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PERIODIC SOLUTIONS FOR SYSTEMS OF TIME-DELAYED PARABOLIC EQUATIONS IN $\mathbb{R}^n$

C.V. Pao
Dept. of Mathematics
College of Physical and Mathematical Sciences
North Carolina State University
Box 8205, Raleigh, NC 27695–8205, USA
e-mail: cvpao@math.ncsu.edu

Abstract: This paper is concerned with the existence and uniqueness of periodic solutions for a coupled system of time-delayed nonlinear parabolic equations in the space $\mathbb{R}^n$. The approach to the problem is by the method of upper and lower solutions which leads to the existence of maximal and minimal periodic solutions for a class of nonlinear reaction functions. A sufficient condition for the uniqueness of the periodic solution is obtained. Applications are given to three reaction-diffusion models from ecology where conditions are obtained for the coexistence of the competing species.

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1. Introduction

Periodic behavior of solutions of parabolic equations occurs in many reaction-diffusion type of chemical, physical and ecological problems, and various methods have been proposed for the investigation of the existence and qualitative properties of periodic solutions. Most of the discussions in the earlier literature

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are for scalar parabolic equations in bounded domains without time delay, and the main concern in these discussions is the existence and stability of periodic solutions (cf. [2]-[5, [8], [9], [14]). In recent years, the existence problem of periodic solution has been extended to coupled system of parabolic equations in both bounded and unbounded domains and mostly are without time delay (cf. [1], [10]-[13], [15]-[17], [20]-[23]). In particular, the papers in [9], [13] are dealt with some reaction-diffusion models in $\mathbb{R}^n$ without time delay. On the other hand, the recent work in [16] is devoted to the existence of periodic solutions for a class of parabolic equations with time delay in bounded domains. In this paper we extend the existence problem in [16] to a coupled system of parabolic equations in the whole space $\mathbb{R}^n$, and apply the results to the coexistence problem of some reaction-diffusion models arising from ecology.

The system of equations under consideration is given in the form

$$\begin{aligned} &\partial u_i/\partial t - L_i u_i = f_i(t, x, \mathbf{u}, \mathbf{u}_\tau), && (t > 0, x \in \mathbb{R}^n)\,, \\ &u_i(t, x) = u_i(t + T, x) \quad (-\tau_i \le t \le 0, x \in \mathbb{R}^n), && (i = 1, \cdots, N)\,, \end{aligned}$$

where $\mathbf{u} \equiv (u_1(t, x), \cdots, u_N(t, x))$, $\mathbf{u}_\tau \equiv (u_1(t - \tau_1, x), \cdots, u_N(t - \tau_N, x))$, T is the period of the solution, and for each $i = 1, \cdots, N$, L_i is a uniformly elliptic operator in the form

$$L_i u_i = \sum_{j,k=1}^{n} a_{j,k}^{(i)} \partial^2 u_i/\partial x_j \partial x_k + \sum_{j=1}^{n} b_j^{(i)} \partial u_i/\partial x_j.$$

The constants $\tau_1, \cdots, \tau_N$ are positive representing the time delays of the solution. The uniform ellipticity of L_i is in the sense that for some positive constants μ_i and ν_i, the coefficients $a_{j,k}^{(i)}$ satisfy the conditions

$$\mu_i |\xi|^2 \ge \sum_{j,k=1}^{n} a_{j,k}^{(i)} \xi_j \xi_k \ge \nu_i |\xi|^2, \quad (i = 1, \cdots, N) \tag{1.1}$$

for every $\xi = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^n$ and every $(t, x) \in \mathcal{D}$, where $\mathcal{D} = \mathbb{R}^+ \times \mathbb{R}^n$.

The purpose of this paper is to show the existence (and uniqueness) of a T-periodic solution of (1.1) for a certain class of reaction functions $f_i(t, x, \mathbf{u}, \mathbf{u}_\tau)$, including the scalar problem $N = 1$. Our approach to the problem is by the method of upper and lower solutions and its associated monotone iterations which lead to the existence of maximal and minimal periodic solutions between upper and lower solutions. This result and some other related conclusions are given in Section 2. In Section 3 we give some applications of the existence theorem to a periodic logistic reaction diffusion equation and two Volterra-Lotka competition models in ecology, where sufficient conditions are obtained to ensuring the coexistence of the competing species.

2. Maximal and Minimal Solutions

Let $\mathcal{D} = [0,\infty)\times\mathbb{R}^n, \mathcal{D}_0^{(i)} = [-\tau_i, 0]\times\mathbb{R}^n, Q^{(i)} = [-\tau_i,\infty)\times\mathbb{R}^n, Q = Q^{(0)}\times\cdots\times Q^{(N)}$, and let $C^{m+\alpha}(\mathcal{D}^*)$, $m = 0, 1, 2, \cdots$, be the set of functions in $C^m(\mathcal{D}^*)$ that are Holder continuous in $\mathcal{D}^*$ with exponent $\alpha \in (0, 1)$, where $C^m(\mathcal{D}^*)$ is the set of m-times continuously differentiable functions in $\mathcal{D}^*$ and $\mathcal{D}^*$ is any one of the above domains. Denote by $C^{1,2}(\mathcal{D})$ the set of functions that are once continuously differentiable in t and twice continuously differentiable in x for $(t, x) \in \mathcal{D}$. The set of N-vector functions $\mathbf{u} = (u_1, \cdots , u_N)$ with u_i in $C^{m+\alpha}(\mathcal{D})$ or in $C^{1,2}(\mathcal{D})$ for all i are denoted, respectively, by $\mathcal{C}^{m+\alpha}(\mathcal{D})$ and $\mathcal{C}^{1,2}(\mathcal{D})$. We say that the coefficients of L_i are smooth functions in $\mathcal{D}$ if $a_{kj}^{(i)} \in C^{2+\alpha}(\mathcal{D})$ and $b_j^{(i)} \in C^{1+\alpha}(\mathcal{D})$ for all k, j and i. As usual, we write $|\mathbf{u}| = |u_1| + \cdots + |u_N|$ for any $\mathbf{u} = (u_1, \cdots , u_N)$ in $\mathbb{R}^N$.

To ensure the existence of a classical T-periodic solution of (1.1) we impose the following main hypotheses:

(H_1) For each i, the coefficients $a_{jk}^{(i)}$, $b_j^{(i)}$ of L_i are smooth T-periodic functions in $\mathcal{D}$ and satisfy condition (1.2).

