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GENERALIZED CAUCHY DISTRIBUTIONS

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Abstract: In this article we generalize the characterization of Cauchy distributions by Knight and Meyer to other probability measures on $\mathbf{R}^n$. The common feature of these distributions is that they arise from the standard volume form of compactifications of $\mathbf{R}^n$ to semi-simple flat homogeneous spaces. Generalized Cauchy distributions are relevant in case the statistical problem is not invariant under the full general linear group.

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1. Introduction

The classical Cauchy distribution on $\mathbf{R}^n$, $\mathrm{const}(1+|x|^2)^{-\frac{n+1}{2}}dx_1 \wedge \cdots \wedge dx_n$, is the standard volume form on the real projective space $\mathbf{R}P^n$ in terms of Cartesian coordinates on $\mathbf{R}^n \subset \mathbf{R}P^n$. The affine group and the larger group of projective transformations acting on the classical Cauchy distribution produce

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the same space of Cauchy distributions. Knight [2] observed that this property, in case $n = 1$, characterizes the space of Cauchy distributions. Knight and Meyer [3] proved this fact for general dimension n. More recently, Dunau and Sénateur [1] presented a new proof of this characterization. In this paper we show that the proof of Dunau and Sénateur [1] can be adapted to obtain a similar characterization of a large class of probability distributions which we call generalized Cauchy distributions.

As in the case of Cauchy distributions, the generalized Cauchy distributions are related to certain compactifications of $\mathbf{R}^n$ to compact symmetric spaces. Aside from real projective space which is related to Cauchy distributions, the best known example is probably the Riemann sphere ($n = 2$) or, more generally, the Moebius space, i.e. the standard sphere S^n with the conformal group, also called Moebius group, $SO(n + 1, 1)$ acting as group of isomorphisms. In differential geometry, these compactifications of $\mathbf{R}^n$ are known under the name "semi-simple flat homogeneous spaces" and are classified by Kobayashi and Nagano [5].

The common feature of these compactifications M of $\mathbf{R}^n$ is that the Lie algebra $\mathfrak{l}$ of the isomorphism group L is a semi-simple graded Lie algebra $\mathfrak{l} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ (vector space direct sum). Let $A \subset L$ denote the subgroup of L with Lie algebra $\mathfrak{a} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0$, $K \subset L$ the maximal compact subgroup of L. As in the case of Cauchy distributions, where $L = SL(n + 1)$, $A = \mathrm{Aff}(n)$ (affine group), the actions of L and A on the standard volume form of M produce the same space of probability measures. Conversely, given L, the generalized Cauchy distributions under a regularity assumption are characterized by this property.

The proof of the characterization of generalized Cauchy distributions is based on the grading $\mathfrak{l} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ of the Lie algebra $\mathfrak{l}$ of the isomorphism group L of M: Let μ be a volume form on M such that the action of any $\ell \in L$ on μ can be produced by the action of some $a \in A$. This implies that the projection of the Lie algebra of the isotropy group of μ onto $\mathfrak{g}_1$ is surjective. On the other hand, the regularity assumption implies that the isotropy group is compact. Combining these two facts we prove that the isotropy group of μ is transitive on M, and therefore, that μ is a generalized Cauchy distribution.

The generalized Cauchy distributions may be of interest in statistics in case the problems at hand are not invariant under the full affine group, but only the conformal group in case of the Moebius space, or the complex affine group in case of even dimensions. Another point we observe is that generalized Cauchy distributions are parametrized by non-compact symmetric spaces, and that the standard metric coincides with the Fisher information metric. In case $n =$

1, where Cauchy distributions are parametrized by the hyperbolic space H^2, additional work was done by Mc Cullagh [6]. Perhaps the parametrization of generalized Cauchy distributions will make it possible to generalize his results.

2. Definitions and results

Let $M = \mathbf{R}P^n$ denote the real projective space, $M = S^n$ the standard sphere or, more generally, anyone of the semi-simple flat homogeneous spaces classified by Kobayashi and Nagano [5]. The isomorphism group L of M has a semi-simple graded Lie algebra $\mathfrak{l}$, and M is on the one hand a compactification of $\mathbf{R}^n$, and on the other hand a compact symmetric space whose isometry groups is a maximal compact subgroup K of L. As L is semi-simple all maximal compact subgroups are conjugate and we may choose K in standard position. $K = SO(n+1)$ in case $\mathbf{R}P^n$ or S^n. The assumption of grading implies that $\mathfrak{l} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ (i, j mod 3), $\mathfrak{g}_{-1} = \mathbf{R}^n$ and $\mathfrak{g}_1 = (\mathbf{R}^n)^*$. Compare Kobayashi [4] for details and a treatment of the well known examples $\mathbf{R}P^n$ and S^n.

Definition. Let μ be a volume form on M and $\ell \cdot \mu$, for $\ell \in L$, the volume form on M induced by the operation of ℓ on M. The space of *generalized Cauchy distributions* is $\{\ell \cdot \mu_0, \ell \in L,\ \mu_0$ the standard volume form on the symmetric space $M\}$.

Remark. As $\ell \cdot \mu_0$ assigns measure zero to the complement of $\mathbf{R}^n \subset M$, we may consider Cauchy distributions as measures on $\mathbf{R}^n$.

Definition. Let μ denote a measure on M. μ is **regular** if the subgroup of L which leaves μ invariant (isotropy group) is compact.

Remarks. The assumption, μ does not charge a hyperplane, of Dunau and Sénateur [1] and Knight and Meyer [3] in case of $\mathbf{R}P^n$ implies that μ is regular. In case of S^n the corresponding assumption would be, that μ does not charge an equator. In the general case, M does not have totally geodesic hypersurfaces and a corresponding assumption is less intuitive.

