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REDUCIBLE PROJECTIVE CURVES: POSTULATION AND HYPERPLANE SECTION

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Abstract: Here we study the postulation of sufficiently general reducible connected nodal curves $T \subset \mathbf{P}^n$, $n \geq 3$, such that every irreducible component of T is a line. We will also consider the postulation of the general hyperplane section of T and study reducible connected curves Y which are union of a rational normal curve of $\mathbf{P}^n$ and $\deg(Y) - n$ lines.

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1. Introduction

The main aim of this paper is the study of the postulation of sufficiently general reducible connected nodal curves $T \subset \mathbf{P}^n$, $n \geq 3$, such that every irreducible component of T is a line. We will also consider the postulation of the general hyperplane section of T (see Theorem 6 and Proposition 7). We will also

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consider other reducible curves $Y \subset \mathbf{P}^n$ which are union of a rational normal curve of $\mathbf{P}^n$ and $\deg(Y) - n$ lines (see Definitions 5 and 7). Quite often results on reducible curves are very useful to obtain results on smooth curves (see, for instance [8], [1], [2], [3], [4], [9] and [5]).

Let $A \subset \mathbf{P}^n$ be a closed subscheme. For any integer t let $\rho_{A,t,n} : H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(t)) \to H^0(A, \mathcal{O}_A(t))$ be the restriction map. We will say that A has *maximal rank* if for all integers t the restriction map $\rho_{A,t.n}$ has maximal rank, i.e. it is injective or surjective. A reduced and connected curve $T \subset \mathbf{P}^n$, $n \geq 3$, will be called a *degree d tree* or a *tree of d lines* if $\deg(T) = d$, $p_a(T) = 0$, T has only nodes as singularities and each irreducible component of T is a line. A degree d *bamboo* $Y \subset \mathbf{P}^n$, $n \geq 3$, is a degree d tree $Y \subset \mathbf{P}^n$ such that there is an ordering $A_1, \dots, A_d$ of the lines of Y such that $A_i \cap A_j \neq \emptyset$ if and only if $|i - j| \leq 1$. A_1 (resp. A_d) will be called the initial (resp. final) line of the bamboo Y. Obviously, if $d \geq 2$ there is another ordering of the lines of the bamboo Y, which exchanges the initial and the final lines of Y, while if $d = 1$ the line A_1 is simultaneously the initial and the final line of Y.

Definition 1. Fix integers n, d with $n \geq 3$ and $d \geq 2$. Fix a map $\tau : \{2, \dots d\} \to \{1, \dots, d-1\}$, such that $\tau(i) < i$ for every i. Let $T(n, d, \tau)$ be the set of all degree d trees $T \subset \mathbf{P}^n$ for which there is an ordering, say $A_1, \dots, A_d$, of the irreducible components of T such that for all integers i, j with $1 \leq j < i \leq d$ we have $A_i \cap A_j \neq \emptyset$ if and only if $j = \tau(i)$.

Remark 1. The set $T(n, d, \tau)$ is an irreducible locally closed subset of the Hibert scheme of $\mathbf{P}^n$. Every tree has a type. A tree may have different types, but if there is one degree d tree with types τ and ψ, then all trees with type τ have type ψ (and conversely), i.e. $T(n, d, \tau) = T(n, d, \psi)$.

In Section 4 we will prove the following result.

Theorem 1. *For all integers n, d with $d \geq n \geq 4$ there is a degree d bamboo $T \subset \mathbf{P}^n$ with maximal rank.*

Definition 2. Fix integers n, d with $n \geq 3$, $d \geq 2$ and a reduced connected degree d nodal curve $T \subset \mathbf{P}^n$, such that every irreducible component of T is a line. We will say that T is a dismantled curve if we may order the irreducible components $A_1, \dots, A_d$ of T and find two functions $\alpha : \{2, \dots, d\} \to \{1, \dots, d-1\}$ and $\beta : \{2, \dots, d\} \to \{-1, 1, \dots, d-1\}$ such that for every integer i with $2 \leq i \leq d$ the following properties hold:

(i) $\alpha(i) < i$ and $\beta(i) < i$ for every i with $2 \leq i \leq d$;

(ii) if $\beta(i) \neq -1$, then $\mathrm{card}(A_i \cap (A_1 \cup \cdots \cup A_{i-1})) = 2$ and $A_i \cap A_j \neq \emptyset$ for some $j \in \{1, \ldots, i-1\}$ if and only if $j \in \{\alpha(i), \beta(i)\}$;

(iii) if $\beta(i) = -1$, then $\mathrm{card}(A_i \cap (A_1 \cup \cdots \cup A_{i-1})) = 1$ and $A_i \cap A_j \neq \emptyset$ for some $j \in \{1, \ldots, i-1\}$ if and only if $j = \alpha(i)$;

(iv) if $\beta(i) \neq -1$ we have $(\alpha(i), \beta(i)) \neq (\alpha(j), \beta(j))$ for every j such that $2 \leq j < i$;

(v) if $\beta(i) \neq -1$, then $A_{\alpha(i)} \cap A_{\beta(i)} = \emptyset$.

Remark 2. Take $T = A_1 \cup \cdots \cup A_d$, α and β as in Definition 2 with $d \geq 2$. The number of integers i with $2 \leq i \leq d$ and $\beta(i) \neq -1$ is the arithmetic genus $p_a(T)$ of T. For every integer i with $2 \leq i \leq d$ the curve $A_1 \cup \cdots \cup A_i$ is a dismantled curve with respect to the functions $\alpha|\{2, \ldots, i\}$ and $\beta|\{2, \ldots, i-1\}$. In particular, the function α shows that $A_1 \cup \cdots \cup A_i$ is connected. Since $d = \deg(T)$, T is nodal and any line is uniquely determined by any two of its points, we have $\beta(i) \neq \alpha(i)$ for every i and $\beta(2) = -1$. Using $d-1$ Mayer - Vietoris exact sequences

$$0 \to \mathcal{O}_{A_1 \cup \cdots \cup A_i}(1) \to \mathcal{O}_{A_1 \cup \cdots \cup A_{i-1}}(1) \oplus \mathcal{O}_{A_i}(1) \\ \to \mathcal{O}_{(A_1 \cup \cdots \cup A_{i-1}) \cap A_i}(1) \to 0\,, \quad (1)$$

$2 \leq i \leq d$, $A_i \cong \mathbf{P}^1$ and the assumption $\mathrm{card}(A_i \cap (A_1 \cup \cdots \cup A_{i-1})) \leq 2$ we obtain $h^1(T, \mathcal{O}_T(1)) = 0$. By [7] or [9] the curve T is in the closure in the Hilbert scheme of $\mathbf{P}^n$ of the set of all smooth curves in $\mathbf{P}^n$ with degree d, genus $p_a(T)$ and non-special hyperplane line bundle. A curve $Y \subset \mathbf{P}^n$ is a tree if and only if it is a dismantled curve and $p_a(Y) = 0$.

Definition 3. For all positive integers n, d with $n \geq 3$ and all functions α, β as in Definition 2 let $B(d, n, \alpha, \beta)$ denote the set of all degree d dismantled curves in $\mathbf{P}^n$, such that there is an ordering of their irreducible components for which α and β have the properties (i), (ii), (iii), (iv) and (v) of Definition 2.

The set $B(d, n, \alpha, \beta)$ is an irreducible locally closed subset of the Hilbert scheme of $\mathbf{P}^n$.

Remark 3. Let E be a curve which is a union of some irreducible components of a dismantled curve. Every connected component of E is a dismantled curve.

Remark 4. For all integers n, d, g with $n \geq 3$, $g \geq 0$ and $d \geq g + n$ there is a dismantled curve $T \subset \mathbf{P}^n$ such that $\deg(T) = d$, $p_a(T) = g$ and T spans $\mathbf{P}^n$.

For all integers n, k, g with $n \geq 3$, $k \geq 1$ and $g \geq 0$ define the integers $r(k,g,n)$ and $q(k,g,n)$ by the relations

$$kr(k,g,n) + 1 - g + q(k,g,n) = \binom{n+k}{k}, 0 \leq q(k,g,n) \leq k-1\,. \qquad (2)$$

For all triples (d,g,n) with $n \geq 3$, $g \geq 0$ and $d \geq g+n$ the critical value of the triple (d,g,n) (or the critical value of a non-special curve of degree d and genus g in $\mathbf{P}^n$) is the minimal integer $k \geq 1$ such that $\binom{n+k}{n} \geq dk+1-k$, i.e. such that $h^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(k)) \geq h^0(Y, \mathcal{O}_Y(k))$ for any reduced and connected curve $Y \subset \mathbf{P}^n$ such that $\deg(Y) = d$, $p_a(Y) = g$ and $h^1(Y, \mathcal{O}_Y(1)) = 0$. For the explict values of the integers $r(k,0,3)$ and $q(k,0,3)$, see [8] or 4.2 in [1].

In Section 3 we will prove the following results.

Theorem 2. *Fix integers d, g with $d \geq g+4 \geq 4$ and let k be the critical value of the triple $(d,g,4)$. Then there is a dismantled curve $Y \in B(d,4,\alpha,\beta)$ (for some α and β) such that $p_a(Y) = g$ and $h^1(\mathbf{P}^4, \mathcal{I}_Y(t)) = 0$ for every $t \geq k$.*

Theorem 3. *Fix integers d, g with $d \geq g+4 \geq 4$ and let k be the critical value of the triple $(d,g,4)$. Then there is a dismantled curve $Y \in B(d,4,\alpha,\beta)$ (for some α and β) such that $p_a(Y) = g$ and $h^0(\mathbf{P}^4, \mathcal{I}_Y(k-1)) = 0$.*

We were unable to prove that for all integers d, g with $d \geq g+4 \geq 4$ we may find a dismantled curve of degree d and arithmetic genus g in $\mathbf{P}^4$ with maximal rank because writing down the inductive proofs of Theorems 2 and 3 we often met dismantled curves with the same degree and genus but combinatorially different. We will give an example (see Example 1) to show that some weak form of condition (iv) in Definition 2 is essential to have Theorems 2 and 3 and to control the postulation of the general hyperplane section of a sufficiently general reducible curve.

