overline{\Omega}$ is connected. It is proved that if S is a closed subset of $(n-1)$-dimensional Lebesgue zero measure on Σ, and if f and $\tilde{f}$ are continuous non-homogeneous differential forms on S, then there exists a non-homogeneous differential form U on Ω which is self-conjugate in Ω and such that U and its adjoint form extend f and $\tilde{f}$ respectively. This result holds for any $n \geq 2$ and it generalizes the classical Rudin-Carleson theorem.

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1. Introduction

W. Rudin (1956, [8]) and L. Carleson (1957, [2]) independently proved the following theorem related to the theory of holomorphic functions of one complex variable:

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Theorem 1.1. *If S is a closed set of Lebesgue zero measure on the unit circle $\partial U = \{z \in \mathbb{C} : |z| = 1\}$ and if f is a continuous function on S, then there exists a continuous function F on $\overline{U} = \{z \in \mathbb{C} : |z| \leq 1\}$ which is holomorphic on the unit open disk U and such that $F(z) = f(z)$ for all z in S.*

The aim of this paper is to prove a general Rudin-Carleson theorem for non-homogeneous differential forms in $\mathbb{R}^n$ which is a generalization of Theorem 1.1.

The next section summarizes some results on k-forms and k-measures which we shall use. In Section 3, we prove a general Rudin-Carleson result in $\mathbb{R}^n$ which is obtained by using the approach of Bishop [1]. In the last Section the Rudin-Carleson theorem for non-homogeneous differential forms is proved.

2. Definitions and Notation

In this section we recall some definitions on k-forms and k-measures which we shall need in the sequel. Let $W^r \subset \mathbb{R}^n$ be an oriented r-dimensional differential simple manifold of class p. A *differential form of degree k* (briefly a *k-form*) on W^r is a function defined on W^r which values are in the k-covectors space Σ_k on $\mathbb{R}^n$. A k-form is represented in an admissible coordinate system $(x_1, \ldots, x_n)$ as

$$u = \frac{1}{k!} u_{s_1 \ldots s_k} dx^{s_1} \ldots dx^{s_k} \, ,$$

where $u_{s_1 \ldots s_k}$ are the components of a k-covector, i.e. the components of a skew-symmetric covariant tensor.

$C^q_k(W^r)$ denotes the space of k-forms which components are continuously differentiable up to the order q in a coordinate system of class C^{q+1} (and then in every coordinate system of class C^{q+1}). $L^p_k(W^r)$ denotes the space of k-forms which components are L^p real functions in a coordinate system of class C^1 (and then in every coordinate system of class C^1).

Let Ω be a domain of $\mathbb{R}^n$; if $u \in C^1_k(\Omega)$, the *differential* of u is the following $(k+1)$-form

$$du = \frac{1}{k!} \frac{\partial}{\partial x^j} u_{s_1 \ldots s_k} dx^j dx^{s_1} \ldots dx^{s_k}.$$

If $u \in C_k(\Omega)$ the *adjoint* of u is the $(n-k)$-form defined as

$$*u = \frac{1}{k!(n-k)!} \delta^{1 \ldots\ldots\ldots\ldots n}_{s_1 \ldots s_k i_1 \ldots i_{n-k}} u_{s_1 \ldots s_k} dx^{i_1} \ldots dx^{i_{n-k}}.$$

We remark that $**u = (-1)^{k(n-k)}u$. If $u \in C^1_k(\Omega)$ the following $(k-1)$-form

$$\delta u = (-1)^{n(k+1)+1} * d * u$$

defines the *co-differential* of u. If $u \in C^2_k(\Omega)$, the *Laplacian* of u is

$$\Delta_2 u = -(d\delta + \delta d)u.$$

A k-form $u \in C^2_k(\Omega)$ is *harmonic* (i.e. $\Delta_2 u = 0$) if, and only if, all the coefficients $u_{s_1 \dots s_k}$ of u are harmonic functions.

