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STABLE FUNCTIONS AND THE LAWSON TOPOLOGY ON SCOTT DOMAINS

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Abstract: The purpose of this paper is to discuss the relation between stable functions and the Lawson topology on Scott domains. For this aim, we first give a characterization theorem of stable function on Scott domains. Then we discuss the relation between stable function and the lower topology on Scott domains, and finally, we obtain a theorem relating stable functions and the Lawson topology on Scott domains.

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1. Introduction

Domain theory provides a mathematical foundation for the denotational semantics of computer programming languages, and belongs to the cross field of lattice theory, general topology, category theory, logic and theoretical computer science. In some earlier studies of domain theory, it was important to explore various categories of domains. For example, Scott [1], [2] investigated

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the class of all ω–algebraic lattices and of all consistently complete ω–algebraic cpo's. Plotkin [3] introduced the class of bifinite domains. Coquand [4] and Jung [5] studied L–domains and bifinite L–domains, etc. In each of these cases, the morphisms were continuous functions and the resulting categories were cartesian–closed. This closure property was very useful to obtain models of the untyped λ–calculus, see, e.g. [2]. However, as Plotkin [6] and Milner [7] showed, continuous function models did not capture all operational properties of ALGOL–like sequential languages, e.g. PCF. This led Berry [8], [9] to investigate the category of dI–domains, with stable functions as morphisms, in order to obtain models of typed λ–calculi. The stable functions also found use in modelling concurrency. Intuitively, stable functions reflect not only the continuity of computations, but also that definite information is needed from the argument in order to obtain a given approximation of the result. Soon after, various categories of domains with stable functions were intensively studied [10], [11], [12]. As was well known, there was an elegant topological characterization of continuous functions on Scott domains which made it convenient to study the relationship of denotational semantics and program logic. But there was no topology characterizing stable functions [13]. In this paper we consider some properties of stable functions on Scott domains and find that there is a close relation between stable functions and the Lawson topology. Perhaps, this may prove useful for understanding the relationship between denotational semantics and program logics.

2. Preliminaries

In this section we present some basic concepts and results on domain theory. Please see [14], [15] for more information about this topic.

Let $(D, \leq)$ be a partially ordered set (a poset). An nonempty subset $A \subseteq D$ is called directed, if for any $a, b \in A$ there exists $c \in A$, $a \leq c$ and $b \leq c$. An element $x \in D$ is compact, if whenever $A \subseteq D$ is directed, $\sup A \in D$ exists and $x \leq \sup A$, then $x \leq a$ for some $a \in A$. We write $K(D)$ for the set of all compact elements of D. $(D, \leq)$ is algebraic if for each $x \in D$ the set $\{d \in K(D) : d \leq x\}$ is directed and has x as supremum. $(D, \leq)$ is called a dcpo if each directed subset of D has a supremum in D. A dcpo D is called a cpo if it contains a smallest element. An algebraic cpo is called a domain. An algebraic complete lattice is called an algebraic lattice. A function $f : D \to E$ between two posets $(D, \leq)$ and $(E, \leq)$ is called continuous if f preserves all (existing) suprema of directed subsets of $(D, \leq)$.

Definition 1. (see [8], [9]) Let $(D, \leq)$ and $(E, \leq)$ be two posets. A continuous function $f : D \to E$ is called stable if for all $x \in D$ and $y \in E$, with $y \leq f(x)$, there exists $m \in D$ with the following property:

$$m \leq x, \quad y \leq f(m), \text{ and whenever } d \in D, \qquad d \leq x \text{ and } y \leq f(d), \text{ then } m \leq d.$$

Definition 2. (see [16]) Let D be a dcpo and $U \subseteq L$. If U satisfies the following conditions:

(1) $U = \uparrow U$,

(2) $\sup D \in U$ implies $D \cap U \neq \emptyset$ for all directed sets $D \subseteq L$,

then U is called a Scott open set of D. The collection of all Scott open subsets of D forms a topology. It is called the Scott topology of D and is denoted $\sigma(D)$.

Proposition 1. (see [16]) *Let D, E be two dcpo and $f : D \to E$ be a function. Then f is a continuous function if and only if f is continuous with respect to the Scott topologies, i.e. $\forall U \in \sigma(E)$, $f^{-1}(U) \in \sigma(D)$.*

Definition 3. A domain D is said to be Scott domain if the dcpo D^T obtained by adding the top 1 is an algebraic lattice. Naturally, D^T is also a Scott domain.