2. Preliminary Results

For all integers $N \geq 3$ and $k \geq 1$ define the integers $a(k,N)$ and $q(k,N)$ using the following relations:

$$(k+1)a(k,N) + q(k,N) = \binom{N+k}{N}, 0 \leq q(k,N) \leq k\,. \qquad (3)$$

From Lemma 1 to Lemma 5 we will collect all the lemmas we need to control the general hyperplane section of a sufficiently general reducible curve $Y \subset \mathbf{P}^n$

and the general intersection of a sufficiently general tree $T \subset \mathbf{P}^3$ with a smooth quadric surface $Q \subset \mathbf{P}^3$.

Lemma 1. *Fix a positive integer k, a hyperplane H of $\mathbf{P}^4$, a plane M of H and a finite set $S \subset H$. Set $Z := S \backslash M \cap S$. Assume $h^1(H, \mathcal{I}_S(k)) = h^1(H, \mathcal{I}_Z(k-1)) = h^1(M, \mathcal{I}_{S\cap M,M}(k)) = 0$ and card$(S \cap M) < (k+2)(k+1)/2$. Then for a general $P \in M$ we have $h^1(H, \mathcal{I}_{S\cup\{P\}}(k)) = 0$.*

Proof. Since card$(S \cap M) < (k+2)(k+1)/2$, we have

$$h^0(M, \mathcal{I}_{S\cap M,M}(k)) > 0.$$

Since P is general in M, we have

$$h^0(M, \mathcal{I}_{S\cap M\cup\{P\},M}(k)) = h^0(M, \mathcal{I}_{S\cap M,M}(k)) - 1.$$

Since $h^1(M, \mathcal{I}_{S\cap M,M}(k)) = 0$, we obtain $h^1(M, \mathcal{I}_{S\cap M\cup\{P\},M}(k)) = 0$. The set $Z \cup \{P\}$ is the residual scheme of $S \cap \{P\}$ with respect to the Cartier divisor M of H. Hence we have the exact sequence

$$0 \to \mathcal{I}_{Z\cup\{P\}}(k-1) \to \mathcal{I}_{S\cup\{P\}}(k) \to \mathcal{I}_{S\cap M\cup\{P\},M}(k) \to 0\,. \tag{4}$$

The cohomology exact sequence of (4) proves the lemma. □

From Lemma 1 we immediately obtain the following corollaries.

Corollary 1. *Fix a positive integer k, a hyperplane H of $\mathbf{P}^4$, a plane M of H and a finite set $S \subset H$. Set $Z := S \backslash M \cap S$. Assume $h^1(H, \mathcal{I}_S(k)) = h^1(H, \mathcal{I}_Z(k-1)) = 0$ and card$(S \cap M) \leq 2$. Then for a general $P \in M$ we have $h^1(H, \mathcal{I}_{S\cup\{P\}}(k)) = 0$.*

Corollary 2. *Fix a positive integer k, a hyperplane H of $\mathbf{P}^4$, a plane M of H and a finite set $S \subset H$. Set $Z := S \backslash M \cap S$. Assume $h^1(H, \mathcal{I}_S(k)) = h^1(H, \mathcal{I}_Z(k-1)) = 0$ and card$(S \cap M) \leq k$. Then for a general $P \in M$ we have $h^1(H, \mathcal{I}_{S\cup\{P\}}(k)) = 0$.*

Lemma 2. *Fix integers k, t with $k < t > 0$, a hyperplane H of $\mathbf{P}^4$, an irreducible hypersurface E of H with deg$(E) = t$ and a finite subset S of H. Set $Z := S \backslash E \cap S$. Assume $h^1(H, \mathcal{I}_S(k)) = h^1(H, \mathcal{I}_Z(k-t)) = 0$ and card$(S \cap E) \leq k$. Then for a general $P \in E$ we have $h^1(H, \mathcal{I}_{S\cup\{P\}}(k)) = 0$.*

Proof. Since card$(S \cap E) \leq k$, we have $h^1(E, \mathcal{I}_{S\cap E}(k)) = 0$ and $h^0(E, \mathcal{I}_{S\cap E}(k-t)) > 0$. Copy the proof of Lemma 1. □

The following three remarks will allow us to use Lemmas 1 and 2 and Corollaries 1 and 2.

Remark 5. Fix two disjoint lines D, R of $\mathbf{P}^4$ and let N be the hyperplane of $\mathbf{P}^4$ spanned by $D \cup R$. Take a hyperplane H of $\mathbf{P}^4$ with $H \neq N$ and set $M = H \cap N$. For every $P \in N$ with $P \notin D \cup R$ there is a line L with $P \in L$, $L \cap D \neq \emptyset$ and $L \cap R \neq \emptyset$. Thus for a general $P \in M$ there is a line T with $T \cap D \neq \emptyset$, $T \cap R \neq \emptyset$ and $T \cap H = \{P\}$.

Remark 6. Let $C \subset \mathbf{P}^4$ be a rational normal curve, $D \subset \mathbf{P}^4$ a line and $H \subset \mathbf{P}^4$ a hyperplane intersecting transversally $C \cup D$. Let F be the join of C and D, i.e. the closure in $\mathbf{P}^4$ of the union of all lines intersecting C and D at different points. F is an irreducible hypersurface of $\mathbf{P}^4$. Taking the linear projection of $\mathbf{P}^4 \backslash D$ onto $\mathbf{P}^2$ from D we obtain

$$\deg(F) = \deg(C) - \text{lenght}(C \cap D) = 4 - \text{lenght}(C \cap D) \geq 2.$$

For a general $P \in F$ there is a line L with $L \cap D \neq \emptyset$, $L \cap C \neq \emptyset$, $L \cap D \cap C = \emptyset$ and $P \in L$. The same is true if P is a general point of $H \cap F$.

Remark 7. Let $C \subset \mathbf{P}^4$ be a rational normal curve and $F \subset \mathbf{P}^4$ its secant variety, i.e. the closure in $\mathbf{P}^4$ of the union of all lines intersecting C at two distinct points. F is an irreducible hypersurface of $\mathbf{P}^4$. Take a general line $D \subset \mathbf{P}^4$ and consider the linear projection of $\mathbf{P}^4 \backslash D$ onto $\mathbf{P}^2$ from D. Using the genus formula for plane curves (i.e. that a nodal rational plane quartic has three nodes) we obtain $\deg(F) = 3$. Take a hyperplane H of $\mathbf{P}^4$ intersecting transversally C. Hence $F \cap H$ has no multiple component. For every point P in a Zariski open dense subset of $F \cap H$ there is a line L with $L \cap H = \{P\}$, $\text{card}(L \cap C) = 2$ and L intersecting quasi-transversally C.

Lemma 3. (see [3], 2.1) *Fix integers $n \geq 3$ and $k \geq 1$. Let $C \subset \mathbf{P}^n$ be a non-degenerate irreducible curve and $H \subset \mathbf{P}^n$ a hyperplane. Fix a linear subspace V of $H^0(H, \mathcal{O}_H(k))$. For any curve $A \subset \mathbf{P}^n$ such that A intersects transversally H set $V(-A \cap H) := \{f \in V : f(P) = 0$ for each $P \in H \cap A\}$. Then for a general reducible conic T such that each irreducible component of T intersect C at a different point we have $\dim(V(-T \cap H)) = \max\{\dim(V) - 2, 0\}$.*

Lemma 4. *Fix non-negative integers a,b, a smooth quadric surface $Q \subset \mathbf{P}^3$, a line $D \subset \mathbf{P}^3$ intersecting transversally Q and a finite subset S of Q. Set $Z := S \backslash S \cap D$. Assume $h^1(Q, \mathcal{I}_S(a,b)) = 0$, $\text{card}(S) \leq (a+1)(b+1) - 2$ and $h^0(Q, \mathcal{I}_Z(a-1, b-1)) \leq (a+1)(b+1) - \text{card}(S) - 2$. Then for a general line $R \subset \mathbf{P}^3$ with $R \cap D \neq \emptyset$ we have $h^1(Q, \mathcal{I}_{S \cup (R \cap Q)}(a,b)) = 0$, i.e.*

$h^0(Q, \mathcal{I}_{S\cup(R\cap Q)}(a,b)) = (a+1)(b+1) - card(S) - 2$.

Proof. For a general $P \in Q$ we have

$$h^0(Q, \mathcal{I}_{S\cup\{P\}}(a,b)) = h^0(Q, \mathcal{I}_S(a,b)) - 1 = (a+1)(b+1) - \mathrm{card}(S) - 1.$$

Fix a general $P \in Q$ and let H be the plane spanned by D and P. Set $E := H \cap H$. For general P we have $E \cap S = E \cap D$, i.e. $E \cap Z = \emptyset$. Thus E is a smooth curve of type $(1,1)$ containing $D \cap Q$ and P. For any $A \in E$ with $A \notin D \cap H$ and $A \neq P$ the line $\langle\{P,A\}\rangle$ intersects D and $\langle\{P,A\}\rangle \cap Q = \{P,A\}$. Thus to prove the lemma it is sufficient to prove that for a general $A \in E$ we have

$$h^0(Q, \mathcal{I}_{S\cup\{P\}\cup\{A\}}(a,b)) = h^0(Q, \mathcal{I}_{S\cup\{P\}}(a,b)) - 1.$$

Suppose that this is not true. Since A is general in E, we obtain $(a+1)(b+1) - \mathrm{card}(S) - 1 = h^0(Q, \mathcal{I}_{S\cup\{P\}}(a,b)) = h^0(Q, \mathcal{I}_{S\cup\{P\}\cup E}(a,b)) = h^0(Q, \mathcal{I}_Z(a-1, b-1)) \leq (a+1)(b+1) - \mathrm{card}(S) - 2$, contradiction. □

We recall a few results proved in [1].