We consider the linear space $\Sigma_{(n)} = \Sigma_0 \oplus \dots \oplus \Sigma_n$ obtained by the direct sum of k-covector spaces Σ_k $(k = 0, \dots n)$. By a *non-homogeneous differential form* we mean a member U of $\Sigma_{(n)}$: $U = \sum_{k=0}^n u_k$, where u_k are k-forms. By $C^q(\Omega)$ we denote the space $C^q_0(\Omega) \oplus \dots \oplus C^q_n(\Omega)$ of non homogeneous differential forms of class C^q in Ω. Likewise to the k-forms, if $U \in C^1(\Omega)$ we define the *differential*, the *adjoint*, the *co-differential*, the *Laplacian* of U respectively as

$$dU = \sum_{h=0}^{n-1} du_h, \quad *U = \sum_{h=0}^{n} *u_h, \quad \delta U = \sum_{h=1}^{n} \delta u_h, \quad \Delta_2 U = \sum_{h=0}^{n} \Delta_2 u_h.$$

A non-homogeneous differential form $U \in C^1(\Omega)$ is called *self-conjugate* (Cialdea [3]) in Ω if $dU = \delta U$ in Ω, i.e. if

$$\delta u_1 = 0, \qquad du_k = \delta u_{k+2}, \ \ (k = 0, \dots, n-2) \qquad du_{n-1} = 0.$$

Note that if U is self-conjugate then U is harmonic, and therefore all the coefficients of u_k are harmonic functions.

Let us consider now some interesting examples. For $n = 2$, if $U = u_0 + u_2$, where $u_0 \equiv u$ is a scalar and $u_2 = v dx dy$ is a 2-form, we have that U is self-conjugate if, and only if, the function $f(z) = u + iv$ is holomorphic.

For $n = 3$, if $U = u_0 + u_2$, where $u_0 \equiv u$, and $u_2 = v_1 dx^2 dx^3 + v_2 dx^3 dx^1 + v_3 dx^1 dx^2$, U is self-conjugate if, and only if, (u, v_1, v_2, v_3) is solution of the *Moisil-Theodorescu system* (Moisil et al [6]):

$$\begin{cases} grad\ u = curl\ (v_1, v_2, v_3)\,, \\ div\ (v_1, v_2, v_3) = 0\,. \end{cases} \tag{2.1}$$

For $n = 4$, if $U = u_0 + u_2 + u_4$, where $u_0 = f_0$,

$$u_2 = f_1(dx^1 dx^2 - dx^3 dx^4) + f_2(dx^1 dx^3 - dx^4 dx^2) + f_3(dx^1 dx^4 - dx^2 dx^3),$$

$$u_4 = f_0 dx^1 dx^2 dx^3 dx^4$$

is self-conjugate if, and only if, (f_0, f_1, f_2, f_3) is solution of the *Fueter system* (Fueter [5] and Sudbery [10]):

$$\begin{cases} \frac{\partial f_0}{\partial x_1} - \frac{\partial f_1}{\partial x_2} - \frac{\partial f_2}{\partial x_3} - \frac{\partial f_3}{\partial x_4} = 0\,, \\ \frac{\partial f_0}{\partial x_2} + \frac{\partial f_1}{\partial x_1} - \frac{\partial f_2}{\partial x_4} + \frac{\partial f_3}{\partial x_3} = 0\,, \\ \frac{\partial f_0}{\partial x_3} + \frac{\partial f_1}{\partial x_4} + \frac{\partial f_2}{\partial x_1} - \frac{\partial f_3}{\partial x_2} = 0\,, \\ \frac{\partial f_0}{\partial x_4} - \frac{\partial f_1}{\partial x_3} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_1} = 0\,. \end{cases} \tag{2.2}$$

For any $n \geq 2$, if $U = u_1$, where $u_1 = v_i dx^i$, U is self-conjugate if, and only if, $(v_1, \ldots, v_n)$ is a *harmonic vector*, i.e. it is solution of the following system (Sudbery [10] and Stein et al [11]):

$$\begin{cases} curl\ (v_1, \ldots, v_n) = 0\,, \\ div\ (v_1, \ldots, v_n) = 0\,. \end{cases} \tag{2.3}$$

More generally, if $U = \omega_k$, ω_k being a differential form of degree k, U is self-conjugate if, and only if, ω_k is a harmonic form, i.e.