Proposition 2. (see [17]) *A domain is Scott domain if and only if every nonempty subset has a meet.*

3. The Main Results

Theorem 1. *Let D, E be two Scott domains, $f : D^T \to E$ a continuous function. Then f is a stable function if and only if f preserves nonempty meets.*

Proof. Firstly, we prove the sufficiency. For any $x \in D^T$ and $y \in E$, with $y \leq f(x)$, the set $m(f, x, y) = \{x' \in D^T : x' \leq x, \quad y \leq f(x')\}$ is a nonempty set. By Proposition 2, $m(f, x, y)$ has a meet. Let $x_0 = \bigwedge m(f, x, y)$, then

$$\begin{aligned} f(x_0) &= f(\bigwedge m(f, x, y)) \\ &= f(\bigwedge\{x' \in D^T : x' \leq x,\ y \leq f(x')\}) \\ &= \bigwedge\{f(x') \in E : x' \in D^T, x' \leq x,\ y \leq f(x')\} \\ &\geq y. \end{aligned}$$

This implies that x_0 is the smallest element of the set $m(f,x,y)$. By Definition 1, f is a stable functions.

Now we proof the necessity. Suppose A is a nonempty subset of D^T. By Proposition 2 we know, that $\bigwedge A$ and $\bigwedge f(A)$ exist. Because f is a continuous function, it is monotone. Hence $f(\bigwedge A) \leq \bigwedge f(A)$. If $p \leq \bigwedge f(A)$, then $\forall a \in A,\ p \leq f(a) \leq f(1)$. Since f is a stable function, then the set $m(f,1,p) = \{x \in D^T : x \leq 1,\ p \leq f(x)\}$ has a smallest element m. Since $A \subseteq m(f,1,p)$, we know $m \leq \bigwedge A$, which implies that $p \leq f(m) \leq f(\bigwedge A)$. Thus $f(\bigwedge A) = \bigwedge f(A)$, i.e. f preserves nonempty meets.

Definition 4. (see [16]) *Let D be a dcpo. We call the topology generated by the complements $L\backslash \uparrow x$ of principal filters (as subbasic open sets) the lower topology and denote it $\omega(L)$.*

Theorem 2. *Let D, E be two Scott domains, $f : D^T \to E$ be a stable function. Then f is continuous relative to the lower topologies, i.e. $\forall U \in \omega(E),\ f^{-1}(U) \in \omega(D^T)$.*

Proof. We only need to show $\forall e \in E,\ f^{-1}\ (E\backslash \uparrow e) \in \omega(D^T)$. Actually, $f^{-1}(E\backslash \uparrow e) = f^{-1}(E)\backslash\ f^{-1}(\uparrow e) = D^T\backslash f^{-1}(\uparrow e)$. If $f^{-1}(\uparrow e) = \emptyset$, then $f^{-1}(E\backslash \uparrow e) = D^T \in \omega(D^T)$. If $f^{-1}(\uparrow e) \neq \emptyset$, we denote $s = \inf f^{-1}(\uparrow e)$. By Theorem 1, $f(s) = f(\inf f^{-1}(\uparrow e)) = \inf(ff^{-1}(\uparrow e)) \geq \inf \uparrow e = e$, i.e. $s \in f^{-1}(\uparrow e)$. $\forall x \in f^{-1}(\uparrow e)$ and $x \leq y \in D^T,\ f(y) \geq f(x) \geq e$, then $y \in f^{-1}(\uparrow e)$. Thus $f^{-1}(\uparrow e)$ is an upper set, which implies that $f^{-1}(\uparrow e) = \uparrow s$. Hence $f^{-1}(E\backslash \uparrow e) = f^{-1}(E) \setminus f^{-1}(\uparrow e) = D^T\backslash \uparrow s \in \omega(D^T)$. This shows that f is continuous relative to the lower topologies.

Definition 5. (see [16]) Let D be a dcpo. Then the common refinement $\sigma(D) \vee \omega(D)$ of the Scott topology and the lower topology is called the Lawson topology and is denoted by $\lambda(D)$. It is clear that the set

$$\{U\backslash \uparrow F : U \in \sigma(D), F \text{ is finite in } D\}$$

form a base for $\lambda(D)$.

Proposition 3. (see [16]) *Let D be a dcpo. Then an upper set U is Lawson open iff it is Scott open.*

Theorem 3. *Let D, E be two Scott domains, $f : D^T \to E$ be a stable function, then f is continuous relative to the Lawson topologies, i.e. $\forall U \in \lambda(E),\ f^{-1}(U) \in \lambda(D^T)$.*

Proof. Since f is a stable function, then f is a continuous function. By Proposition 1, $\forall\ V \in \sigma(E),\ f^{-1}(V) \in \sigma(D^T)$. But Theorem 2 shows that $\forall\ U \in \omega(E),\ f^{-1}(U) \in \omega(D^T)$. Hence, f is continuous relative to the Lawson topologies.