Definition 4. Fix non-negative integers γ, δ and a finite subset S of a smooth quadric surface $Q \subset \mathbf{P}^3$. We will say that S is (γ,δ)-independent if $h^0(Q, \mathcal{I}_S(\gamma,\delta)) = \max\{0, (\gamma+1)(\delta+1) - \mathrm{card}(S)\}$. We will say that S has general position if for every subset A of S and all non-negative integers x, y such that $(x, y, \mathrm{card}(A)) \neq (1,1,4)$ the set A is (x,y)-independent.

Lemma 5. ([1], Lemma 6.2) *Let $C \subset \mathbf{P}^3$ be an irreducible curve intersecting transversally a smooth quadric surface $Q \subset \mathbf{P}^3$ and $S \subset Q$ a finite set which has general position. Assume card$(S \cap C) \leq 3$. Then for a general line $D \subset \mathbf{P}^3$ intersecting C the set $S \cup (D \cap Q)$ has general position.*

We will apply Lemma 5 when C is a line. A reader of [5] may wish to apply it taking as C a rational normal curve of $\mathbf{P}^3$.

We will use the following elementary forms of Horace lemma introduced in [8] (see Lemma 1 for a related situation with a full proof).

Lemma 6. *Let $H \subset \mathbf{P}^n$ be a hyperplane and $C \subset \mathbf{P}^n$ a curve; we allow that some irreducible component of C is contained in H. Let A be the residual scheme of C with respect to H; if C is reduced, A is just the union of all irreducible components of C not contained in H. Fix a positive integer k. If the restrictiom maps*

$$\rho_{A,k-1,n} : H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(k-1)) \to H^0(A, \mathcal{O}_A(k-1))$$

and

$$\rho_{C\cap H,k,n} : H^0(H, \mathcal{O}_H(k)) \to H^0(C \cap H, \mathcal{O}_{C\cap H}(k))$$

are bijective (resp. injective, resp. surjective), then the restriction map $\rho_{A,k,n}$: $H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(k)) \to H^0(A, \mathcal{O}_A(k))$ *is bijective (resp. injective, resp. surjective).*

Lemma 7. *Let* $Q \subset \mathbf{P}^3$ *be a smooth quadric surface and* $C \subset \mathbf{P}^3$ *a curve. Let* A *be the residual scheme of* C *with respect to the Cartier divisor* Q *of* $\mathbf{P}^3$*. Fix an integer* $k \geq 2$*. If the restriction maps*

$$\rho_{A,k-2,3} : H^0(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(k-2)) \to H^0(A, \mathcal{O}_A(k-2))$$

and

$$\rho_{C\cap Q,k,Q} : H^0(Q, \mathcal{O}_Q(k)) \to H^0(C \cap Q, \mathcal{O}_{C\cap Q}(k))$$

are bijective (resp, injective, resp. surjective), then the restriction map $\rho_{C,k,3}$: $H^0(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(k)) \to H^0(C, \mathcal{O}_C(k))$ *is bijective (resp. injective, resp. surjective).*

Proposition 1. *For all positive integers* d*,* k *with* $d \leq a(k, 3)$ *there is a degree* d *tree* $T \subset \mathbf{P}^3$ *such that* $\rho_{T,k,3}$ *is surjective.*

Proof. Fix a smooth quadric surface $Q \subset \mathbf{P}^3$. First, we assume $d = a(k, 3)$. If $k = 1$, just take a reducible conic. However, for related matters it is worthwhile to remark that more is true because there is a degree 3 linearly normal tree. For $k = 2$ use that any degree 3 linearly normal tree is arithmetically Cohen - Macaulay. More is true; take a smooth quadric surface Q and $E = A_1 \cup A_2 \cup A_3 \cup A_4 \subset Q$ with A_1 a line of type $(1, 0)$ on Q and A_2, A_3 and A_4 different lines of type $(0, 1)$ on Q; E is a degree 4 tree; since Q is the only quadric containing E, the map $\rho_{E,2,3}$ is surjective. Now assume $k \geq 3$ and that the result is true for the integers $k' = k - 2$ and $d' = a(k - 2, 3)$. Let $Z \subset \mathbf{P}^3$ be a degree $a(k - 2, 3)$ tree such that $\rho_{Z,k-2,3}$ is surjective. Notice that $2a(k-2, 3) - a(k, 3) + a(k-2, 3) < (k+1)(k+1 - a(k, 3) + a(k-2, 3))$; here we use that $|q(k - 2, 3) - q(k, 3)| < a(k, 3) - a(k - 2, 3)$ by the explicit values of the integers $a(x, 3)$ and $q(x, 3)$ given Remark 16. Hence, by Lemmas 4 and 5 moving if necessary the lines of Z, we may assume that Z is transversal to Q, that no two points of $Z \cap Q$ are contained in a line of type $(0, 1)$ on Q and that the union S of $2a(k - 2, 3) - a(k, 3) + a(k - 2, 3)$ points of $Z \cap Q$ satisfies $h^1(Q, \mathcal{I}_S(k, k - a(k, 3) + a(k - 2, 3))) = 0$. We add in Q the union W of $a(k, 3) - a(k - 2, 3)$ lines of type $(0, 1)$, each of them containing one of the points of $Z \cup Q \backslash S$. Set $T = Z \cup W$. Since no two points of $Z \cap Q$ are contained in a line of type $(0, 1)$ of Q, T is a degree $a(k, 3)$ tree. By Horace lemma 7 $\rho_{T,k,3}$ is surjective. Now assume $d < a(k, 3)$. By the first part of the proof there is a degree $a(k,3)$ tree Y such that $\rho_{Y,k,3}$ is surjective. Write $Y = A_1 \cup \cdots \cup A_{a(k,3)}$ with respect to a good ordering of the lines of Y. Set $T := A_1 \cup \cdots \cup A_d$. Hence T is a degree d tree. Using $a(k,3) - d$ Mayer - Vietoris exact sequences as in Remark 2 we obtain the surjectivity of the restriction map $H^0(Y, \mathcal{O}_Y(k)) \to H^0(T, \mathcal{O}_T(k))$. Thus the surjectivity of $\rho_{Y,k,3}$ implies the surjectivity of $\rho_{T,k,3}$. □

Theorem 4. *Take d, α and β as in Definition 2. Fix a hyperplane H of $\mathbf{P}^4$. Then for a general $T \in B(d,4,\alpha,\beta)$ the set $T \cap H$ has maximal rank.*

Proof. Let k be the critical value of the triple $(d,0,3)$. If $k = 1$, then $d \leq 4$, $p_a(T) = 0$ and the result is obvious. Assume $k \geq 2$. We need to check that $h^1(H, \mathcal{I}_{T\cap H}(t)) = 0$ for every $t \geq k$ and $h^0(H, \mathcal{I}_{T\cap H}(k-1)) = 0$ for a general $T \in B(d,4,\alpha,\beta)$. Since $h^1(T, \mathcal{O}_T(1)) = 0$ for every dismantled curve T (Remark 2), if $h^1(H, \mathcal{I}_{T\cap H}(k)) = 0$, then $h^1(H, \mathcal{I}_{T\cap H}(t)) = 0$ for every $t > k$ (Castelnuovo - Mumford 's Lemma). Since $B(d,4,\alpha,\beta)$ is irreducible, it is sufficient to find $A \in B(d,4,\alpha,\beta)$ and $B \in B(d,4,\alpha,\beta)$. such that

$$h^0(H, \mathcal{I}_{A\cap H}(k-1)) = 0 \quad \text{and} \quad h^1(H, \mathcal{I}_{B\cap H}(k-1)) = 0\,.$$

We will prove the existence of B leaving to the reader the very similar proof of the existence of the curve A; indeed, the existence of A also follows at once (just adding $d - \binom{k+2}{3}$ suitable lines) from the existence of the curve B for the integer $d' = \binom{k+2}{3}$. Fix a general $T = A_1 \cup \cdots \cup A_d \in B(d,4,\alpha,\beta)$. Set $T(i) := A_1 \cup \cdots \cup A_i$, $1 \leq i \leq d$. Let $D \subset \mathbf{P}^4$ be a general line such that $D \cap A_j \neq \emptyset$ if and only if $A_{i+1} \cap A_j \neq \emptyset$.

Claim. *We have $h^1(H, \mathcal{I}_{H\cap(T(i)\cup D)}(k)) = 0$.*

By the irreducibility of $B(d,4,\alpha,\beta)$ and induction on i the Claim implies the existence of the curve B and, hence it implies the theorem.