$$d\omega_k = 0, \qquad \delta\omega_k = 0. \tag{2.4}$$

We give now the concept of a k-measure which was introduced by Fichera [4]. A *k-measure* μ is the object determined (in a fixed coordinate system $(t^1, \ldots, t^r)$) by the coefficients $\mu_{s_1 \ldots s_k}(A)$, where $\mu_{s_1 \ldots s_k}$ is a measure on the family $\{A\}$ of all the Borel sets of W^r, and $\mu_{s_1 \ldots s_k}(A)$ depends skew-symmetrically on the indices $s_1, \ldots, s_k$. Moreover, if $\tilde{\mu}_{i_1 \ldots i_k}(A)$ are the components of the k-measure μ in the coordinate system $(\tau^1, \ldots, \tau^r)$, we have for each Borel set $A \in \{A\}$

$$\tilde{\mu}_{i_1 \ldots i_k}(A) = \int_A \left[det \frac{\partial(\tau^1, \ldots, \tau^r)}{\partial(t^1, \ldots, t^r)} \right] \frac{\partial t^{s_1}}{\partial \tau^{i_1}} \cdots \frac{\partial t^{s_k}}{\partial \tau^{i_k}} d\mu_{s_1 \ldots s_k}.$$

The space of all the k-measures on W^r is denoted by $M_k(W^r)$.

If $u \in C_k(W^r)$ and $v \in C_h(W^r)$, the *exterior product* between u and v is defined as the following $(k+h)$-form:

$$u \wedge v = \frac{1}{k!h!} u_{s_1 \dots s_k} v_{i_1 \dots i_h} dx^{s_1} \dots dx^{s_k} dx^{i_1} \dots dx^{i_h}.$$

If $u \in C_k(W^r)$ and $\mu \in M_{r-k}(W^r)$, we set

$$\int_{W^r} u \wedge \mu = \frac{1}{k!(r-k)!} \delta^{s_1 \dots \dots s_n}_{1 \dots \dots \dots n} \int_{W^r} u_{s_1 \dots s_k} d\mu_{s_{k+1} \dots s_n}$$

By definition, $\mu \wedge u = (-1)^{k(r-k)}(u \wedge \mu)$.

We remember that if $\mu \in M_k(W^r)$, there exist a k-form $g \in L^1_k(W^r)$ and a singular k-measure $\mu^* \in M_k(W^r)$ such that

$$\mu_{s_1 \dots s_k}(A) = \mu^*_{s_1 \dots s_k}(A) + \int_A g_{s_1 \dots s_k}\, dx\,,$$

where $x(A)$ is the r-dimensional Lebesgue measure on W^r. We say that μ is *absolutely continuous* if $\mu^* = 0$.

We introduce now a *non-homogeneous measure*. Obviously this is generalization of the concept of non-homogeneous differential form. In fact, by a non-homogeneous measure α we mean $\sum_{k=0}^r \alpha^k$, where α^k are k-measures. $M(W^r)$ denotes the direct sum $M_0(W^r) \oplus \dots \oplus M_r(W^r)$.

If W^r is a compact oriented r-dimensional differential simple manifold, we can define in $C_k(W^r)$ the following norm

$$\|u\| = \sum_{h=1}^{l} \sum_{s_1 \dots s_k} \max_{C_h} |u_{s_1 \dots s_k}|\,,$$

where $C_1, \dots, C_l$ are closed sets covering W^r. If $U \in C(W^r)$, the norm of U is the sum of the component norms of U.

We give now a theorem which is the extension of the classical Riesz representation theorem related to the non-homogenous differential forms:

Theorem 2.1. *If $\Lambda(U)$ is a continuous linear functional on $C(W^r)$, there exists one and only one measure $\mu \in M(W^r)$ such that $\Lambda(U) = \int U \wedge \mu$.*

Proof. This result follows immediately from Fichera [4, 1.VI]. Note that $\int U \wedge \mu$ is the integral of U with respect to the non-homogeneous measure μ. It is defined as

$$\int_{W_r} u_0 \wedge \mu^r + \dots + u_k \wedge \mu^{r-k} + \dots + u_r \wedge \mu^0$$

if $U = u_0 + \ldots + u_r$ and $\mu = \mu^0 + \ldots + \mu^r$. $\square$

3. Preliminary Results

In this section, Ω is a bounded domain of $\mathbb{R}^n$, and Σ is its boundary $\partial\Omega$. We denote by $[C(\Sigma)]^2$ the cartesian product $C(\Sigma) \times C(\Sigma)$. If $(f, g) \in [C(\Sigma)]^2$, we define $\|(f, g)\| = \|f\| + \|g\|$.