(H_2) (i) For each i, $f_i(t, x, \cdot) \in C^\alpha(\mathcal{D}), f_i(\cdot, \mathbf{u}, \mathbf{v}) \in C^{1+\alpha}(\mathcal{S})$ and $f_i(t, x, \cdot)$ is T-periodic in t and bounded in x as $|x| \to \infty$, where $\mathcal{S}$ is a subset of $\mathbb{R}^N \times \mathbb{R}^N$.

(ii) The vector function $\mathbf{f}(t, x, \mathbf{u}, \mathbf{v}) = (f_1(t, x, \mathbf{u}, \mathbf{v}), \cdots , f_N(t, x, \mathbf{u}, \mathbf{v}))$ is quasimonotone nondecreasing in $\mathcal{S}$.

Recall that a vector function $\mathbf{f}(\cdot, \mathbf{u}, \mathbf{v})$ is said to be quasimonotone nondecreasing in $\mathcal{S}$, if for each $i = 1, \cdots , N$ and each $(\mathbf{u}, \mathbf{v}) \in \mathcal{S}$, where $\mathbf{u} = (u_1, \cdots , u_N)$ and $\mathbf{v} = (v_1, \cdots , v_N)$, the function $f_i(\cdot, \mathbf{u}, \mathbf{v})$ is nondecreasing in u_j for $j \neq i$ and is nondecreasing in v_j for all $j = 1, \cdots , N$. Since the function $\mathbf{f}(\cdot, \mathbf{u}, \mathbf{v})$ is a C^1-function of $(\mathbf{u}, \mathbf{v})$, the definition of quasimonotone nondecreasing of $\mathbf{f}(\cdot, \mathbf{u}, \mathbf{v})$ is equivalent to

$$\begin{aligned}(\partial f_i/\partial u_j)(t, x, \mathbf{u}, \mathbf{v}) &\geq 0 \quad \text{for } j \neq i\,,\\ (\partial f_i/\partial v_j)(t, x, \mathbf{u}, \mathbf{v}) &\geq 0 \quad \text{for all } j, \quad ((\mathbf{u}, \mathbf{v}) \in \mathcal{S}), \quad i = 1, \cdots , N.\end{aligned}$$

The set $\mathcal{S}$ is taken as the sector between a pair of ordered upper and lower solutions given by (2.4) below. It is obvious from (H_2)-(i) that for each i there exists a constant K_i such that $f_i(\cdot, \mathbf{u}, \mathbf{v})$ satisfies the Lipschitz condition

$$|f_i(t,x,\mathbf{u},\mathbf{v}) - f_i(t,x,\mathbf{u}',\mathbf{v}')| \le K_i(|\mathbf{u}-\mathbf{u}'| + |\mathbf{v}-\mathbf{v}'|) \quad \text{for } (\mathbf{u},\mathbf{v}), (\mathbf{u}',\mathbf{v}') \text{ in } \mathcal{S}. \tag{2.1}$$

Under the hypothesis (H_2)-(ii) we have the following definition of upper and lower solutions.

Definition 2.1. A function $\tilde{\mathbf{u}} \equiv (\tilde{u}_1, \cdots, \tilde{u}_N) \in \mathcal{C}^{1,2}(\mathcal{D}) \cap \mathcal{C}^{\alpha}(Q)$ is called an upper solution of (1.1) if

$$\begin{aligned} &\partial\tilde{u}_i/\partial t - L_i\tilde{u}_i \ge f(t,x,\tilde{\mathbf{u}}\tilde{\mathbf{u}}_\tau), && (t>0, x\in\mathbb{R}^n), \\ &\tilde{u}_i(t,x) \ge \tilde{u}_i(t+T,x), && (-\tau_i \le t \le 0, x \in \mathbb{R}^n), \quad i=1,\cdots,N, \end{aligned} \tag{2.2}$$

and there exist positive constants A_i, γ_i with $\gamma_i < (4\nu_i T)^{-1}$ such that

$$|\tilde{u}_i(t,x)| \le A_i \exp(\gamma_i |x|^2) \quad \text{as } |x| \to \infty, \tag{2.3}$$

where ν_i is the constant appeared in (1.2).

Similarly, $\hat{\mathbf{u}} \equiv (\hat{u}_1, \cdots, \hat{u}_N)$ is called a lower solution of (1.1) if it satisfies (2.3) and the inequalities in (2.2) in reversed order. The pair $\tilde{\mathbf{u}}, \hat{\mathbf{u}}$ are said to be ordered if $\tilde{\mathbf{u}} \ge \hat{\mathbf{u}}$, that is, $\tilde{u}_i \ge \hat{u}_i$ for every $i = 1, \cdots, N$. For a given pair of ordered upper and lower solutions $\tilde{\mathbf{u}}, \hat{\mathbf{u}}$ we set

$$\begin{aligned} \mathcal{S}^{(1)} &= \{\mathbf{u} \in \mathcal{C}(Q); \hat{\mathbf{u}} \le \mathbf{u} \le \tilde{\mathbf{u}}\} \\ \mathcal{S}^{(2)} &= \{\mathbf{v} \in \mathcal{C}(Q); \hat{\mathbf{u}}_\tau \le \mathbf{v}_\tau \le \tilde{\mathbf{u}}_\tau\} \\ \mathcal{S} &= \mathcal{S}^{(1)} \times \mathcal{S}^{(2)}. \end{aligned} \tag{2.4}$$

Define

$$\begin{aligned} &\mathcal{L}_i u_i = \partial u_i/\partial t - L_i u_i + K_i u_i \\ &F_i(t,x,\mathbf{u},\mathbf{v}) = K_i u_i + f_i(t,x,\mathbf{u},\mathbf{v}), \quad (i=1,\cdots,N), \end{aligned} \tag{2.5}$$

where K_i is the Lipschitz constant in (2.1). Then the differential equations in (1.1) may be written as

$$\mathcal{L}_i u_i = F_i(t,x,\mathbf{u},\mathbf{u}_\tau), \quad (t>0, x\in\mathbb{R}^n), \quad i=1,\cdots,N. \tag{2.6}$$