Proof of the Claim. If $\beta(i+1) = -1$, then $\mathrm{card}(T(i) \cup D) = 1$ and we may take as $D \cap H$ any general point of H. Thus $h^1(H, \mathcal{I}_{H\cap(T(i)\cup D)}(k)) = 0$ in this case. Assume $\beta(i+1) \neq -1$. Hence D is a general line intersecting the lines $A_{\alpha(i+1)}$ and $A_{\beta(i+1)}$. By condition (iv) of Definition 2 $A_{\alpha(i+1)} \cap A_{\beta(i+1)} = \emptyset$. Thus $A_{\alpha(i+1)} \cup A_{\beta(i+1)}$ spans a 3-dimensional linear space N. For general $T(i)$ we have $N \neq H$. Hence we may take as $D \cap H$ a general point of the plane $M := N \cap H$. By condition (iv) of Definition 2 we may deform $T(i)$ keeping fixed the linear span of $A_{\alpha(i+1)} \cup A_{\beta(i+1)}$ in such a way that $A_j \cap M = \emptyset$ if $j \notin \{\alpha(i+1), \beta(i+1)\}$. Hence we may apply Lemma 1 taking $S = H \cap T(i)$. □

Definition 5. Fix integers n, d with $d \geq n \geq 3$ and a reduced connected degree d nodal curve $T \subset \mathbf{P}^n$, such that one irreducible component of T is a rational normal curve of $\mathbf{P}^n$, while the other irreducible components of T are lines. We will say that T is a semi-dismantled curve if we may order the irreducible components $A_0, A_1, \dots, A_d$ of T (with A_0 the rational normal curve) and find two functions $\alpha : \{1, \dots, d-n\} \to \{0, \dots, d-n-1\}$ and $\beta : \{1, \dots, d-n\} \to \{-1, 0, 1, \dots, d-n-1\}$ such that for every integer i with $1 \leq i \leq d-n$ the following properties hold:

(i) $\alpha(i) < i$ and $\beta(i) < i$;

(ii) if $\beta(i) \neq -1$, then $\mathrm{card}(A_i \cap (A_0 \cup A_1 \cup \dots \cup A_{i-1})) = 2$ and $A_i \cap A_j \neq \emptyset$ for some $j \in \{0, \dots, i-1\}$ if and only if $j \in \{\alpha(i), \beta(i)\}$;

(iii) if $\beta(i) = -1$, then $\mathrm{card}(A_i \cap (A_0 \cup A_1 \cup \dots \cup A_{i-1})) = 1$ and $A_i \cap A_j \neq \emptyset$ for some $j \in \{0, 1, \dots, i-1\}$ if and only if $j = \alpha(i)$;

(iv) if $\beta(i) \neq -1$ and $(\alpha(i), \beta(i)) \neq (0, 0)$ we have $(\alpha(i), \beta(i)) \neq (\alpha(j), \beta(j))$ for every j such that $1 \leq j < i$;

(v) if $\beta(i) \notin \{-1, 0\}$, then $A_{\alpha(i)} \cap A_{\beta(i)} = \emptyset$;

(vi) there are at most n integers $h \in \{1, \dots, d-n\}$ such that $(\alpha(h), \beta(h)) = (0, 0)$.

Remark 8. Take $T = A_0 \cup A_1 \cup \dots \cup A_d$, α and β as in Definition 5. The number of integers i with $1 \leq i \leq d-n$ and $\beta(i) \neq -1$ is the arithmetic genus $p_a(T)$ of T. For every integer i with $1 \leq i \leq d-n$ the curve $A_0 \cup A_1 \cup \dots \cup A_i$ is a dismantled curve with respect to the functions $\alpha|\{1, \dots, i\}$ and $\beta|\{1, \dots, i\}$. In particular the function α shows that $A_0 \cup A_1 \cup \dots \cup A_i$ is connected. As in Remark 2 we obtain $h^1(T, \mathcal{O}_T(1)) = 0$. By [7] or [9] the curve T is in the closure in the Hilbert scheme of $\mathbf{P}^n$ of the set of all smooth curves in $\mathbf{P}^n$ with degree d, genus $p_a(T)$ and non-special hyperplane line bundle.

Definition 6. For all positive integers n, d with $d \geq n \geq 3$ and all functions α and β as in Definition 5, let $S(d, n, \alpha, \beta)$ be the set of all degree d semi-dismantled curves in $\mathbf{P}^n$ such that there is an ordering of their irreducible components for which α and β have the properties (i), (ii), (iii), (iv), (v) and (vi) of Definition 5.

The set $S(d, n, \alpha, \beta)$ is an irreducible locally closed subset of the Hilbert scheme of $\mathbf{P}^n$. The semi-dismantled curves are the curves used in [5].

Remark 9. Let E be a curve which is a union of some of the irreducible components of a semi-dismantled curve. Every connected component of E is either a dismantled curve or a semi-dismantled curve.

Remark 10. For all integers n, d and g with $n \geq 3$, $g \geq 0$ and $d \geq g+n$ there is a semi-dismantled curve $T \subset \mathbf{P}^n$ such that $\deg(T) = d$, $p_a(T) = g$ and T spans $\mathbf{P}^n$.

The proof of Theorem 4 and a use of Lemma 2 and Remarks 5 and 6 gives the following result.

Theorem 5. *Fix a hyperplane H of $\mathbf{P}^4$, an integer $d \geq 4$ and functions α, β as in Definition 5. Then the set $T \cap H$ has maximal rank for a general $T \in S(d, 4, \alpha, \beta)$.*

The results correspondings to Theorems 4 and 5 are very easy for dismantled curves and semi-dismantled curves in $\mathbf{P}^3$ (see Proposition 2 for their statements) because, instead of Lemmas 1 and 2 and Remarks 5 and 6, we may use the following observation.

Remark 11. Let $C \subset \mathbf{P}^3$ be a rational normal curve and E, L, R lines in $\mathbf{P}^3$ such that $L \cap R = \emptyset$ and E is not tangent to C. Let $H \subset \mathbf{P}^3$ be a plane intersecting transversally C, E, L and R. The secant variety of C is $\mathbf{P}^3$, while the union of all tangent lines of C is a surface W; for every $P \in \mathbf{P}^3 \backslash W$ there is a line D with $P \in D$, $\mathrm{card}(D \cap C) = 2$ and D intersecting quasi-transversally C; in particular there is such a line D for a general $P \in H$. The join of C and E in $\mathbf{P}^3$ is $\mathbf{P}^3$, and hence for a general $P \in H$ there is a line T with $P \in T$, $\mathrm{card}(T \cap C) = \mathrm{card}(E \cap T) = 1$ and $C \cup E \cup T$ nodal. The join of D and L is $\mathbf{P}^3$, and hence for a general $P \in H$ there is a line T with $P \in T$, $\mathrm{card}(T \cap D) = \mathrm{card}(L \cap T) = 1$ and $D \cup L \cup T$ nodal.

Proposition 2. *Let $B(d, 3, \alpha, \beta)$ be an irreducible component of the set of all dismantled curves in $\mathbf{P}^3$, $S(d_1, 3, \alpha_1, \beta_1)$ an irreducible component of the set of all semi-dismantled curves in $\mathbf{P}^3$ and $H \subset \mathbf{P}^3$ a plane. Then for a general $T \in B(d, 3, \alpha, \beta)$ and a general $Y \in S(d_1, 3, \alpha_1, \beta_1)$ the sets $T \cap H$ and $Y \cap H$ have maximal rank. For a general $S \subset H$ with card$(S) = d$ and a general $F \subset H$ with card$(F) = d_1$ there are $U \in B(d, 3, \alpha, \beta)$ and $V \in S(d_1, 3, \alpha_1, \beta_1)$ such that $U \cap H = S$ and $V \cap H = F$.*

The following example shows that some weak form of condition (iv) in Definitions 2 and 5 is essential to have this type of results.

Example 1. Fix integers n, d with $n \geq 4$ and $d \geq (n+1)^2/2$. Let $M \subset \mathbf{P}^n$ be a 3-dimensional linear space and A_1, A_3 disjoint lines in M. Every line D with $D \cap A_1 \neq \emptyset$ and $D \cap A_3 \neq \emptyset$ is contained in M. Let

$$\alpha : \{2, \dots, d\} \to \{1, \dots, d-1\} \text{ and } \beta : \{2, \dots, d\} \to \{1, \dots, d-1\}$$

be defined by $\alpha(i) = i-1$ for $2 \leq i \leq n$, $\alpha(i) = 1$ for $n+1 \leq i \leq d$, $\beta(i) = -1$ for $2 \leq i \leq n$ and $\beta(i) = 3$ for $n+1 \leq i \leq d$. Let $T \subset \mathbf{P}^n$ be the general union of lines A_i, $1 \leq i \leq d$, such that $A_j \cap A_i \neq \emptyset$ for $1 \leq j < i$ if and only if $j \in \{\alpha(i), \beta(i)\}$. T is a connected non-degenerate curve, but for a general hyperplane H of $\mathbf{P}^n$ the hyperplane section $T \cap H$ has not maximal rank, because it contains $d-n+3$ points contained in the plane $M \cap H$, and hence $T \cap H$ is contained in a reducible quadric hypersurface of H, while $d \geq h^0(H, \mathcal{O}_H(2))$.

For the proof of Theorems 2 and 3 we will need the following results similar to Proposition 1.