Lemma 3.1. *Let D be a linear closed subspace of $[C(\Sigma)]^2$. Let S be a closed subset of Σ such that if $(\mu, \tilde{\mu}) \in D^\perp$ then $(\mu, \tilde{\mu})|_S = 0$. If $(f, \tilde{f}) \in [C(S)]^2$ is such that $\|(f, \tilde{f})\| \leq r < 1$, then there exists $(F, \tilde{F}) \in D$ such that $\|(F, \tilde{F})\| \leq 1$ and $(F\big|_S, \tilde{F}\big|_S) = (f, \tilde{f})$.*

Proof. Let C_r be the space of all the D elements which have the norm less then $r \leq 1$. We consider the restriction mapping Ψ of D into $[C(S)]^2$. We suppose *ab absurdo* that $(f, \tilde{f}) \notin \overline{\Psi(C_r)}$; thus there exists a continuous linear functional A on $[C(S)]^2$ such that $A[(f, \tilde{f})] > 1$ and $|A(g, \tilde{g})| < 1, \forall (g, \tilde{g}) \in \overline{\Psi(C_r)}$. It follows from Theorem 2.1 that there exists $(\nu, \tilde{\nu}) \in [M(S)]^2$ such that

$$\int_S g \wedge \nu + \tilde{g} \wedge \tilde{\nu} = A[(g, \tilde{g})], \qquad \forall (g, \tilde{g}) \in [C(S)]^2.$$

We define the continuous linear functional B as the composition of A and Ψ. Furthermore $\|B\| \leq \frac{1}{r}$. It follows from the classical Hahn-Banach theorem that there exists a continuous linear functional $\tilde{B}$ on $[C(\Sigma)]^2$ which extends B and $\|\tilde{B}\| = \|B\|$. From Theorem 2.1 there exists $(\vartheta, \tilde{\vartheta}) \in [C(\Sigma)]^2$ such that

$$\int_\Sigma g \wedge \vartheta + \tilde{g} \wedge \tilde{\vartheta} = \tilde{B}[(g, \tilde{g})], \qquad \forall (g, \tilde{g}) \in [C(\Sigma)]^2$$

and $\|\tilde{B}\| = \|(\vartheta, \tilde{\vartheta})\|$. We can extend $(\nu, \tilde{\nu})$ on Σ setting $(\delta, \tilde{\delta})(C) = (\nu, \tilde{\nu})(C \cap S)$ for every subset C of Σ and $(\delta, \tilde{\delta}) \equiv 0$ outwards of S. Thus, the measure $(\mu, \tilde{\mu}) = (\delta - \vartheta, \tilde{\delta} - \tilde{\vartheta})$ is in $D^\perp$; in fact for each $(h, \tilde{h}) \in [C(\Sigma)]^2$

$$\int_\Sigma h \wedge \mu + \tilde{h} \wedge \tilde{\mu} = \int_\Sigma h \wedge \delta + \tilde{h} \wedge \tilde{\delta} - \int_\Sigma h \wedge \vartheta + \tilde{h} \wedge \tilde{\vartheta} =$$

$$\int_S [h|_S \wedge \nu + \tilde{h}|_S \wedge \tilde{\nu}] - \tilde{B}(h, \tilde{h}) = 0.$$

Therefore $(\mu,\tilde\mu)\big|_S=0$, but for $(f,\tilde f)\in[C(S)]^2$ we obtain a contradiction:

$$\int_S f\wedge\mu+\tilde f\wedge\tilde\mu>1-\int_S f\wedge\vartheta+\tilde f\wedge\tilde\vartheta\ge 1-\sup_{\|(f,\tilde f)\|\le r}\Big|\int f\wedge\vartheta+\tilde f\wedge\tilde\vartheta\Big|\ge$$

$$1-\|(f,\tilde f)\|\ \|(\vartheta,\tilde\vartheta)\|=1-\|(f,\tilde f)\|\ \|B\|\ge 0.$$