Using $\mathbf{u}^{(0)} = \tilde{\mathbf{u}}$ or $\mathbf{u}^{(0)} = \hat{\mathbf{u}}$ as the initial iteration we construct a sequence $\{\mathbf{u}^{(m)}\} \equiv \{u_1^{(m)}, \cdots, u_N^{(m)}\}$ from the linear iteration process

$$\begin{aligned} &\mathcal{L}_i u_i^{(m)} = F_i(t,x,\mathbf{u}^{(m-1)}, \mathbf{u}_\tau^{(m-1)}), && (t>0, x\in\mathbb{R}^n) \\ &u_i^{(m)}(t,x) = u_i^{(m-1)}(t+T,x), && (-\tau_i \le t \le 0, x\in\mathbb{R}^n), \end{aligned} \tag{2.7}$$

where $m = 1, 2, \cdots$. Since for each i and each m, the above problem is a linear uncoupled initial-value problem with known initial function $u_i^{(m)}(0,x) = u_i^{(m-1)}(T,x)$, the sequence $\{\mathbf{u}^{(m)}\}$ is well-defined (see Lemma 3.1 of [18]). We denote the sequence by $\{\bar{\mathbf{u}}^{(m)}\}$ if $\mathbf{u}^{(0)} = \tilde{\mathbf{u}}$ and by $\{\underline{\mathbf{u}}^{(m)}\}$ if $\mathbf{u}^{(0)} = \hat{\mathbf{u}}$, and refer to them as maximal and minimal sequences, respectively. The following lemma gives the monotone property of these sequences.

Lemma 2.1. *The sequences* $\{\bar{\mathbf{u}}^{(m)}\} \equiv \{\bar{u}_1^{(m)}, \cdots, \bar{u}_N^{(m)}\}$, $\{\underline{\mathbf{u}}^{(m)}\} \equiv \{\underline{u}_1^{(m)}, \cdots, \underline{u}_N^{(m)}\}$ *are well defined and possess the monotone property*

$$\hat{\mathbf{u}} \le \underline{\mathbf{u}}^{(m-1)} \le \underline{\mathbf{u}}^{(m)} \le \bar{\mathbf{u}}^{(m)} \le \bar{\mathbf{u}}^{(m-1)} \le \tilde{\mathbf{u}} \quad \text{in } \mathcal{D}. \tag{2.8}$$

Moreover, $\bar{\mathbf{u}}^{(m)}$ *and* $\underline{\mathbf{u}}^{(m)}$ *are ordered upper and lower solutions for every* m.

Proof. The existence of the sequences $\{\bar{\mathbf{u}}^{(m)}\}$, $\{\underline{\mathbf{u}}^{(m)}\}$ and the growth property

$$|u_i^{(m)}| \le A' \exp(\gamma' |x|^2) \quad \text{as } |x| \to \infty \tag{2.9}$$

for $u_i^{(m)} = \bar{u}_i^{(m)}$ and $u_i^{(m)} = \underline{u}_i^{(m)}, i = 1, \cdots, N$, follow from the proof of Lemma 3.1 in [18], where A' and γ' are some positive constants. We show the monotone property (2.8). Let $\bar{w}_i^{(0)} = \bar{u}_i^{(0)} - \bar{u}_i^{(1)} \equiv \tilde{u}_i - \bar{u}_i^{(1)}, i = 1, \cdots, N$. By (2.2), (2.5) and (2.7),

$$\begin{aligned}
\mathcal{L}_i \bar{w}_i^{(0)} &= (\partial \tilde{u}_i/\partial t - L_i \tilde{u}_i + K_i \tilde{u}_i) - (K_i \bar{u}_i^{(0)} + f_i(t, x, \bar{\mathbf{u}}^{(0)}, \bar{\mathbf{u}}_\tau^{(0)})) \\
&= \partial \tilde{u}_i/\partial t - L_i \tilde{u} - f_i(t, x, \tilde{\mathbf{u}}, \tilde{\mathbf{u}}_\tau) \ge 0, \\
\bar{w}_i^{(0)}(t,x) &= \tilde{u}_i(t,x) - \tilde{u}_i(t+T, x) \ge 0.
\end{aligned}$$

In view of $|\bar{w}_i^{(0)}| \le |\tilde{u}_i| + |\bar{u}_i^{(1)}|$, the relations (2.3) and (2.9) imply that for any constant $\delta_i > \bar{\gamma}_i \equiv \max\{\gamma_i, \gamma_i{}'\}$

$$\limsup_{R\to\infty} \left[e^{-\delta_i R^2} \min_{|x|=R} \bar{w}_i^{(0)}(t,x) \right] \ge \lim_{R\to\infty} e^{-\delta_i R^2} (-\bar{A}_i e^{\bar{\gamma}_i R^2}) = 0, \tag{2.10}$$

where $\bar{A}_i = A_i + A_i'$. It follows from the Phragman-Lindelof principle that $w_i^{(0)} \ge 0$ in $\mathcal{D}$ (cf. [14], [19]). This proves $\bar{\mathbf{u}}^{(0)} \ge \bar{\mathbf{u}}^{(1)}$. A similar argument gives $\underline{\mathbf{u}}^{(1)} \ge \underline{\mathbf{u}}^{(0)}$. Moreover, by (2.7), (2.1) and the quasimonotone nondecreasing property of $\mathbf{f}(\cdot, \mathbf{u}, \mathbf{v})$ we have

$$\begin{aligned}
\mathcal{L}_i(\bar{u}_i^{(1)} - \underline{u}_i^{(1)}) = K_i(\bar{u}_i^{(0)} - \underline{u}_i^{(0)}) + f_i(t, x, \bar{\mathbf{u}}^{(0)}, \bar{\mathbf{u}}_\tau^{(0)}) \\
- f_i(t, x, \underline{\mathbf{u}}^{(0)}, \underline{\mathbf{u}}_\tau^{(0)}) \ge 0, \\
\bar{u}_i^{(1)}(t,x) - \underline{u}_i^{(1)}(t,x) = \tilde{u}_i(t+T, x) - \hat{u}(t+T, x) \ge 0.
\end{aligned}$$

Since $w_i^{(1)} \equiv (\bar{u}_i^{(1)} - \underline{u}_i^{(1)})$ satisfies also the relation (2.10), we conclude that $\bar{u}_i^{(1)} - \underline{u}_i^{(1)} \geq 0$. The above conclusions show that $\underline{\mathbf{u}}^{(0)} \leq \underline{\mathbf{u}}^{(1)} \leq \bar{\mathbf{u}}^{(1)} \leq \bar{\mathbf{u}}^{(0)}$. An induction argument as that in [18] gives the monotone property (2.8).

