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CLOSED ORBITS FOR ACTIONS OF INFINITE-DIMENSIONAL LINEAR GROUPS

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Abstract: Let V be an infinite-dimensional locally convex topological vector space. For any $d \geq 2$ the projective linear group $PGL(V)$ acts on the set of all degree d hypersurfaces of $\mathbf{P}(V)$. Here we determine the unique closed orbit for this action and the good orbits whose closure is the union of two orbits.

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1. The Statements

Let $\mathbf{K}$ be either the field of complex numbers or the field of real numbers. We equip $\mathbf{K}$ with the usual Euclidean topology. In this paper we will say vector space instead of $\mathbf{K}$-vector space. If V is any infinite-dimensional topological vector space, then $\mathbf{P}(V)$ is an infinite-dimensional $\mathbf{K}$-manifold. Let V be an infinite-dimensional $\mathbf{K}$-vector space equipped with a Hausdorff locally convex vector space topology τ. For any integer $d \geq 1$, let $P^d(V)$ be the set of all continuos $\mathbf{K}$-valued homogeneous polynomials of degree d on V. Hence $P^1(V)$ is the topological dual of V. For every $f \in P^d(V)\backslash\{0\}$ such that no irreducible factor of f is a multiple factor, let $X(f) = \{f = 0\} \subset \mathbf{P}(V)$ be the associated

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closed degree d hypersurface. If $f \in P^d(V)\backslash\{0\}$ and $f = f_1^{a_1} \dots f_s^{a_s}$, $s \geq 1$, is a decomposition (unique up to non-zero multiplicative constants) of f into positive degree polynomials such that $a_i > 0$ for all i, each f_j is irreducible, f_i and f_j are not proportional if $i \neq j$, set $X(f) := a_1X(f_1) + \cdots + a_sX(f_s)$. Even in this case we will say that $X(f)$ is a closed degree d hypersurface of $\mathbf{P}(V)$, i.e. we will allow that a hypersurface has multiple components. With this convention we may identify $\mathbf{P}(P^d(V))$ with the set of all degree d hypersurfaces (even with multiple components) of $\mathbf{P}(V)$. The linear group $GL(V)$ acts on each vector space $P^d(V)$. The projective linear group $PGL(V) = GL(V)/\mathbf{K}^*Id$ acts on $\mathbf{P}(V)$ and on each $\mathbf{P}(P^d(V))$. One can put several natural vector space topologies on $P^d(V)$ (see [1] for the complex case), and hence on $\mathbf{P}(P^d(V))$. However, we would like to obtain a partial substitute of the theory of the Hilbert scheme of a finite-dimensional projective space, while if V is infinite-dimensional no such compactness result may be true in full generality: the Hilbert scheme parametrizing one point of $\mathbf{P}(V)$ is $\mathbf{P}(V)$ itself which is not compact. Hence we will introduce the following definition.

Definition 1. We will say that a net $\{X_\alpha\}_{\alpha \in I}$ in $\mathbf{P}(P^d(V))$ ψ-converges to the hypersurface $X \in \mathbf{P}(P^d(V))$ if for every finite-dimensional linear subspace M of $\mathbf{P}(V)$ either $M \subset X$ or M is not eventually contained in X_α, and the net $\{X_\alpha \cap M\}_{\alpha \in I}$ converges to the degree d hypersurface $X \cap M$ of M. The closure of a subset A of $\mathbf{P}(P^d(V))$ will be the union of all ψ-limits of nets of elements of A.

In the set-up of Definition 1 let M be a finite-dimensional linear subspace of $PGL(V)$ in $\mathbf{P}(P^d(V))$ is not contained in X. Even if X has no multiple component, the degree d hypersurface $X \cap M$ may have multiple components. Thus it seems essential to allow hypersurfaces with multiple components. In this paper we study the closure of orbits for the action of $PGL(V)$ in $\mathbf{P}(P^d(V))$ and prove the following results.