This shows that $(f,\tilde f)\in\overline{\Psi(C_r)}$. Hence, there exists $(F_1,\tilde F_1)\in D$, where $F_1=\sum_{k=0}^{n-1}F_{1,k}$ and $\tilde F_1=\sum_{k=0}^{n-1}\tilde F_{1,k}$, such that $\|(F_1,\tilde F_1)\|\le r$ and the absolute value of the difference of all the coefficients of $f-F_1$ and $\tilde f-\tilde F_1$ is $\le\frac{1-r}{2^{n+1}l}$ (note that n is fixed) on S. So $\|f-F_1\|=\sum_{h=1}^{l}\sum_{s_1\dots s_k}\max_{C_h\cap S}|((f_k)_{s_1\dots s_k}-(F_{1,k})_{s_1\dots s_k})|\le\frac{1-r}{4}$, where $C_1,\dots,C_l$ are the closed sets covering Σ. Likewise

$$\|\tilde f-\tilde F_1\|=\sum_{h=1}^{l}\sum_{s_1\dots s_k}\max_{C_h\cap S}|((\tilde f_k)_{s_1\dots s_k}-(\tilde F_{1,k})_{s_1\dots s_k})|\le\frac{1-r}{4},$$

then $\|(f-F_1,\tilde f-\tilde F_1)\|\le\frac{1-r}{2}$.

We set $(h,\tilde h)=(f-F_1,\tilde f-\tilde F_1)\in[C(S)]^2$, it follows from the result just proved that there exists $(F_2,\tilde F_2)\in D$ such that $\|F_2,\tilde F_2\|\le\frac{1-r}{2}$ and the absolute value of the difference of all the coefficients of $(f-F_1-F_2)$ and $(\tilde f-\tilde F_1-\tilde F_2)$ is $\le\frac{1-r}{2^{n+1}l}\frac{1}{2^2}$ on S. Therefore $\|(f-F_1-F_2,\tilde f-\tilde F_1-\tilde F_2)\|\le\frac{1-r}{2}\frac{1}{2^2}$.

By induction, we find a sequence $\{(F_m,\tilde F_m)\}\in D$ such that

$$\|(F_1,\tilde F_1)\|\le r,\quad \|(F_m,\tilde F_m)\|\le\frac{1-r}{2^{(m-1)}},$$

$m\ge 2$ and the absolute value of the difference of all the coefficients of $[f-\sum_{j=1}^m F_j]$ and $[\tilde f-\sum_{j=1}^m\tilde F_j]$ is $\le\frac{1-r}{l2^{n+1}}\frac{1}{2^m}$ on S. So $\|(f-\sum_{j=1}^m F_j,\tilde f-\sum_{j=1}^m\tilde F_j)\|\le\frac{1-r}{2}\frac{1}{2^m}$. Then the series $\sum_{m=1}^\infty(F_m,\tilde F_m)$ converges, and we say $(F,\tilde F)\in[C(\Sigma)]^2$ its sum. Since D is a closed subspace, $(F,\tilde F)\in D$ and

$$\|(F,\tilde F)\|\le\sum_{m=1}^\infty\|(F_m,\tilde F_m)\|<r+(1-r)\sum_{m=2}^\infty\frac{1}{2^{m-1}}=1.$$

It is evident that $(F\big|_S,\tilde F\big|_S)=(f,\tilde f)$. □

From the previous result the next theorem follows:

Theorem 3.1. *Let D be a linear closed subspace of $[C(\Sigma)]^2$. Let S be a closed subset of Σ such that if $(\mu,\tilde\mu)\in D^\perp$ then $(\mu,\tilde\mu)|_S=0$. If $(f,\tilde f)\in[C(S)]^2$ and γ is a positive continuous function on Σ such that $\|(f,\tilde f)\|\le\gamma$ on S, then there exists $(F,\tilde F)\in D$ such that $(F\big|_S,\tilde F\big|_S)=(f,\tilde f)$.*

Proof. Let $E = \left\{(g, \tilde{g}) \in [C(\Sigma)]^2 \,/\, \gamma(g, \tilde{g}) \in D\right\} \subset [C(\Sigma)]^2$. The orthogonal of E consists of $(\gamma\nu, \gamma\tilde{\nu}) \in [M(\Sigma)]^2$ such that $(\nu, \tilde{\nu}) \in D^\perp$. We observe that if $(\mu, \tilde{\mu}) \in E^\perp$ then $(\mu, \tilde{\mu})\big|_S = 0$. So, it follows from Lemma 3.1 that there exists $(G, \tilde{G}) \in E$ such that $G\big|_S = \frac{f}{\gamma}$ and $\tilde{G}\big|_S = \frac{\tilde{f}}{\gamma}$. Then, there exists $(F, \tilde{F}) \in D$ such that $(\gamma G, \gamma\tilde{G}) = (F, \tilde{F})$. Therefore $(F, \tilde{F})$ extends $(f, \tilde{f})$. □

We remark that this theorem could also deduced from a general result Saab [9, Corollary 3].