To show that $\bar{\mathbf{u}}^{(m)}$ and $\underline{\mathbf{u}}^{(m)}$ are upper and lower solutions for every m we observe from (2.7) and (2.8) that

$$\bar{\mathbf{u}}^{(m)}(t,x) = \bar{\mathbf{u}}^{(m-1)}(t+T,x) \geq \bar{\mathbf{u}}^{(m)}(t+T,x),$$
$$\underline{\mathbf{u}}^{(m)}(t,x) = \underline{\mathbf{u}}^{(m-1)}(t+T,x) \leq \underline{\mathbf{u}}^{(m)}(t+T,x).$$

It follows again from the proof of Lemma 3.1 in [18] that $\bar{\mathbf{u}}^{(m)}$ and $\underline{\mathbf{u}}^{(m)}$ are ordered upper and lower solutions. This proves the lemma. □

In view of the monotone property (2.8) the pointwise limits

$$\lim_{m\to\infty} \bar{\mathbf{u}}^{(m)}(t,x) = \bar{\mathbf{u}}(t,x), \quad \lim_{m\to\infty} \underline{\mathbf{u}}^{(m)}(t,x) = \underline{\mathbf{u}}(t,x) \tag{2.11}$$

exist and satisfy the relation $\bar{\mathbf{u}}(t,x) \geq \underline{\mathbf{u}}(t,x)$. Moreover

$$\begin{aligned} &\bar{u}_i(t,x) = \bar{u}_i(t+T,x) \quad \text{and} \\ &\underline{u}_i(t,x) = \underline{u}_i(t+T,x) \quad \text{in} \quad D_0^{(i)}, \end{aligned} \qquad i = 1,\cdots,N. \tag{2.12}$$

In the following theorem we show that $\bar{\mathbf{u}}(t,x)$ and $\underline{\mathbf{u}}(t,x)$ are the respective maximal and minimal T-periodic solutions of (1.1) in $\mathcal{S}^{(1)}$.

Theorem 2.1. *Let $\tilde{\mathbf{u}}, \hat{\mathbf{u}}$ be ordered upper and lower solutions of (1.1), and let hypotheses (H_1), (H_2) hold. Then the sequence $\{\bar{\mathbf{u}}^{(m)}\}$ given by (2.7) with $\bar{\mathbf{u}}^{(0)} = \tilde{\mathbf{u}}$ converges monotonically to a maximal T-periodic solution $\bar{\mathbf{u}}$ in $\mathcal{S}^{(1)}$, while the sequence $\{\underline{\mathbf{u}}^{(m)}\}$ with $\underline{\mathbf{u}}^{(0)} = \hat{\mathbf{u}}$ converges monotonically to a minimal T-periodic solution $\underline{\mathbf{u}}$ in $\mathcal{S}^{(1)}$. Moreover,*

$$\hat{\mathbf{u}} \leq \underline{\mathbf{u}}^{(m)} \leq \underline{\mathbf{u}}^{(m+1)} \leq \underline{\mathbf{u}} \leq \bar{\mathbf{u}} \leq \bar{\mathbf{u}}^{(m+1)} \leq \bar{\mathbf{u}}^{(m)} \leq \tilde{\mathbf{u}} \quad \text{in} \quad Q \tag{2.13}$$

for every $m = 1, 2, \cdots$.

Proof. Let $\Gamma_i(t,x;s,\xi), i = 1,\cdots,N$, be the fundamental solution of $\mathcal{L}_i$, and let

$$J_{i,0}^{(m-1)}(t,x) = \int_{\mathbb{R}^n} \Gamma_i(t,x,0,\xi) u_i^{(m-1)}(T,\xi) d\xi, \ \ m = 1,2,\cdots.$$

By the integral representation for solutions of linear initial-value problems in $\mathbb{R}^n$, we may express the solution $u_i^{(m)}(t,x)$ of (2.7) in the form

$$u_i^{(m)}(t,x) = J_{i,0}^{(m-1)}(t,x) + \int_0^t ds \int_{\mathbb{R}^n} \Gamma_i(t,x;s,\xi) F_i(s,\xi,\mathbf{u}^{(m-1)}(s,\xi),\mathbf{u}_\tau^{(m-1)}(s,\xi)) d\xi, \tag{2.14}$$

where $\mathbf{u}^{(m)} = (u_1^{(m)}, \cdots, u_N^{(m)})$ stands for either $\bar{\mathbf{u}}^{(m)}$ or $\underline{\mathbf{u}}^{(m)}$ (cf. [6], [14]). It is easy to see from (2.3) and hypothesis (H_2) that for any $(\mathbf{u}, \mathbf{v}) \in \mathcal{S}$, $F_i(\cdot, \mathbf{u}, \mathbf{v})$ possesses the growth property

$$|F_i(s, \xi, \mathbf{u}(s,\xi), \mathbf{v}(s,\xi))| \leq A_i'' \exp(\gamma_i''|\xi|^2) \quad \text{as} \quad |\xi| \to \infty,$$

where A_i'' and γ_i'' are some positive constants with $\gamma'' < (4\nu_i T)^{-1}$. This implies that for each fixed $(t,x) \in \mathcal{D}$, the function $q_i(s,\xi) \equiv \Gamma_i(t,x;s,\xi)F_i(s,\xi,\mathbf{u}(s,\xi),\mathbf{v}(s,\xi))$ in (2.14) is integrable in $\mathcal{D}$ (cf. [6], [14]). Hence, by letting $m \to \infty$ in (2.14) and applying the dominated convergence theorem we obtain the relation

$$u_i(t,x) = J_{i,0}(t,x) + \int_o^t ds \int_{\mathbb{R}^n} \Gamma_i(t,x;s,\xi) F_i(s,\xi,\mathbf{u}(s,\xi),\mathbf{u}_\tau(s,\xi)) d\xi,$$

where $u_i(t,x)$ is either $\bar{u}_i(t,x)$ or $\underline{u}_i(t,x)$, and

$$J_{i,0}(t,x) = \int_{\mathbb{R}^n} \Gamma(t,x;0,\xi) u_i(T,\xi) d\xi, \quad i = 1, \cdots, N.$$

A standard regularity argument for parabolic initial-value problems shows that u_i satisfies the equation in (2.6) (cf. [14]). The periodic initial condition of u_i is given by (2.12). This shows that $\bar{\mathbf{u}}$ and $\underline{\mathbf{u}}$ are solutions of (1.1).