Theorem 4. *Let D, E be two Scott domains and $f : D^T \to E$ be a function of preserving finite meets. If f is continuous relative to the Lawson topologies, then f is a stable function.*

Proof. First, we show that f is a continuous function. $\forall\ U \in \sigma(E) \subset \lambda(E)$, since f is continuous relative to the Lawson topologies, then $f^{-1}(U) \in \lambda(D^T)$. Since f preserves finite meets we know that f is monotone. Because $f^{-1}(U)$ is an upper set since U is, we have therefore the conclusion $f^{-1}(U) \in \sigma(L)$ by Proposition 3. Hence, f is continuous by Proposition 1.

We will show that f preserves nonempty meets by the following three steps:

(1) For any filtered subset F of D^T, $\inf F = \lim F$ with respect to the Lawson topology, and this limit is unique (where F is naturally taken as a net).

Actually, $\mu = \{D^T \setminus \uparrow x : x \in D^T \text{and } x \not\leq \inf F\}$ is a local subbase of $\inf F$ with respect to the lower topology, and $\forall D^T \setminus \uparrow x \in \mu$, since $\inf F \in D^T \setminus \uparrow x$ and $D^T \setminus \uparrow x$ is a lower set, we know that F is eventually contained in $D^T \setminus \uparrow x$. Hence, F converges to $\inf F$ with respect to $\omega\left(D^T\right)$. It also converges to $\inf F$ with respect to $\sigma\left(D^T\right)$ trivially, since every Scott neighbourhood of $\inf F$ is an upper set, and hence contains F. As a consequence, F converges to $\inf F$ with respect to $\lambda\left(D^T\right)$.

Now let y be any limit of F with respect to $\lambda\left(D^T\right)$. If $u \in F$, then $\downarrow u$ is $\lambda\left(D^T\right)$–closed since $D^T \setminus \downarrow u$ is a Scott open set, and F is eventually in $\downarrow u$. Hence, $y \leq u$ for all $u \in F$, and so $y \leq \inf F$. Conversely, $\uparrow \inf F$ is $\lambda\left(D^T\right)$–closed, since $D^T \setminus \uparrow \inf F \in \omega\left(D^T\right)$, and contains F. Thus $y \in \uparrow \inf F$, and this proves $y = \inf F$.

(2) f preserves infs of filtered sets.

Let F be a filtered set in D^T. Then $\inf F = \lim F$ (with respect to $\lambda\left(D^T\right)$) by (1). Since f is monotone, we know that $f\left(F\right)$ is a filtered set of E. Since f is Lawson–continuous, we have $f(\inf\ F) = f(\lim F) = \lim f(F) = \inf f(F)$. Hence, f preserves infs of filtered sets.

(3) f preserves nonempty meets.

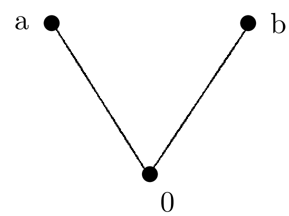
Since f preserves infs of filtered sets and finite meets, then f preserves nonempty meets.

By Theorem 1 we know that f is a stable function. The proof is complete.

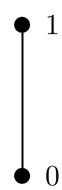
Theorem 5. *Let D, E be two Scott domains, and $f : D^T \to E$ be a function preserving finite meets. Then f is a stable function if and only if f is continuous relative to the Lawson topologies.*

Proof. This follows directly from Theorem 3 and Theorem 4.

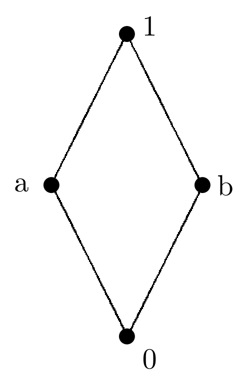
Remark. (1) If f doesn't preserve finite meets, then a Lawson continuous function does not need to be a stable function. For example, let D be the truth value domain [18]



and E be the two–point domain:



Then D^T is the following domain:



Taking $f : D^T \to E$ as $f(a) = f(b) = f(1) = 1$, and $f(0) = 0$, then f is continuous relative to the Lawson topology. Clearly, $1 \leq f(a), 1 \leq f(b), 1 \leq f(1)$, but there is not a smallest element in the set $\{a, b, 1\}$. Hence, f is not a stable function.

(2) The main results of the present paper are valid for D^T. We will discuss whether the results are valid for D in the other papers.

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