4. The Rudin-Carleson Theorem in $\mathbb{R}^n$

From now on, Ω is assumed to be a bounded domain of $\mathbb{R}^n$ such that its boundary $\partial\Omega$ is a Lyapunov hypersuface Σ (i.e. Σ has a uniformly Hölder continuous normal field of some exponent $\lambda \in (0, 1]$) and such that $\mathbb{R}^n - \overline{\Omega}$ is connected.

At first let us prove the following theorem:

Theorem 4.1. *If $\alpha = \alpha^0 + \ldots + \alpha^{n-1}$, $\tilde{\alpha} = \tilde{\alpha}^n + \ldots + \tilde{\alpha}^1 \in M(\Sigma)$ are such that*

$$\int_\Sigma \left[\alpha \wedge *U + U \wedge \tilde{\alpha}\right] = 0 \tag{4.1}$$

for every non-homogeneous differential form $U = \sum_{j=0}^n u_j \in C^\infty(\Omega) \cap C^0(\overline{\Omega})$ which is self-conjugate in Ω. Then α and $\tilde{\alpha}$ are absolutely continuous (i.e. all the components of α and $\tilde{\alpha}$ are absolutely continuous k-measures).

Proof. We set $U = u_{k-1} + u_{k+1}$ $(k = 1, \ldots, n-1)$ with

$$u_{k-1} = -\delta\omega_h^{i_1 \ldots i_k}, \quad u_{k+1} = d\omega_h^{i_1 \ldots i_k}\,, \tag{4.2}$$

where $\omega_h^{i_1 \ldots i_k}$ is the k-form $\omega_h dx^{i_1} \ldots dx^{i_k}$ and ω_h are harmonic polynomials of degree h. U is self-conjugate, in fact

$$du_{k-1} = -d\delta\omega_h^{i_1 \ldots i_k} = \delta d\omega_h^{i_1 \ldots i_k} = \delta u_{k+1}$$

and

$$du_{k+1} = d^2\omega_h^{i_1 \ldots i_k} = 0, \quad \delta u_{k-1} = \delta^2\omega_h^{i_1 \ldots i_k} = 0,$$

for $k = 1, \ldots, n-1$. Substituting (4.2) in (4.1) we obtain

$$\int_\Sigma \Big[\alpha^k \wedge *d\omega_h^{i_1 \ldots i_k} - \alpha^{k-2} \wedge *\delta\omega_h^{i_1 \ldots i_k} - \delta\omega_h^{i_1 \ldots i_k} \wedge \tilde{\alpha}^k + d\omega_h^{i_1 \ldots i_k} \wedge \tilde{\alpha}^{k+2}\Big] = 0 \tag{4.3}$$

for $k = 1, \dots, n-1$. Equality (4.3) are the conditions of Cialdea [3, Theorem X] which assure the absolute continuity of α and $\tilde{\alpha}$. $\square$

We observe that the previous result in the case $n = 2$ gives the classical Brothers Riesz theorem [7].

We show now the Rudin-Carleson theorem for non-homogeneous differential forms:

Theorem 4.2. *If S is a closed subset of $(n-1)$-dimensional Lebesgue zero measure on Σ and if f and $\tilde{f}$ are continuous non-homogeneous differential forms on S, then there exists a non-homogeneous differential form U on Ω which is self-conjugate in Ω and such that U and its adjoint form $*U$ extend f and $\tilde{f}$, respectively.*

Proof. Let

$$D = \left\{ (U, *U)|_\Sigma \ / \ U \in C^\infty(\Omega) \cap C^0(\overline{\Omega}),\ dU = \delta U \text{ in } \Omega \right\} \tag{4.4}$$

be the subspace of $[C_0(\Sigma) \oplus \dots \oplus C_{n-1}(\Sigma)]^2$. Its orthogonal is

$$D^\perp = \left\{ (\mu, \tilde{\mu}) \ / \int_\Sigma [\mu \wedge *U + U \wedge \tilde{\mu}] = 0, \quad \forall (U, *U) \in D \right\}. \tag{4.5}$$