Lemma 8. *Fix an integer $k \geq 3$. There is a tree $T \subset \mathbf{P}^3$ such that $deg(T) = a(k,3)-1$, $\rho_{T,k,3}$ is surjective and $T = E \cup D \cup F$ with D a line, $E \cup D$ a tree and F the union of $a(k,3) - a(k-2,3) - 2$ disjoint lines, each of them intersecting D but not E.*

Proof. Let $Q \subset \mathbf{P}^3$ be a smooth quadric surface. Let E be a general tree of degree $a(k-2,3)$. By Proposition 1 the map $\rho_{E,k-2,3}$ is surjective. Let $D \subset Q$ be a line of type $(1,0)$ intersecting E and $F \subset Q$ union of $a(k,3) - a(k-2,3) - 2$ lines of type $(0,1)$ not intersecting E. Then apply Horace Lemma 7 as in the part $k \geq 3$ of the proof of Proposition 1. □

Lemma 9. *Fix an integer $k \geq 5$. There is a tree $T \subset \mathbf{P}^3$ such that $deg(T) = a(k,3)-2$, $\rho_{T,k,3}$ is surjective and $T = E \cup D \cup R \cup F \cup G$ with D and R lines, $E \cup D \cup R$ a tree, F the union of $a(k-2,3) - a(k-4,3) - 2$ disjoint lines, each of them intersecting D but not $E \cup R \cup G$, and G the union of $a(k,3) - a(k-2,3) - 2$ disjoint lines, each of them intersecting R but not $E \cup D \cup F$.*

Proof. Copy the proof of Lemma 8 with the following modification: make two steps with two different smooth quadrics, the first one from the critical value $k-4$ to the critical value $k-2$ and the second one from the critical value $k-2$ to the critical value k. □

3. Proofs of Theorems 2 and 3

For every positive integer k define the integers $g(k)$ and $f(k)$ by the relations

$$k(g(k)+4)+1-g(k)+f(k)=\binom{k+4}{4}, 0 \le f(k) \le \max\{0, k-2\}. \qquad (5)$$

As in [4] we define the following assertion H_k:

H_k: There is a dismantled curve $Y \subset \mathbf{P}^4$ such that $\deg(Y) = g(k)+4$, $p_a(Y) = g(k) - f(k)$, $\rho_{Y,k,4}$ is bijective and there is an ordering $A_1, \dots, A_{g(k)+4}$ of the lines of Y such that $A_1 \cup \dots \cup A_{g(k)+4-f(4)}$ is a dismantled curve of degree $g(k)+4-f(k)$ and arithmetic genus $g(k)-f(k)$ spanning $\mathbf{P}^4$, while the lines A_j, $g(k)+5-f(k) \le j \le g(k)+4$, are disjoint and each of them intersects $A_1 \cup \dots \cup A_{g(k)+4-f(4)}$ at exactly one point.

The assertion H_k makes sense because $g(k) \ge f(k)$ (see [2], Lemma 2, or [4], 2.6 and 4.3).

Proposition 3. *For every integer $k \ge 1$ the assertion H_k is true.*

Proof. H_1 is true because a linearly normal degree 4 bamboo of $\mathbf{P}^4$ is arithmetically Cohen - Macaulay. The proof of H_2 in [2], end of §1, does not give a semi-dismantled curve and its most obvious modification does not give a dismantled curve. We start with a linearly normal bamboo $A_1 \cup A_2 \cup A_3 \cup A_4 \subset \mathbf{P}^4$. Let $H \subset \mathbf{P}^4$ be a hyperplane intersecting transversally $A_1 \cup A_2 \cup A_3 \cup A_4$. Set $P_i := A_i \cap H$, $1 \le i \le 4$. We will add six lines A_j, $5 \le j \le 10$, contained in H. Let A_5 be the line spanned by P_1 and P_3, A_6 a general line containing P_2 and intersecting A_5, A_7 a general line containing P_4 and intersecting A_6, A_8 a general line intersecting A_5 and A_7, A_9 a general line intersecting A_6 and A_8 and A_{10} a general line intersecting A_7 and A_9. The curve $A_5 \cup A_6 \cup A_7 \cup A_8 \cup A_9$ is contained in a unique quadric surface of H and hence, $A_5 \cup A_6 \cup A_7 \cup A_8 \cup A_9 \cup A_{10}$ is contained in no quadric surface of H. By Horace lemma 6 $A_1 \cup \dots \cup A_{10}$ is not contained in a quadric hypersurface and hence it gives a solution of H_2.

Now we fix an integer $k \ge 3$ and assume that H_{k-1} is true. Take a solution Y of H_{k-1} and a general hyperplane H of $\mathbf{P}^4$. In particular we assume $\text{card}(Y \cap H) = \deg(Y)$. By [2], Lemma 3, we have $g(k) - g(k-1) \le a(k,3)$. Thus there is a tree $Z \subset H$ with $\deg(Z) = g(k) - g(k-1)$ and $\rho_{Z,k,3}$ surjective (Proposition 1).

First Claim. *If $f(k-1) \ge f(k)$ there is a tree Z as above such that $\text{card}(Z \cap Y) = \deg(Z) + f(k-1) - f(k)$, $Y \cup Z$ is a dismantled curve and for* *a fixed ordering of the lines of Y compatible with the statement of H_{k-1} the curve E does not intersect the last $f(k)$ lines of Y.*

Proof of the First Claim. Take an ordering of the lines A_i, $1 \leq i \leq g(k-1)+4$ of Y as in the statement of H_{k-1}. Set $P_j := A_j \cap H$. Take as $A_{g(k-1)+5}$ a line spanned by two points P_i and P_j belonging to lines A_i and A_j with $i < j \leq g(k-1)+4-f(k-1)$ and $A_i \cap A_j = \emptyset$. Suppose to have defined the lines A_x for all integers x with $g(k-1)+5 \leq x \leq y < g(k)+4$. We take as line A_{y+1} a general line intersecting A_y and containing one of the points P_j with $j \leq g(k-1)+4-f(k-1)$ and $P_j \notin A_x$ for all integers x with $g(k-1)+5 \leq x \leq y$. In this way we may obtain a tree Z with $\deg(Z) = g(k) - g(k-1)$ because $g(k) - g(k-1) < g(k-1) - f(k-1)$ (e.g. by Lemma 15 and Remark 17 or by [2], Lemma 3). Notice that we may find Z passing through $f(k-1) - f(k)$ general points of H. Hence, without loosing generality we may find a pair (Y_1, Z) with Y_1 isotrivial deformation of Y, say $Y_1 = B_1 \cup \cdots \cup B_{g(k-1)+4}$ with B_j corresponding to A_j, $\rho_{Y_1,k-2,4}$ bijective, such that $B_i \cap Z \neq \emptyset$ for every integer i with $g(k-1)+3-f(k-1) \leq i \leq g(k-1)+3-f(k)$. By construction $Y_1 \cap Z$ is a dismantled curve with degree $g(k)+4$ and arithmetic genus $g(k)-f(k)$.

Second Claim. *If $f(k-1) < f(k)$ there is a tree Z as above such that card$(Z \cap Y) =$ deg$(Z) - f(k-1) + f(k)$ and $Y \cup Z$ is a dismantled curve which has an ordering compatible with the statement of H_k in which exactly $f(k-1)$ of the last $f(k)$ lines of $Y \cup Z$ are the last lines of Y.*

Proof of the Second Claim. First assume $f(k) - f(k-1) \leq a(k,3) - a(k-2,3) - 1$ and $k \geq 6$ or $k = 4$. By [2], Lemma 3 we have $g(k) - g(k-1) < a(k,3)$. Take a tree $T = E \cup D \cup F$ satisfying the thesis of Lemma 8 and $Y \subseteq T$ a degree $g(k) - g(k-1)$ tree containing D and at least $f(k) - f(k-1)$ of the lines of F. The existence of such a tree Y follows from the inequalities $g(k) - g(k-1) \geq 2k-3 \geq k-1 \geq 1 + f(k-1)$, the first one being [2], Lemma 2, while the second one following from the inequality in (3). Copy the proof of the First Claim. Now assume $f(k) - f(k-1) \geq a(k,3) - a(k-2,3)$ and $k \geq 7$. By Lemma 15 we have $g(k) - g(k-1) \leq a(k-2,3) - 2$. We apply Lemma 9 instead of Lemma 8 and conclude as in the previous case. For low values of k use the values of $g(x)$, $x \leq 15$, given in Remark 17 and the value of $a(k-2,3)$ given in Remark 16.

Set $X := Y \cup Z$. We have $\deg(X) = g(k)+4$ and $p_a(X) = g(k) - f(k)$. By the First or the Second Claim X is a dismantled curve. Hence, it is sufficient to prove that we may find curves Y, Z as above such that $\rho_{X,k,4}$ is bijective. By Horace lemma 6 it is sufficient to prove that we may find curves Y, Z as above with $\rho_{Z\cup(Y\cap H),k,3}$ surjective. By Lemmas 1 and 2, Corollaries 1 and 2 and Remark 5 this is true because there is such a curve Z with $\rho_{Z,k,3}$ surjective (Proposition 1). □

Proofs of Theorems 2 and 3. We will write down only the proof of Theorem 2 because the same proof works for Theorem 3, too. Since the case $k = 1$ is trivial, we may assume $k \geq 2$. Thus $h^2(\mathbf{P}^4, \mathcal{I}_Y(k-1)) = 0$ for any dismantled curve Y with critical value k (Remark 2); by Castelnuovo-Mumford's lemma it is sufficient to prove the existence of a curve $Y \in B(d, 4, \alpha, \beta)$ such that $p_a(Y) = g$ and $h^1(\mathbf{P}^4, \mathcal{I}_Y(k)) = 0$. We try to follow as much as possible [2] and [4]. Let s be the maximal integer such that $g(s) \leq g$. For all integers $x > s$ consider the following assertion $R(x)$:

$R(x)$, $x > s$: there exists a triple (X, Z, T) such that:

1) $X = Z \cup T$, $Z \cap T = \emptyset$ and $\rho_{X,k,4}$ is bijective;

2) Z is a dismantled curve of degree $r(k, g, 4) - q(k, g, 4)$ and arithmetic genus g;

3) T is the disjoint union of $q(k, g, 4)$ lines.