To show the T-periodic property $\mathbf{u}(t,x) = \mathbf{u}(t+T,x)$ for every $t > 0$ we observe from hypotheses (H_1), (H_2) and the mean-value theorem that the function $w_i(t,x) \equiv u_i(t,x) - u_i(t+T,x), i = 1, \cdots, N$, satisfies the relation

$$\begin{aligned} &\partial w_i/\partial t - L_i w_i = f_i(t,x,\mathbf{u}(t,x),\mathbf{u}_\tau(t,x)) \\ &\qquad - f_i(t+T,x,\mathbf{u}(t+T,x),\mathbf{u}_\tau(t+T,x)) \\ &= \sum_{j=1}^N \frac{\partial \mathrm{f_i}}{\partial \mathrm{u_j}}(t,x,\xi,\eta) w_j(t,x) + \sum_{j=1}^N \frac{\partial \mathrm{f_i}}{\partial \mathrm{v_j}}(t,x,\xi,\eta)(w_\tau)_j(t,x) \text{in } D \\ &w_i(t,x) = 0 \qquad \text{in } D_0^{(i)}, \qquad i = 1, \cdots, N, \end{aligned} \tag{2.15}$$

where $(\xi,\eta) \equiv (\xi(t,x),\eta(t,x))$ is an intermediate value in $\mathcal{S}$, and $(w_\tau)_j(t,x) = w_j(t-\tau_j,x)$. Let

$$D_k \equiv (0, k\underline{\tau}] \times \mathbb{R}^n, \quad D_k^{(i)} = [-\tau_i, k\underline{\tau}], \quad k = 1, 2, \cdots,$$

and consider problem (2.15) in the domain $\mathcal{D}_1 = (0, \underline{\tau}] \times \mathbb{R}^n$, where $\underline{\tau} = \min\{\tau_1, \cdots, \tau_N\} > 0$. By the initial condition in (2.15), $(w_\tau)_j(t,x) = 0$ in $\mathcal{D}_1$, for every $j = 1, \cdots, N$. This implies that

$$\partial w_i/\partial t - L_i w_i \quad = \quad \sum_{j=1}^N c_{i,j}(t,x) w_j(t,x) \quad \text{in } \mathcal{D}_1 \tag{2.16}$$

and $w_i(0,x)=0$, where

$$c_{ij}(t,x)=\frac{\partial \mathrm{f_i}}{\partial \mathrm{u_j}}(t,x,\xi(t,x),\eta(t,x)),\quad i,j=1.\cdots,N.$$

Since by (H_2), $c_{ij}\in C^{\alpha}_{loc}(\mathcal{D}_1)$ and is bounded as $|x|\to\infty$, and since by the quasimonotone nondecreasing property of $f(\cdot,\mathbf{u},\mathbf{v})$, $c_{ij}\geq 0$ for $j\neq i$, we conclude from Lemma 5.2 in [18] that $w_i\geq 0$ on $\mathcal{D}_1$ for every i. Replacing w_i by $-w_i$ in (2.15) yields $w_i\leq 0$ on $\mathcal{D}_1$. This shows that $w_i=0$ on $\mathcal{D}_1$ for every i. We next consider problem (2.15) in the domain $\mathcal{D}_2=(0,2\underline{\tau}]\times\mathbb{R}^n$. Since the above conclusion ensures $(w_\tau)_j(t,x)=0$ in $\mathcal{D}_2$ for all j, we see that $w_i(t,x)$ satisfies (2.16) in $\mathcal{D}_2$. It follows from the same reasoning as that for $\mathcal{D}_1$ that $w_i=0$ on $\mathcal{D}_2$ for every i. An induction argument leads to $w_i=0$ on $\mathcal{D}_k$ for every k and every i. This proves that $\bar{\mathbf{u}}(t,x)=\bar{\mathbf{u}}(t+T,x)$ for every $(t,x)\in\mathbb{R}^+\times\mathbb{R}^n$. Finally, the maximal and minimal property of $\bar{\mathbf{u}}(t,x)$ and $\underline{\mathbf{u}}(t,x)$ follows from the argument in the proof of Theorem 2.1 in [16]. This proves the theorem. □

In general, problem (1.1) may not have a unique solution. A simple example is the linear scalar problem, where $T=2\pi$ and $f(t,x,u)=(\cos t)u$ (without time delay). It is easy to verify that for any constant $\rho_0>0$ the pair $\tilde{u}=\rho_0 e^{\sin t}$ and $\hat{u}=0$ are ordered upper and lower solutions. However, there are infinite number of solutions in the form $u=\rho e^{\sin t}$ and $\hat{u}\leq u\leq\tilde{u}$ whenever $0\leq\rho\leq\rho_0$. In the following theorem we give a sufficient condition for the uniqueness of the periodic solution.

Theorem 2.2. *Let the conditions in Theorem 2.1 hold. If, in addition,* $\tilde{\mathbf{u}}(0,x)=\hat{\mathbf{u}}(0,x)$ *or* $\bar{\mathbf{u}}^{(m)}(0,x)=\underline{\mathbf{u}}^{(m)}(0,x)$ *for some* $m\geq 1$, *then* $\bar{\mathbf{u}}(t,x)=\underline{\mathbf{u}}(t,x)$ $(\equiv\mathbf{u}^*(t,x))$ *and* $\mathbf{u}^*(t,x)$ *is the unique T-periodic solution of (1.1) in* $\mathcal{S}^{(1)}$.

Proof. By considering problem (1.1) as an initial-value problem with the initial function $\mathbf{u}(0,x)$ given by $\tilde{\mathbf{u}}(0,x)$ (or $\hat{\mathbf{u}}(0,x)$) the existence-uniqueness theorem for initial-value parabolic systems in $\mathbb{R}^n$ ensures that $\bar{\mathbf{u}}(t,x)=\underline{\mathbf{u}}(t,x)$ in $\mathcal{D}$ (cf. [18]). It follows from the maximal and minimal property of $\bar{\mathbf{u}}(t,x)$ and $\underline{\mathbf{u}}(t,x)$ that $\mathbf{u}^*(t,x)$ is the unique solution of (1.1) in $\mathcal{S}^{(1)}$. In the case of $\bar{\mathbf{u}}^{(m)}(0,x)=\underline{\mathbf{u}}^{(m)}(0,x)$ for some $m\geq 1$ then since $\bar{\mathbf{u}}^{(m)}$ and $\underline{\mathbf{u}}^{(m)}$ are also ordered upper and lower solutions and are in $\mathcal{S}^{(1)}$ the same argument as that for $\tilde{\mathbf{u}},\hat{\mathbf{u}}$ leads to $\bar{\mathbf{u}}(t,x)=\underline{\mathbf{u}}(t,x)$. This proves the theorem. □

In the special case of scalar parabolic equations, where $N=1$, the quasimonotone nondecreasing property of $f_1(t,x,u,v)$ with respect to u is trivially satisfied. As a consequence of Theorems 2.1 and 2.2 we have the following conclusion.