$D^\perp \subseteq [M_0(\Sigma) \oplus \dots \oplus M_{n-1}(\Sigma)]^2$. Hence, if $(\mu, \tilde{\mu}) \in D^\perp$, from Theorem 4.1, μ and $\tilde{\mu}$ are absolute continuous (that is, all the components of every k-measure of μ and $\tilde{\mu}$ are absolutely continuous). Moreover, if S is a closed subset of $(n-1)$-dimensional Lebesgue zero measure on Σ, we have that $(\mu, \tilde{\mu})|_S = 0$. Hence, it follows from Theorem 3.1 that there exists $(F, *F) \in D$ such that $F\ |_S = f$ and $*F\ |_S = \tilde{f}$, therefore there exists $U \in C^\infty(\Omega) \cap C^0(\overline{\Omega})$ self-conjugate in Ω for that holds the thesis. $\square$

Let us consider now some applications. For $n = 2$, we get the Rudin-Carleson theorem in the real form. It is enough to take D as the subspace of the restrictions to Σ of all the conjugate functions u and v in Ω, i.e. the functions u and v satisfying the *Cauchy-Riemann equations*: $u_x = v_y,\ u_y = -v_x$.

For $n = 3$, we achieve the following theorem providing an interesting property of solutions of the Moisil-Theodorescu system (2.1):

Theorem 4.3. *Let $f_0 \in C_0(S)$, $f_1 \in C_1(S)$, $f_2 \in C_2(S)$, where S is a closed of 2-dimensional Lebesgue zero measure on $\Sigma = \partial\Omega$, there exists a non-homogeneous form $U = u_0 + u_2 \in C^\infty(\Omega) \cap C^0(\overline{\Omega})$, where $u_0 \equiv u$ and*

$u_2 = v_1dx^2dx^3 + v_2dx^3dx^1 + v_3dx^1dx^2$ such that (u, v_1, v_2, v_3) is solution of (2.1) and such that $u_0\big|_S = f_0$, $u_2\big|_S = f_2$ and $*u_2\big|_S = f_1$.

Proof. It is enough to take $D = \big\{(U, *U)\big|_\Sigma \ / \ U = u_0 + u_2 \in C^\infty(\Omega) \cap C^0(\overline{\Omega}), \ \ du_0 = \delta u_2, \ \ du_2 = 0 \text{ in } \Omega\big\}$. □

For $n = 4$, let Ω be a domain of $\mathbb{R}^4$. We say that a form $U = u_0 + u_2 + u_4 \in C^1(\Omega)$ *satisfies the Fueter system* if there exists $(f_0, f_1, f_2, f_3) \in [C^1(\Omega)]^4$ such that $u_0 = f_0$, $u_2 = f_1(dx^1dx^2 - dx^3dx^4) + f_2(dx^1dx^3 - dx^4dx^2) + f_3(dx^1dx^4 - dx^2dx^3)$, $u_4 = f_0dx^1dx^2dx^3dx^4$ and if $(f_0, f_1, f_2, \ f_3)$ is the solution of (2.2). Evidently, a form satisfying the Fueter system is self-conjugate, but generally speaking, the converse is not true. However, the following result holds true:

Theorem 4.4. *A form $U = u_0 + u_2 + u_4$ of class C^1 on a domain $\Omega \subset \mathbb{R}^4$ such that $u_0 = *u_4$ and $u_2 = - * u_2$ is self-conjugate in Ω if, and only if, U satisfies the Fueter system in Ω.*

Proof. Let $U = u_0 + u_2 + u_4 \in C^1(\Omega)$ be a self-conjugate form in Ω where

$$u_0 = g_0, \quad u_2 = \frac{1}{2}u_{ij}dx^idx^j \quad u_4 = h_0dx^1 \dots dx^4.$$