Main Theorem. *Let M denote a semi-simple flat homogeneous space with isomorphism group L and Lie algebra $\mathfrak{l} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, $A \subset L$ is the subgroup with Lie algebra $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0$. If μ is a regular volume form on M such that $\{\ell \cdot \mu, \ell \in L\}$ and $\{a \cdot \mu, a \in A\}$ coincide, then μ is a generalized Cauchy distribution.*

Remarks. The converse is fairly obvious since the maximal compact subgroup K of L leaves the standard volume form invariant, and $L(\bmod K)$ coincides with $A\ (\bmod K)$. We note also, that the main theorem is not empty: The maximal compact subgroup $K \subset L$ is the isotropy group of the standard volume form μ_0 of M, (compare Nagano [7], Lemma 2.4).

The above remark shows that the space of generalized Cauchy distributions is parametrized by the non-compact symmetric space L/K, and we have the following result.

Proposition. *The Fisher information metric on the space of generalized Cauchy distributions coincides, up to a scalar factor, with the standard metric of the non-compact symmetric space L/K.*

The proof of this proposition is straight forward. Both, the Fisher metric as well as the standard metric are invariant under the action of L on L/K. As L is semi-simple, the two metrics differ at most by a scalar factor.

3. Proof of the Main Theorem

To prepare for the proof we recall some facts concerning semi-simple flat homogeneous spaces M. We refer to Kobayashi [4] and Section 2 for details. The Lie algebra $\mathfrak{l}$ of the isomorphism group L of M is semi-simple and graded, i.e. $\mathfrak{l} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ (vector space direct sum). The component $\mathfrak{g}_{-1}$ may be identified with the tangent space T_0M. As $\mathbf{R}^n \subset M$, $T_0M = T_0\mathbf{R}^n \cong \mathbf{R}^n$. The Killing form of $\mathfrak{l}$ induces the usual pairing of $\mathbf{R}^n = \mathfrak{g}_{-1}$ with $\mathfrak{g}_1 = (\mathbf{R}^n)^*$. Since the maximal compact subalgebra $\mathfrak{k}$ of a semi-simple Lie algebra $\mathfrak{l}$ is unique up to conjugation, we may assume that $\mathfrak{k} \subset \mathfrak{l}$ is in standard position, i.e. that $\mathfrak{k}$ is generated by the skew symmetric matrices in $\mathfrak{g}_0$ and the elements $x - x^*$, $x \in \mathbf{R}^n$. By the assumption of the Main Theorem, the isotropy group K_μ of the volume form μ is compact, and in view of the above remark, we may assume that the Lie algebra of K_μ is contained in $\mathfrak{k}$.

To prove the Main Result it suffices to show that K_μ is transitive on M, since in that case μ is proportional to the standard volume form. (At this point μ is no longer the general μ of the Main Theorem since a possible conjugation with $\ell \in L$ moved $K_\mu \subset K$ into general position.) Now, the fact that the sets $\{a \cdot \mu, a \in A\}$ and $\{\ell \cdot \mu, \ell \in L\}$ coincide, implies that the action of any element $\ell \in L$ on μ can be reversed by the action of a well chosen $a \in A$, i.e. that

$a^{-1}\ell \in K_\mu$. Let us now consider the Lie algebra $\mathfrak{k}_\mu$ of K_μ. On the one hand, the above observation on $a^{-1}\ell$ with $\ell \in L$ arbitrary, shows that the projection of $\mathfrak{k}_\mu \subset \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ to $\mathfrak{g}_1$ is surjective. On the other hand, the intersection of the Lie algebra $\mathfrak{k}$ of K with $\mathfrak{g}_{-1} \oplus \mathfrak{g}_1$ is $\{x - x^*, x \in \mathfrak{g}_{-1}\}$. By the above observation on the projection $\mathfrak{k}_\mu \to \mathfrak{g}_1$ we infer that $\mathfrak{k}_\mu \cap \mathfrak{g}_{-1} \oplus \mathfrak{g}_1 = \{x - x^*, x \in \mathfrak{g}_{-1}\}$. This fact in return implies that K_μ is transitive in a neighborhood of $0 \in M$. As M is compact and connected, the action of K_μ on M is transitive, and hence μ is proportional to the standard volume form.

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References

[1] J.-L. Dunau, H. Sénateur, An elementary proof of the Knight-Meyer characterization of the Cauchy distribution, *J. Multivariate Analysis*, **22** (1987), 74-78.

[2] F. B. Knight, A characterization of the Cauchy type, *Proc. AMS*, **55** (1976), 130-135.

[3] F. B. Knight, P. A. Meyer, Une caractérisation de la loi de Cauchy, *Z. Wahrscheinlichkeitstheorie verw. Gebiete*, **34** (1976), 129-134.

[4] S. Kobayashi, *Transformation Groups in Differential Geometry*, Berlin, Heidelberg, New York, Springer Verlag (1972).

[5] S. Kobayashi, T. Nagano, On filtered Lie algebras and geometric structures I, *J. Mathematics and Mechanics*, **13** (1964), 875-907.

[6] P. Mc Cullagh, Moebius transformations and Cauchy parameter estimates, *Ann. of Statistics*, **24** (1996), 787-808.

[7] T. Nagano, Transformation groups on compact symmetric spaces, *Trans. AMS*, **118** (1965), 428-453.

[8] T. Ochiai, Geometry associated with semi-simple flat homogeneous spaces, *Trans. AMS*, **152** (1970), 159-193.