Corollary 2.1. *Let $\tilde{u}, \hat{u}$ be ordered upper and lower solutions of (1.1) for $N = 1$, where $f_1(\cdot, u, v)$ is nondecreasing in v, and let hypotheses (H_1) and (H_2)-(i) be satisfied. Then all the conclusions in Theorem 2.1 hold. If, in addition, $\tilde{u}(0,x) = \hat{u}(0,x)$ or $\bar{u}^{(m)}(0,x) = \underline{u}^{(m)}(0,x)$ for some $m \geq 1$, then $\bar{u}(t,x) = \underline{u}(t,x) (\equiv u^*(t,x))$ and $u^*(t,x)$ is the unique solution in $\mathcal{S}^{(1)}$.*

3. Applications

It is seen from Theorem 2.1 that the main requirement for the existence of a T-periodic solution to (1.1) is the quasimonotone nondecreasing property of $\mathbf{f}(\cdot, \mathbf{u}, \mathbf{v})$ and the existence of a pair of ordered upper and lower solutions. In this section we consider three reaction-diffusion models from ecology where the reaction function $\mathbf{f}(\cdot, \mathbf{u}, \mathbf{v})$ is quasimonotone nondecreasing. Two of these models are the Volterra-Lotka competition systems, and our main concern is to find sufficient conditions for the existence of positive periodic solutions so that the competing species can coexist without extinction.

3.1. A Periodic Logistic Diffusion Equation

As a first example we consider the scalar logistic diffusion equation (without time delay) in the form

$$\begin{aligned} \partial u/\partial t - D^* \nabla^2 u &= u(a - bu) \quad \text{in } \mathcal{D}, \\ u(0,x) &= u(T,x) \quad \text{in } \mathbb{R}^n, \end{aligned} \tag{3.1}$$

where $D^* \equiv D^*(t,x)$, $a \equiv a(t,x)$ and $b \equiv b(t,x)$ are bounded positive functions in $C^\alpha(\mathcal{D})$ and are T-periodic in t. It is obvious, that this problem has always the trivial solution $u = 0$. To show the existence of a positive T-periodic solution it suffices to find a pair of positive ordered upper and lower solutions. Indeed, the constants $\tilde{u} = M$ and $\hat{u} = \delta$ satisfy the inequalities and the reversed inequalities, respectively, in (2.2) if

$$0 \geq M(a - bM) \quad \text{and} \quad 0 \leq \delta(a - b\delta).$$

It is obvious that the above conditions are satisfied by any positive constants $M \geq M^*$ and $\delta \leq m^*$, where

$$\begin{aligned} M^* &\equiv \sup\{a(t,x)/b(t,x); (t,x) \in \mathcal{D}\} \\ m^* &\equiv \inf\{a(t,x)/b(t,x); (t,x) \in \mathcal{D}\}. \end{aligned} \tag{3.2}$$

This construction leads to a pair of constant upper and lower solutions.

Using $u^{(0)} = M$ (resp. $u^{(0)} = \delta$) as the initial iteration in the iteration process (2.7) (with $N = 1$) the first iteration $\bar{u}^{(1)}$ (resp. $\underline{u}^{(1)}$) is governed by the linear initial-value problem

$$\begin{aligned} \partial u^{(1)}/\partial t - D^* \nabla^2 u^{(1)} + K u^{(1)} &= K u^{(0)} + u^{(0)}(a - b u^{(0)}) \quad \text{in } \mathcal{D}, \\ u^{(1)}(0, x) &= u^{(0)}(T, x) \qquad\qquad\qquad \text{in } \mathbb{R}^n, \end{aligned} \tag{3.3}$$

where K is any nonnegative constant satisfying

$$K \geq \sup\{2Mb(t, x) - a(t, x); (t, x) \in \mathcal{D}\}.$$

In view of Lemma 2.1, the pair $\bar{u}^{(1)}$ and $\underline{u}^{(1)}$ are nonconstant upper and lower solutions of (3.1). As a consequence of Corollary 2.1 with $f(\cdot, u, v) = u(a - bu)$, which is independent of v, we have the following conclusion.

Theorem 3.1. *Let $D^* \equiv D^*(t, x)$, $a \equiv a(t, x)$ and $b \equiv b(t, x)$ be bounded positive T-periodic functions in $C^\alpha(\mathcal{D})$. Then problem (3.1) has the trivial solution $u = 0$ and a positive T-periodic solution $u^*(t, x)$. Moreover,*

$$m^* \leq \underline{u}^{(1)}(t, x) \leq u^*(t, x) \leq \bar{u}^{(1)}(t, x) \leq M^* \quad \text{on } \mathcal{D}, \tag{3.4}$$

where M^* and m^* are given by (3.2), and $\bar{u}^{(1)}$ and $\underline{u}^{(1)}$ are governed by (3.3) with $u^{(0)} = M^*$ and $u^{(0)} = m^*$, respectively.

3.2. A Volterra-Lotka Competition Model

In the Volterra-Lotka competition model with two competing species (and with possible time delays) the population densities u, v with periodic reaction rates are governed by the system

$$\begin{aligned} &\partial u/\partial t - D_1^* \nabla^2 u = u(a_1 - b_1 u - c_1 v_{\tau_2}), \\ &\partial v/\partial t - D_2^* \nabla^2 v = v(a_2 - b_2 u_{\tau_2} - c_2 v), \qquad \text{in } \mathcal{D}, \\ &u(t, x) = u(t + T, x) \qquad \text{in } D_0^{(1)}, \\ &v(t, x) = v(t + T, x) \qquad \text{in } D_0^{(2)}, \end{aligned} \tag{3.5}$$

where for each $i = 1, 2$, $D_i^* \equiv D_i^*(t, x)$, $a_i \equiv a_i(t, x)$, $b_i \equiv b_i(t, x)$ and $c_i \equiv c_i(t, x)$ are bounded T-periodic functions in $C^\alpha(\mathcal{D})$. The existence of periodic solutions of the above system in bounded domains (and without time delays) has been investigated in [7], [8], [12], [16], [20], [22], and one of the main concerns in these works is the coexistence (or permanence) of the two competing species. It is obvious, from Theorem 3.1 that problem (3.5) has the trivial solution $(0,0)$ and the two semitrivial solutions in the form $(u^*,0)$ and $(0,v^*)$, where u^* and v^* are some positive T-periodic solutions of the respective scalar equations

$$\partial u/\partial t - D_1^*\nabla^2 u = u(a_1 - b_1 u) \quad \text{in } \mathcal{D} \tag{3.6}$$

and

$$\partial v/\partial t - D_2^*\nabla^2 v = v(a_2 - c_2 v) \quad \text{in } \mathcal{D}. \tag{3.7}$$

To ensure the existence of a positive T-periodic solution we assume that there exist positive constants M_1, M_2 and ϵ_0 such that

$$\begin{aligned} &[a_1(t,x)/b_1(t,x)] \le M_1 \le [a_2(t,x)/b_2(t,x)] - \epsilon_0, \\ &[a_2(t,x)/c_2(t,x)] \le M_2 \le [a_1(t,x)/c_1(t,x)] - \epsilon_0 \quad ((t,x) \in \mathcal{D}), \end{aligned} \tag{3.8}$$

where ϵ_0 can be arbitrarily small.