By hypothesis

$$*u_4 = h_0 = u_0 = g_0,$$

$$\begin{aligned} *u_2 = \frac{1}{2}(&u_{12}dx^3dx^4 - u_{13}dx^2dx^4 + u_{14}dx^2dx^3 + u_{23}dx^1dx^4 \\ &-u_{24}dx^1dx^3 + u_{34}dx^1dx^2) = -u_2, \end{aligned}$$

we have that

$$u_{12} = -u_{34}, \quad u_{13} = u_{24}, \quad u_{14} = -u_{23}.$$

Thus, substituting in u_2 we obtain that

$$\begin{aligned} u_2 = \frac{1}{2}(&u_{12}(dx^1dx^2 - dx^3dx^4) + u_{13}(dx^1dx^2 - dx^4dx^2) \\ &+ u_{14}(dx^1dx^4 - dx^2dx^3)) = g_1(dx^1dx^2 - dx^3dx^4) \\ &\qquad + g_2(dx^1dx^3 - dx^4dx^2) + g_3(dx^1dx^4 - dx^2dx^3). \end{aligned}$$

The converse is obvious. □

The following theorem shows a property of (2.2) solution:

Theorem 4.5. *If $f = f_0 + f_2$ and $\tilde{f} = f_0 - f_2$ are continuous non-homogeneous differential forms on a closed S of 3-dimensional Lebesgue zero measure on $\Sigma = \partial\Omega$, there exists a non-homogeneous form $W \in C^\infty(\Omega) \cap C^0(\overline{\Omega})$ which satisfies (2.2) and such that W and $*W$ extend f and $\tilde{f}$, respectively.*

Proof. It follows from Theorem 4.2 that there exists $U = u_0 + u_2 + u_4 \in C^\infty(\Omega) \cap C^0(\overline{\Omega})$ self-conjugate in Ω such that $u_0|_S = f_0$, $u_2|_S = f_2$, $*u_2|_S = -f_2$, $*u_4|_S = f_0$.

Let $W = w_0 + w_2 + w_4$ be the form defined as

$$w_0 = \frac{1}{2}(u_0 + *u_4), \qquad w_2 = \frac{1}{2}(u_2 - *u_2), \qquad w_4 = \frac{1}{2}(u_0 + *u_4).$$

$W \in C^\infty(\Omega) \cap C^0(\overline{\Omega})$ is such that $*w_4 = w_0$ and $*w_2 = -w_2$ and it is self-conjugate in Ω, in fact

$$dw_0 = \frac{1}{2}d(u_0 + *u_4) = \frac{1}{2}(du_0 + d * u_4) = \frac{1}{2}(\delta u_2 - * * d * u_4) =$$

$$\frac{1}{2}(\delta u_2 + *\delta u_4) = \frac{1}{2}(\delta u_2 + *du_2) = \frac{1}{2}(\delta u_2 - \delta * u_2) = \delta(\frac{1}{2}(u_2 - *u_2)) = \delta w_2;$$

$$dw_2 = -d * w_2 = (-1)^{1+3(4-3)} * *d * w_2 = - * \delta w_2 = - * dw_0 =$$
$$(-1)^{1-4(4+1)+1} \delta * w_0 = \delta * w_0 = \delta w_4.$$

Therefore, it follows from Theorem 4.4 that W satisfies the Fueter system. Moreover, W extends f and $*W$ extends $\tilde{f}$. ☐

To conclude, we show the following theorem which gives a similar property of harmonic vector (2.3):

Theorem 4.6. *For any $n \geq 2$, let $f_1 \in C_1(S)$, $f_{n-1} \in C_{n-1}(S)$, where S is a closed subset of (n-1)-dimensional Lebesgue zero measure on Σ. There exists a 1-form $u_1 = w_h dx^h \in C_1^\infty(\Omega) \cap C_1^0(\overline{\Omega})$ whose components satisfy (2.3) and such that $u_1|_S = f_1$, $*u_1|_S = f_{n-1}$.*

Proof. We apply Theorem 4.2 when $f = f_1$ and $\tilde{f} = f_{n-1}$. Thus, there exists $U = u_1 = w_h dx^h$ self-conjugate, i.e. $du_1 = 0$, $\delta u_1 = 0$, such that $u_1|_S = f_1$, $*u_1|_S = f_{n-1}$. But $du_1 = 0$ and $\delta u_1 = 0$ are equivalent to (2.3), as one can easily verify. ☐

A similar result can be proved for harmonic forms (2.4).

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