If $g - g(s) - q(s+1, g, 4) \geq 0$ we also need the following assertion $R'(s+1)$:

$R'(s+1)$, if $g - q(s+1, g, 4) \geq 0$: there exist a dismantled curve Y of degree $r(s+1, g, 4)$ and arithmetic genus $g - q(s+1, g, 4)$ such that $\rho_{Y,s+1.4}$ is bijective.

As in [4], Lemmas 3.2 and 3.3, from H_s (proved in Proposition 3) we obtain $R'(s+1)$ if $g - q(s+1, g, 4) \geq 0$ and $R(s+1)$ otherwise. Then as in [4], Lemmas 3.5 and 3.6, we obtain $R(x)$ for all integers $x \geq s+2$. Then as in [4], §5, we obtain the theorem.

Theorem 6. *Fix integers d, g with $d \geq g + 4 \geq 4$ and let k be the critical value of the triple $(d, g, 4)$. Then there is a semi-dismantled curve $Y \in S(d, n, \alpha, \beta)$ (for some α and β) such that $p_a(Y) = g$ and $h^1(\mathbf{P}^4, \mathcal{I}_Y(t)) = 0$ for every $t \geq 4$.*

Proof. Since the case $k = 1$ is trivial, we may assume $k \geq 2$. Thus $h^2(\mathbf{P}^4, \mathcal{I}_Y(k)) = 0$ for every semi-dismantled curve Y with critical value k (Remark 8). By Castelnuovo-Mumford's lemma it is sufficient to prove the existence of $Y \in S(d, n, \alpha, \beta)$ such that $p_a(Y) = g$ and $h^1(\mathbf{P}^4, \mathcal{I}_Y(k)) = 0$.

Change in a straightforward way the assertion H_k to an assertion H_k^* for semi-dismantled curves; for instance for H_1^* take as solution any rational normal curve of $\mathbf{P}^4$ (it is arithmetically Cohen - Macaulay). In the proof of H_2 given in the proof of Proposition 3 to obtain a proof of H_2^* take a rational normal curve of $\mathbf{P}^4$ instead of $A_1 \cup A_2 \cup A_3 \cup A_4$. Then the proof of Proposition 3 works verbatim, just quoting H_s^* instead of H_s. □

The same proof gives the following result.

Theorem 7. *Fix integers d, g with $d \geq g+4 \geq 4$ and let k be the critical value of the triple $(d,g,4)$. Then there is a semi-dismantled curve $Y \in S(d,n,\alpha,\beta)$ (for some α and β) such that $p_a(Y) = g$ and $h^0(\mathbf{P}^4, \mathcal{I}_Y(k-1)) = 0$.*

4. Proof of Theorem 1

For all positive integers $N, s, d_1, \dots, d_s$ let $B(N,s,d_1,\dots,d_s)$ be the set of all reduced curves $T \subset \mathbf{P}^N$ which are disjoint union of s bamboos of degree $d_1, \dots, d_s$. The set $B(N,s,d_1,\dots,d_s)$ is an irreducible locally closed subset of the Hilbert scheme of $\mathbf{P}^N$.

Remark 12. Let $H \subset \mathbf{P}^N$ be a hyperplane. Fix $S \subset H$ with $\mathrm{card}(S) = d$ and a type τ for degree d trees. Then there exists $T \in T(N,d,\tau)$ such that $H \cap T = S$.

Remark 13. Let $T \subset \mathbf{P}^n$ be a tree of degree $d \geq N$. Let k be the first positive integer such that $r(k,0,n) \geq d$. The integer k is the critical value of the triple $(d,0,n)$. If $k=1$ the tree T has maximal rank if and only if it spans $\mathbf{P}^n$. If $k \geq 2$ by Castelnuovo-Mumford's lemma the tree T has maximal rank if and only if $h^1(\mathbf{P}^n, \mathcal{I}_Y(k)) = 0$ and $h^0(\mathbf{P}^n, \mathcal{I}_T(k-1)) = 0$.

Concerning reducible space curves with arithmetic genus zero we only know very weak forms of Theorem 1. Proposition 4 is one of them. The interested reader may obtain several other similar results (for instance fixing the type τ of the tree) using its proof and the proof of Proposition 1 and Lemmas 8 and 9.

Proposition 4. *Fix integers k, d with $k \geq 2$ and $r(k-1,0,3)+[(k+1)/2] \leq d \leq r(k,0,3) - [(k+1)/2]$. Then there exists a degree d tree $T \subset \mathbf{P}^3$ with maximal rank.*

Proof. We use induction on the integer k, the case $k \le 3$ being easy. Assume $k \ge 4$ and that the result is true for all integers $k' \le k-1$. Fix an integer d such that $r(k-1,0,3) + [(k+1)/2] \le d \le r(k,0,3) - [(k+1)/2]$ and a smooth quadric surface $Q \subset \mathbf{P}^3$. By the inductive assumption for the integer $k' = k-2$ there is a tree $A \subset \mathbf{P}^3$ such that $\deg(A) = d - r(k,0,3) + r(k-2,0,3) + 1$, $h^1(\mathbf{P}^3, \mathcal{I}_A(k-2)) = 0$ and $h^0(\mathbf{P}^3, \mathcal{I}_A(k-3)) = 0$. Let τ be the type of A. By semicontinuity we have $h^1(\mathbf{P}^3, \mathcal{I}_B(k-2)) = 0$ and $h^0(\mathbf{P}^3, \mathcal{I}_B(k-3)) = 0$ for a general $B \in B(3, \deg(A), \tau)$. We have $h^1(Q, \mathcal{I}_{B\cup Q}(k, k - r(k,0,3) + r(k-2,0,3)+1)) = 0$; here we use Lemmas 4 and 5 and the inequality $\deg(A) \le (k+1)(k - r(k,0,3) + r(k-2,0,3)+2)/2 - 2$. Let $D \subset Q$ be the union of $r(k,0,3) - r(k-2,0,3) - 1$ lines of type $(0,1)$ of Q, each of them intersecting a prescribed connected component of B. Hence $B \cup D$ is a degree d tree. Call ψ a type of $B \cup D$. Using Horace lemma 6 as in [1] we obtain $h^1(\mathbf{P}^3, \mathcal{I}_{B\cup D}(k-2)) = 0$ and $h^0(\mathbf{P}^3, \mathcal{I}_{B\cup D}(k-3)) = 0$. By semicontinuity we have $h^1(\mathbf{P}^3, \mathcal{I}_T(k)) = 0$ and $h^0(\mathbf{P}^3, \mathcal{I}_T(k-1)) = 0$ for a general $T \in T(3,d,\psi)$. By Remark 13 the general tree $T \in T(3,d,\psi)$ has maximal rank. □

Consider the following assertion $A(k,N)$, $k \ge 1$, $N \ge 4$:

$A(k,N)$, $k \ge 1$, $N \ge 4$: there is a disjoint union $Y \subset \mathbf{P}^N$ of $q(k,0,N)+1$ bamboos such that $\deg(Y) = r(k,0,N)$ and the restriction map $\rho_{Y,k,N}$ is bijective.

The assertion $A(k,N)$ is well-defined because

$$r(k,0,N) \ge q(k,0,N) + 1 \text{ for all integers } k \ge 1 \text{ and } N \ge 4,$$

for instance use that $q(k,0,N) \le k-1$ and that $k^2 + k - 1 \le \binom{N+k}{N}$, the latter inequality implying $r(k,0,N) \ge k$; alternatively, see [4], Lemma 4.1.

Lemma 10. *Fix an integer $k \ge 3$. Then there exists an integer t with $\max\{1, q(k-1,0,4)\} \le t \le 2 + q(k-1,0,4)$ and a reduced curve $T \subset \mathbf{P}^3$ such that T is the union of t disjoint bamboos, deg$(T) = r(k,0,4) - r(k-1,0,4)$ and $\rho_{T,k,3}$ is surjective.*

Proof. Let $Q \subset \mathbf{P}^3$ be a smooth quadric surface. It is easy to check that $r(k,0,4) - r(k-1,0,4) \ge r(k-1,0,4) - r(k-2,0,4)$ (see Remark 15); however, if this inequality were false for the integer k we are interested in, then the proof would be easier and shorter. Assume for the moment the existence of a curve $E \subset \mathbf{P}^3$ such that $\deg(E) = r(k-1,0,4) - r(k-2,0,4)$, E is a union of x disjoint bamboos and $\rho_{E,k-2,4}$ is surjective. Fix an integer y such that $0 \le y \le \min\{r(k,0,4) + r(k-2,0,4) - 2r(k-1,0,4), 2x\}$. We may deform E so that E intersects transversally Q, no two points of $E \cap Q$ are on the same line of type $(0,1)$ on Q and we may apply Lemmas 4 and 5 to control the postulation of suitable subsets of $E \cap Q$; we only need the existence of a subset of $E \cap Q$ with cardinality $2(\deg(E)) - y$ and with good postulation. Let $F \subset Q$ be the union of $r(k,0,4) + r(k-2,0,4) - 2r(k-1,0,4)$ different lines of type $(0,1)$, exactly y of them intersecting E. We also impose that each point of $E \cap F$ is contained in an initial line of E or in a final line of E and that no initial or final line of a connected component of degree at least two of E intersects two lines of F. To satisfy these conditions we use the inequality $y \leq 2x$. The curve $E \cup F$ has degree $r(k,0,4) - r(k-1,0,4)$ and $x + r(k,0,4) + r(k-2,0.4) - 2r(k-1,0,4) - y$ connected components, each of them being a bamboo. Subtract the inequality in (3) for the integers $k' = k-2$ and $N = 3$ from the same inequality for the integers k and $N = 3$. Using Remark 16 we obtain