By letting $u_1 = u$, $u_2 = M - v$, where $M = M_2 + \epsilon$ for a sufficiently small $\epsilon > 0$, we transform problem (3.5) into the form of (1.1) with $\mathbf{u} = (u_1, u_2)$, $L_i = D_i^*\nabla^2$ and

$$\begin{aligned} f_1(t,x,\mathbf{u},\mathbf{u}_\tau) &= u_1[a_1 - b_1 u_1 - c_1(M - (u_2)_{\tau_2})], \\ f_2(t,x,\mathbf{u},\mathbf{u}_\tau) &= -(M - u_2)[a_2 - b_2(u_1)_{\tau_1} - c_2(M - u_2)], \end{aligned} \tag{3.9}$$

where $(u_i)_{\tau_i} = u_i(t - \tau_i)$, $i = 1,2$. It is clear that the function $\mathbf{f}(\cdot,\mathbf{u},\mathbf{v}) = (f_1(\cdot,\mathbf{u},\mathbf{v}), f_2(\cdot,\mathbf{u},\mathbf{v}))$ given by (3.9) is quasimonotone nondecreasing for $\mathbf{u},\mathbf{v}$ in $\mathbb{R}^+ \times [0,M]$ and satisfies hypothesis (H_2) for every subset $\mathcal{S} \equiv \mathcal{S}^{(1)} \times \mathcal{S}^{(2)}$ with $\mathcal{S}^{(1)}, \mathcal{S}^{(2)}$ in $\mathbb{R}^+ \times [0,M]$. Hence, to show the existence of a positive T-periodic solution it suffices to find a pair of positive upper and lower solutions in $\mathbb{R}^+ \times [0,M]$. We seek a constant pair in the form $(\tilde{u}_1, \tilde{u}_2) = (M_1, M_2)$ and $(\hat{u}_1, \hat{u}_2) = (\delta_1, \delta_2)$, where M_1 and M_2 are the constants satisfying (3.8), and δ_1 and δ_2 are some sufficiently small positive constants such that

$$(b_1/c_1)\delta_1 - \delta_2 + \epsilon \le \epsilon_0 \quad \text{and} \quad \delta_2/\delta_1 \le b_2/c_2. \tag{3.10}$$

Indeed, this pair are upper and lower solutions if

$$\begin{aligned} &0 \ge M_1[a_1 - b_1 M_1 - c_1(M - M_2)], \\ &0 \ge -(M - M_2)[a_2 - b_2 M_1 - c_2(M - M_2)], \\ &0 \le \delta_1[a_1 - b_1\delta_1 - c_1(M - \delta_2)], \\ &0 \le -(M - \delta_2)[a_2 - b_2\delta_1 - c_2(M - \delta_2)]. \end{aligned}$$

It is easily seen from $M - M_2 = \epsilon$, that the first two inequalities are satisfied if

$$(a_1 - c_1\epsilon)/b_1 \le M_1 \le (a_2 - c_2\epsilon)/b_2, \tag{3.11}$$

while the last two inequalities are fulfilled if

$$a_2/c_2 - (b_2\delta_1/c_2 - \delta_2) \le M \le a_1/c_1 - (b_1\delta_1/c_1 - \delta_2). \tag{3.12}$$

In view of condition (3.8) and $M = M_2 + \epsilon$, the inequalities in (3.11) hold for any $\epsilon \le (b_2/c_2)\epsilon_0$ and those in (3.12) are satisfied if

$$b_2\delta_1/c_2 - \delta_2 + \epsilon \ge 0 \quad \text{and} \quad b_1\delta_1/c_1 - \delta_2 + \epsilon \le \epsilon_0.$$

The above requirements are clearly fulfilled by the relation (3.10). This shows that the pair (M_1, M_2) and (δ_1, δ_2) are ordered upper and lower solutions of the transformed problem of (3.5). By Theorem 2.1, the transformed problem has a maximal T-periodic solution $(\bar{u}_1, \bar{u}_2)$ and a minimal T-periodic solution $(\underline{u}_1, \underline{u}_2)$ such that $(\delta_1, \delta_2) \le (\underline{u}_1, \underline{u}_2) \le (\bar{u}_1, \bar{u}_2) \le (M_1, M_2)$. This implies that $(\bar{u}, \underline{v}) = (\bar{u}_1, M - \bar{u}_2)$ and $(\underline{u}, \bar{v}) = (\underline{u}_1, M - \underline{u}_2)$ are T-periodic solutions of the original problem (3.5) such that $\delta_1 \le \underline{u} \le \bar{u} \le M_1$ and $\epsilon \le \underline{v} \le \bar{v} \le M - \delta_2$. Nonconstant upper and lower bounds of $(\bar{u}, \underline{v})$ or $(\underline{u}, \bar{v})$ can be obtained from the iteration process (2.7). To summarize the above conclusions we have the following coexistence theorem.

Theorem 3.2. *Let $D_i^* \equiv D_i^*(t,x)$, $a_i \equiv a_i(t,x)$, $b_i \equiv b_i(t,x)$ and $c_i \equiv c_i(t,x)$, $i = 1, 2$, be bounded positive T-periodic functions in $C^\alpha(\mathcal{D})$. If there exist positive constants M_1, M_2 and ϵ_0 such that condition (3.8) holds, then in addition to the trivial and semitrivial T-periodic solutions $(0,0)$, $(u^*, 0)$, $(0, v^*)$, problem (3.5) has positive T-periodic solutions $(\bar{u}, \underline{v})$ and $(\underline{u}, \bar{v})$ such that $\underline{u} \le \bar{u} \le M_1$ and $\underline{v} \le \bar{v} \le M_2$ on $\mathcal{D}$.*

3.3. A Modified Competition Model

We next consider a modified Volterra-Lotka competition model where the competition between the two competing species follow the hypothesis of the Holling-Tanner interaction mechanism. The equations governing the competing species u, v are given by

$$\begin{aligned}
&\partial u/\partial t - D_1^*\nabla^2 u = u(a_1 - b_1 u - c_1 v_{\tau_2}/(1 + \sigma_1 u_{\tau_1})),\\
&\partial v/\partial t - D_2^*\nabla^2 v = v(a_2 - b_2 u_{\tau_1}/(1 + \sigma_2 u_{\tau_1}) - c_2 v) \quad \text{in} \quad \mathcal{D}\\
&u(t,x) = u(t+T, x) \quad \text{in } D_0^{(1)},\\
&v(t,x) = v(t+T, x) \quad \text{in } D_0^{(2)},
\end{aligned} \tag{3.13}$$

where for each $i = 1, 2$, the diffusion coefficient D_i^* and the reaction rates a_i, b_i and c_i are the same functions as that in (3.5), and the function $\sigma_i \equiv \sigma_i(t, x)$ is a bounded nonnegative T-periodic function in $C^\alpha(\mathcal{D})$. The system in (3.13) in a bounded domain and without time delay has been treated in [1], [15], [20]. It is obvious that this system has also the trivial solution $(0, 0)$ and the semitrivial solutions $(u^*, 0)$ and $(0, v^*)$ as that in (3.5). To ensure the existence of a positive T-periodic solution we assume that there exist positive constants M_1, M_2 and ϵ_0 such that

$$\begin{aligned} &[a_1(t,x)/b_1(t,x)] \le M_1 < M^*, \\ &[a_2(t,x)/c_2(t,x)] \le M_2 \le [a_1(t,x)/c_1(t,x)] - \epsilon_0, \quad ((t,x) \in \mathcal{D}), \end{aligned} \tag{3.14}$$

where M^* is an arbitrary constant if $\sigma_2 \ge b_2/a_2$ on $\mathcal{D}$, and $M^* = \underline{M}$ if $\sigma_2 < b_2/a_2$ for some $(t, x) \in \mathcal{D}$. The constant $\underline{M}$ in the latter case is given by

$$\underline{M} \equiv \inf\{a_2/(b_2 - \sigma_2 a_2); b_2 > \sigma_2 a_2 \text{ on } \mathcal{D}\}. \tag{3.15}$$

By letting $u_1 = u$ and $u_2 = M - v$, where $M = M_2 + \epsilon$ for a small $\epsilon > 0$ we transform problem (3.13) into the form (1.1) with $\mathbf{u} = (u_1, u_2)$ and

$$\begin{aligned} &f_1(t, x, \mathbf{u}, \mathbf{u}_\tau) = u_1[a_1 - b_1 u_1 - c_1(M - (u_2)_{\tau_2})/(1 + \sigma_1(u_1)_{\tau_1})], \\ &f_2(t, x, \mathbf{u}, \mathbf{u}_\tau) = -(M - u_2)[a_2 - b_2(u_1)_{\tau_1}/(1 + \sigma_2(u_1)_{\tau_1}) - c_2(M - u_2)]. \end{aligned} \tag{3.16}$$

It is obvious that the vector function $\mathbf{f}(\cdot, \mathbf{u}, \mathbf{v}) = (f_1(\cdot, \mathbf{u}, \mathbf{v}), f_2(\cdot, \mathbf{u}, \mathbf{v}))$ is quasi-monotone nondecreasing for $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^+ \times [0, M]$. Moreover, the constant pair $(\tilde{u}_1, \tilde{u}_2) = (M_1, M_2)$ and $(\hat{u}_1, \hat{u}_2) = (\delta_1, \delta_2)$ are upper and lower solutions if

$$\begin{aligned} &0 \ge M_1[a_1 - b_1 M_1 - c_1(M - M_2)/(1 + \sigma_1 M_1)] \\ &0 \ge -(M - M_2)[a_2 - b_2 M_1/(1 + \sigma_2 M_1) - c_2(M - M_2)] \\ &0 \le \delta_1[a_1 - b_1\delta_1 - c_1(M - \delta_2)/(1 + \sigma_1\delta_1)] \\ &0 \le -(M - \delta_2)[a_2 - b_2\delta_1/(1 + \sigma_2\delta_1) - c_2(M - \delta_2)]. \end{aligned} \tag{3.17}$$

Since $M = M_2 + \epsilon$ the above inequalities are satisfied if

$$\begin{aligned} &M_1 \ge a_1/b_1, && b_2 M_1/(1 + \sigma_2 M_1) \le a_2 - c_2\epsilon, \\ &c_1(M - \delta_2) \le a_1 - b_1\delta_1, && b_2\delta_1/(1 + \sigma_2\delta_1) \ge a_2 - c_2(M - \delta_2). \end{aligned} \tag{3.18}$$

It is easily seen from condition (3.14) that the requirements in (3.18) are all fulfilled by some sufficiently small δ_1, δ_2, and ϵ (with $(b_1/c_1)\delta_1 - \delta_2 \le \epsilon_0 - \epsilon$

and $c_2\delta_2 \leq b_2\delta_1/(1+\sigma_2\delta_1)$). Hence for these values of δ_1, δ_2, and ϵ, the pair (M_1, M_2) and (δ_1, δ_2) are ordered upper and lower solutions of the transformed problem of (3.13). This construction leads to the following result.

Theorem 3.3. *Let D_i^*, a_i, b_i and c_i, $i = 1, 2$, be the same bounded positive T-periodic functions as that in Theorem 3.2, and let $\sigma_i \equiv \sigma_i(t, x)$ be bounded nonnegative T-periodic functions in $C^\alpha(\mathcal{D})$. Assume that condition (3.14) holds. Then in addition to the trivial and semitrivial T-periodic solutions $(0, 0)$, $(u^*, 0)$, $(0, v^*)$, problem (3.13) has positive T-periodic solutions $(\bar{u}, \underline{v})$ and $(\underline{u}, \bar{v})$ such that $\underline{u} \leq \bar{u} \leq M_1$, and $\underline{v} \leq \bar{v} \leq M_2$ on $\mathcal{D}$.*

It is seen from the definition of M^* that condition (3.14) is reduced to condition (3.8) if $\sigma_2 = 0$. Moreover, the first requirement in (3.14) is satisfied by any choice of M_1 with $M_1 \geq a_1/b_1$ if $\sigma_2 \geq b_2/a_2$ on $\mathcal{D}$.

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