$$2(\deg(E)) \leq (k+1)(k+1-\deg(F))$$

and hence

$$2(\deg(E)) - y \leq (k+1)(k+1-\deg(F)).$$

Thus by Lemmas 4 and 5 for a general such $E \cup F$ we have $h^1(Q, \mathcal{I}_{E\cap(Q\setminus F)}(k - \deg(F), k)) = 0$. Thus by Horace lemma 7 the map $\rho_{E,k-2,4}$ is surjective. Every connected component of $E \cup F$ is a bamboo and we may obtain as number of the connected components of $E \cup F$ an arbitrary integer $w = x + r(k,0,4) + r(k-2,0,4) - 2r(k-1,4) - y$ such that $x \leq w \leq \min\{x + r(k,0,4) + r(k-2,0,4) - 2r(k-1,0,4), 3x\}$. For k odd we start this construction from a linearly normal degree 3 bamboo L: the map $\rho_{L,1,3}$ is bijective. For k even we start from the same degree 3 bamboo: the map $\rho_{L,2,3}$ is surjective. For large k, say $k \geq 10$, for any positive integer $t \leq k+1$ it is easy to obtain T as above with t connected components taking at each inductive step y minimal until we find a pair (x', k') as above such that $x' + r(k',0,4) + r(k'-2,0,4) - 2r(k'-1,0,4) \geq t$, while $t \leq 2 + q(k'-1,0,4) \leq k'$. For small k, say $k \leq 15$, use the values of the integer $r(x,0,4)$ and $q(x,0,4)$ given in Remark 15 for $x \leq 15$. □

Lemma 11. *For all integers k, n with $k \geq 1$ and $N \geq 4$ the assertion $A(k,N)$ is true.*

Proof. The case $k = 1$ is obviously true. Hence, we will assume $k \geq 2$ and that $A(k-1,N)$ is true.

Step 1. The general construction which we will introduce in this step works for any integer $N \geq 4$. However, in this step we will prove two claims only if $N \geq 5$. Take a disjoint union $Z \subset \mathbf{P}^N$ of $q(k-1,0,N)+1$ bamboos with $\deg(Z) = r(k-1,0,N)$ and Z satisfying $A(k-1,N)$, i.e. such that $\rho_{Z,k-1,N}$ is bijective. Fix an ordering of the connected components $E_1, \dots, E_{q(k-1,0,N)}$ of Z and set

$$e_i := \deg(E_i), \qquad 1 \le i \le 1 + q(k-1,0,N).$$

By semicontinuity the restriction map $\rho_{W,k-1,N}$ is bijective for a general $W \in B(N, 1+q(k-1,0,N); e_1, \dots, e_{1+q(k-1,0,N)})$. Fix a hyperplane H of $\mathbf{P}^N$. For a general $W \in B(N, 1+q(k-1,0,N); e_1, \dots, e_{1+q(k-1,0,N)})$ the set $W \cap H$ may be considered as a general subset of H. First assume $q(k,0,N) \ge q(k-1,0,N)$ Call C_i, $1 \le i \le 1+q(k-1,0,N)$, the connected components of W with $\deg(C_i) = e_i$. Fix positive integers a_i, $1 \le i \le 1+q(k,0,N)-q(k-1,0,N)$, such that $a_1 + \dots + a_{1+q(k,0,N)-q(k-1,0,N)} = r(k,0,N) - r(k-1,0,N)$; there is some choice of these integers a_i because $r(k,0,N) - r(k-1,0,N) \ge k$ (Lemma 16). Consider the set Γ of all curves $T \subset H$ which are disjoint unions of $1+q(k,0,N)-q(k-1,0,N)$ disjoint bamboos, say B_i, $1 \le i \le 1+q(k,0,N)-q(k-1,0,N)$, with $\deg(B_i) = a_i$ for every i, $B_j \cap W = \emptyset$ if $j \ge 2$, $B_1 \cap C_i = \emptyset$ if $i \ge 2$, $\mathrm{card}(B_1 \cap C_1) = 1$, $B_1 \cup C_1$ with only nodes as singularities and $B_1 \cap C_1$ belonging to the first line of B_1 and the last line of C_1. The algebraic set Γ is an irreducible subset of the Hilbert scheme of H. Notice that for any $T \in \Gamma$ the curve $W \cup T$ is the disjoint union of $1+q(k,0,N)$ bamboos.

First Claim. *If $N \ge 5$ and $A(k, N-1)$ is true, then for a general $T \in \Gamma$ the map $\rho_{T,k,N-1}$ is surjective.*

Proof of the First Claim. We have $q(k,0,N) - q(k-1,0,N) - 1 \le r(k,0,N-1) - \deg(T)$ (Lemma 16). Hence we may obtain T throwing away $r(k,0,N-1) - \deg(T)$ suitable lines from a degree $r(k,0,N-1)$ bamboo L satisfying $A(k,N-1)$, i.e. such that $\rho_{L,k,N-1}$ is surjective. Hence $\rho_{T,k,N-1}$ is surjective; here we use the surjectivity of the restriction map $H^0(L, \mathcal{O}_L(k)) \to H^0(T, \mathcal{O}_T(k))$ which is proved as in the last part of the proof of Proposition 1.

Since $W \cap H$ is a general subset of H and $h^0(H, \mathcal{O}_H(k)) = \binom{N+k-1}{k}$, the map $\rho_{T \cup (W \cap H),k,N-1}$ is bijective. Hence, by Horace lemma 6 the map $\rho_{W \cup T,k,N}$ is bijective, i.e. $W \cup T$ satisfies $A(k,N)$. Now assume $q(k,0,N) < q(k-1,0,N)$. Fix positive integers b_i, $1 \le i \le q(k-1,0,N) - q(k,0,N)$, such that $b_1 + \dots + b_{q(k-1,0,N)-q(k,0,N)} = r(k,0,N) - r(k-1,0,N)$. This is possible by the easy inequality $r(k,0,N) - r(k-1,0,N) \ge q(k-1,0,N) - q(k,0,N)$, which is true by Lemma 16 because $q(k,0,N) \ge 0$, $q(k-1,0,N) \le k-2$ and $N \ge 4$. Consider the set Φ of all unions $T \subset H$ of $q(k-1,0,N) - q(k,0,N)$ disjoint bamboos, say D_i, $1 \le i \le q(k-1,0,N) - q(k,0,N)$, such that $\deg(D_i) = b_i$, D_i intersects the last line of C_{2i-1} and the first line of C_{2i} for every i, and $\mathrm{card}(T \cap W) = 2(q(k-1,0,N) - q(k,0,N))$. Notice that for every $T \in \Phi$ the curve $W \cup T$ is the union of $1 + q(k, 0, N)$ disjoint bamboos.

Second Claim. *If $N \geq 5$ and $A(k, N-1)$ is true, then for a general $T \in \Phi$ the map $\rho_{T,k,N-1}$ is surjective.*

The proof of the Second Claim. The proof is similar to the one just given for the First Claim, and hence it is omitted. Since $W \cap H$ is a general subset of H, the map $\rho_{T \cup (W \cap H),k,N-1}$ is bijective. Hence by Horace lemma 6 the map $\rho_{W \cup T.k,N}$ is bijective, i.e. $W \cup T$ satisfies $A(k, 0, N)$.

Step 2. By the First and the Second Claim in Step 1 to prove the lemma by induction on N it is sufficient to prove the surjectivity of the map $\rho_{T,k,3}$ when $N = 4$. Just use Lemma 10. □

Proof of Theorem 1. Fix an integer $d \geq n$ and let k be the critical value of the triple $(d, 0, n)$. Since the case $d = n$, is obvious (any linearly normal tree is arithmetically Cohen - Macaulay), we assume $d > n$, i.e. $k \geq 2$. Thus k is the only integer such that $r(k-1, 0, n) < d \leq r(k, 0, n)$. By the semicontinuity theorem for the cohonology groups of fibers of flat families of sheaves and the irreducibility of the algebraic set of all bamboos of degree d, it is sufficient to finds two bamboos A, B such that $\rho_{B,k,n}$ is surjective and $\rho_{A,k-1,n}$ is injective. First assume $d \geq r(k-1, 0, n) + q(k-1, 0, n)$. Let $T \subset \mathbf{P}^n$ be the disjoint union of $1 + q(k-1, 0, n)$ bamboos satisfying $A(k-1, n)$. Thus $\deg(T) = r(k-1, 0, n)$ and $\rho_{T,k-1,n}$ is bijective. Call C_i, $1 \leq i \leq 1 + q(k-1, 0, n)$, the connected components of T. We claim the existence of a curve $E \subset \mathbf{P}^n$ such that E is the disjoint union of $\max\{1, q(k-1, 0, n)\}$ bamboos, say R_j, $1 \leq j \leq \max\{1, q(k-1, 0, n)\}$, $\deg(E) = d - r(k-1, 0, n)$ and $T \cup E$ is a degree d bamboo. If $q(k-1, 0, n) = 0$ to check the claim it is sufficient to take as E the general bamboo, whose initial line (for some ordering) intersects the final line of T. If $q(k-1, 0, n) > 0$ and $d \geq r(k-1, 0, n) + q(k-1, 0, n) - 1$, to check the claim it is sufficient to add $q(k-1, 0, n) - 1$ general bamboos R_j, $1 \leq j \leq q(k-1, 0, n)$, such that R_j intersects the final line of C_j and the initial line of C_{j+1}. Set $A := T \cup E$. By construction A is a degree d bamboo. Since $\rho_{T,k-1,n}$ is injective and $T \subset A$, $\rho_{A,k-1,n}$ is injective. Now assume $r(k-1, 0, n) < d < r(k-1, 0, n) + q(k-1, 0, n) - 1$. We will also assume $k \geq 3$ because the case $k = 2$ is trivial since $q(1, 0, n) \leq 1$. Then as in Proposition 11 (or Prosition 10 if $n = 4$) we apply Horace lemma 6 with respect to the hyperplane H and find a curve $C \subset H$ such that $\deg(C) = d - r(k-1, 0, n) + 1$, $D \cup C$ is a bamboo and $\rho_{D \cup C,k-1,n}$ is injective; here we only need the inequality $\deg(C) \geq q(k-2, 0, n)$ which is trivial by Lemma 14. If $d = r(k-1, 0, n+1)$, set $A := D \cup C$. If $d \geq r(k-1, 0, n) + 2$, then take as A a bamboo union of $D \cup C$ and any bamboo

V of degree $d - r(k-1,0,n) - 1$ such that $\mathrm{card}(V \cap (D \cup C)) = 1$ and the initial line of V intersects the final line of $D \cup C$. The proof of the existence of a degree d bamboo B such that $\rho_{B,k,n}$ is surjective is very similar and left to the reader; now the easy part is when $r(k-1,0,n) < d \le r(k,0,n) - q(k,0,n) + 1$ and the difficult part is when $r(k,0,n) - q(k,0,n) + 1 < d \le r(k,0,n)$.

5. Other Reducible Curves

Definition 7. Let $T \subset \mathbf{P}^n$, $n \ge 3$, be a reduced and connected curve with only nodes as singularities. Set $d := \deg(T)$ and $g := p_a(T)$. We will say that T is a C-curve of degree d and genus g if it has $d - n + 1$ irreducible components, one of its irreducible component, say A_0, is a rational normal curve of $\mathbf{P}^n$, the other $d - n$ irreducible components of T are lines, each of these lines intersects A_0 at exactly one point and there is an ordering $A_1, \dots, A_{d-n}$ of these lines such that $A_i \cap A_j \ne \emptyset$ for some pair (i,j) with $j < i$ if and only if there is an integer s with $1 \le s \le g$ such that $i = 2s$ and $j = 2s - 1$.

Remark 14. $T \subset \mathbf{P}^n$ be a C-curve of degree d and genus g. Notice that $2g \le d - n$. Conversely, for all triples (n,d,g) with $n \ge 3$ and $0 \le 2g \le d - n$ there is a C-curve in $\mathbf{P}^n$ with degree d and genus g.

Let $C(n,d,g)$ be the set of all C-curves of degree d and genus g in $\mathbf{P}^n$. For all triples (n,d,g) with $n \ge 3$ and $0 \le 2g \le d - n$ the set $C(n,d,g)$ is an irreducible locally closed subscheme of the Hilbert scheme of $\mathbf{P}^n$.

Lemma 7 immediately gives the following result.

Proposition 5. *Fix integers n, d, g with $n \ge 3$ and $0 \le 2g \le d - n$ and a hyperplane H of $\mathbf{P}^n$. For a general $Y \in C(n,d,g)$ the set $Y \cap H$ has maximal rank.*

6. Numerical lemmas

In this section we collect the numerical lemmas that we used in the body of the paper.

Lemma 12. *We have $r(k,0,N) \ge k^2 - k + 1$ if $N \ge 6$ and $k \ge 5$, or $N = 5$ and $k \ge 6$, or $N = 4$ and $k \ge 9$.*

Proof. Set $F(k,N) = \binom{N+k}{N} - k^3 + 2k^2 - 3k$. Since $q(k,0,N) \le k - 1$, we have $r(k,0,N) \geq \binom{N+k}{N}/k-1$. Hence it is sufficient to check when $F(k,N) \geq 0$. For $N \geq 4$ and $k \geq 3$ the function $F(k,N)$ is an increasing function of N. We have $F(k,4) \geq 0$ for $k \geq 9$, $F(k,5) \geq 0$ for $k \geq 6$, $F(k,6) \geq 0$ for $k \geq 3$ and hence $F(k,N) \geq 0$ for $k \geq 3$ and $N \geq 7$. □

Lemma 13. *We have $r(k,0,N) - r(k-1,0,N) \leq r(k,0,N-1) - k + 1$ if $N \geq 6$ and $k \geq 3$, or $N = 5$ and $k \geq 6$, or $N = 4$ and $k \geq 9$.*

Proof. Subtracting (2) for the triple $(k-1,0,N)$ from (2) for the triple $(k,0,N)$ we obtain $(k-1)(r(k,0,N) - r(k-1,0,N)) + r(k,0,N) + q(k,0,N) - q(k-1,0,N) = \binom{N+k-1}{N-1}$ Hence by (2) for the triple $(k,0,N-1)$ it is sufficient to have $r(k,0,N) \geq k + (k-1)^2$. By Lemma 12 this is true if $N \geq 6$ and $k \geq 3$ or $N = 5$ and $k \geq 6$ or $N = 4$ and $k \geq 9$. □

Remark 15. We have $r(2,0,4) = 7$, $q(2,0,4) = 0$, $r(3,0,4) = 11$, $q(3,0,4) = 1$, $r(4,0,4) = 17$, $q(4,0,4) = 1$, $r(5,0,4) = 25$, $q(5,0,4) = 0$, $r(6,0,4) = 34$, $q(6,0,4) = 5$, $r(7,0,4) = 48$, $q(7,0,4) = 3$, $r(8,0,4) = 61$, $q(8,0,4) = 6$, $r(9,0,4) = 79$, $q(9,0,4) = 3$, $r(10,0,4) = 100$, $q(10,0,4) = 0$, $r(11,0,4) = 124$, $q(11,0,4) = 0$, $r(12,0,4) = 151$, $q(12,0,4) = 7$, $r(13,0,4) = 183$, $q(13,0,4) = 0$, $r(14,0,4) = 218$, $q(14,0,4) = 7$, $r(15,0,4) = 258$, $q(15,0,4) = 5$, $r(2,0,5) = 10$, $q(2,0,5) = 0$, $r(3,0,5) = 18$, $q(3,0,5) = 1$, $r(4,0,5) = 31$, $q(4,0,5) = 1$, $r(5,0,5) = 50$, $q(5,0,5) = 1$, $q(2,0,n) = 0$ if $n \equiv 2,3 \pmod 4$ and $q(2,0,n) = 1$ if $n \equiv 0,1 \pmod 4$.

Lemma 14. *For all integers $k \geq 3$ we have $r(k,0,4) - r(k-1,0,4) \geq k-1$ and $a(k,4) - a(k-1,4) \geq k-1$.*

Proof. See the proof of [2], Lemma 2. □

Remark 16. By [6] or [2], top of p. 217, we have $a(k,3) = (k+3)(k+2)/6$ and $b(k,3) = 0$ if $k \equiv 0,1 \pmod 3$, $a(k,3) = (k+4)(k+1)/6$ and $b(k,3) = (k+1)/3$ if $k \equiv 2 \pmod 3$. As in Lemma 13 we have $r(k,0,4) - r(k-1,0,4) \leq a(k,3)$ if $k \geq 15$. For $3 \leq k \leq 14$ the same inequality is true by the explicit values of $r(k,0,4)$, $k \leq 15$, given in Remark 15.

Lemma 15. *For all integers $k \geq 7$ we have $g(k) - g(k-1) \leq a(k,3) - 2$.*

Proof. Assume $g(k) - g(k-1) \geq a(k,3) - 1$. Subtracting the equality in (2) for the integer $k-1$ from the same equality for the integer k we obtain

$$(k-1)(g(k) - g(k-1)) + g(k) + f(k) - f(k-1) = \binom{k+3}{3}. \qquad (6)$$

Using the inequality $(k-2)g(k-1) \le \binom{k+3}{4}$ which comes from (2) for the integer $k-1$ and the definitions of the integers $a(k,3)$ and $b(k,3)$ given in (3) and the inequality $b(k) \le (k+1)/3$ (see Remark 16) we obtain $\binom{k+3}{3} - 2a(k,3) - b(k,3) + 8 + g(k-1) \le \binom{k+3}{3}$, $(k-2)g(k-1) + 3g(k-1) \le 2(k+1)a(k,3) + 8(k+1) + (k+1)b(k,3) \le 2\binom{k+3}{3} + 3k^2/2$ and hence $\binom{k+3}{4} - (k-1)k - k + 4 + 3g(k-1) \le \binom{k+3}{3} + 3k^2/2$, which is certainly false for, say, $k \ge 11$. For $7 \le k \le 15$ use the explicit values of the integers $a(k,3)$ and of the integers $g(x)$ for $x \le 15$ given in the next remark. □

Remark 17. We have $g(2) = 6$, $f(2) = 0$, $g(3) = 11$, $f(3) = 0$, $g(4) = 17$, $f(4) = 2$, $g(5) = 26$, $f(5) = 1$, $g(6) = 37$, $f(6) = 0$, $g(7) = 50$, $f(7) = 1$, $g(8) = 46$, $f(8) = 5$, $g(9) = 86$, $f(9) = 4$, $g(10) = 106$, $f(10) = 6$, $g(11) = 132$, $f(11) = 0$, $g(12) = 161$, $f(12) = 0$, $g(13) = 193$, $f(13) = 11$, $g(14) = 231$, $f(14) = 0$, $g(15) = 272$, $f(15) = 7$.

Lemma 16. ([2], Lemma 5, and [4], Lemma 4.4) *For all integers n, k with $n \ge 4$ and $k \ge 2$ we have $r(k,0,n) - r(k-1,0,n) \ge k+1$.*

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