Theorem 1. *For any integer $d \geq 2$, the only closed orbit of $PGL(V)$ in $\mathbf{P}(P^d(V))$ is given by the sets $X(h^d)$ with $h \in V'$.*

Theorem 2. *Fix an integer $d \geq 2$ and a non-closed orbit A of $PGL(V)$ in $\mathbf{P}(P^d(V))$ such that the closure of A is exactly the union of A and of another orbit. Assume the existence of $X \in A$, say $X = \{f = 0\}$ and $Q \in \mathbf{P}(V)$ such that X is smooth at Q. Then there are $z, w \in V'$ such that $f = czw^{d-1}$ with $c \in \mathbf{K}$, $c \neq 0$, and z not proportional to w. The closure of A is the union of A and the unique closed orbit $PGL(V)Y$ with $Y = \{w^d = 0\}$.*

We also consider the action of $PGL(V)$ on the set of all r-dimensional linear subspaces of $P^d(V)$, generalizing the case $r = 1$ which corresponds to the action of $PGL(V)$ on $\mathbf{P}(P^d(V))$. Let M be a Hausdorff topological vector space such that all finite-dimensional vector subspaces are closed and supplemented. For instance, by Hahn - Banach, it is sufficient to assume that M is locally convex. For any positive integer r let $\mathrm{Grass}(r, M)$ be the Grassmannian of all r-dimensional linear subspaces of M. Hence $\mathrm{Grass}(1, M) = \mathbf{P}(V)$. For any $W \in \mathrm{Grass}(r, M)$ and any topological supplement U of W, the set of all $B \in \mathrm{Grass}(r, M)$ such that $B \cap U = \emptyset$ is an open chart for $\mathrm{Grass}(r, M)$ (see [2], §3, or [5], p. 89, for the case of a complex Banach space). With these charts the set $\mathrm{Grass}(r, M)$ is equipped with a structure of analytic manifold. More generally, for all positive integers s, $r_1, \dots, r_s$ with $1 \leq r_1 < \cdots < r_s$, let $\mathrm{Flag}(r_1, \dots, r_s, M)$ be the set of all flags of linear subspaces $M_1 \subset \cdots \subset M_s \subset M$ with $\dim(M_i) = r_i$ for all i. The projective linear group $PGL(V)$ acts on each Grassmannian $\mathrm{Grass}(r, P^d(V))$ and on each flag variety $\mathrm{Flag}(r_1, \dots, r_s, P^d(V))$. In Section 2 we will prove the following result.

Theorem 3. *Fix an integer $d \geq 2$ and a closed orbit A for the action of $PGL(V)$ on Flag$(1, 2, P^d(V))$, say $A = PGL(V)T$ with T represented by a non-zero degree d form f and by the set of all non-zero linear combinations of f and another degee d form g. Assume the existence of a hypersurface $Y = \{\lambda f + \mu g = 0\}$, such that Y has at least one smooth point. Then there are closed hyperplanes H, M of $\mathbf{P}(V)$, say $H = \{z = 0\}$ and $M = \{w = 0\}$ such that $X(g) = dH$ (i.e $g = cz^d$ for some constant $c \neq 0$) and $Y = (d-1)H \cup M$ (i.e. wz^{d-1} is an equation of Y).*

2. The Proofs

In this section we prove Theorems 1, 2, 3.

Example 1. $\mathbf{P}(P^1(V))$ is the set of all closed hyperplanes of $\mathbf{P}(V)$, i.e. $\mathbf{P}(V') = \mathbf{P}(P^1(V))$, where V' is the τ-dual of V. Fix $f, g \in V' \backslash \{0\}$. By Hahn - Banach, there are a topological isomorphism $\phi : \mathrm{Ker}(f) \oplus \mathbf{K} \to V$ and a topological isomorphism $\gamma : \mathrm{Ker}(g) \oplus \mathbf{K} \to V$. The composition of $\gamma^{-1} \circ (\phi | \mathrm{Ker}(f))$ with the projection $\mathrm{Ker}(g) \oplus \mathbf{K} \to \mathrm{Ker}(g)$ induces a continuos linear map $\beta : \mathrm{Ker}(f) \to \mathrm{Ker}(g)$ with closed image. If β is an isomorphism, we obtain that $\mathbf{P}(\mathrm{Ker}(f))$ and $\mathbf{P}(\mathrm{Ker}(g))$ are projectively equivalent. If β is not an isomorphism, we obtain the existence of a closed hyperplane M of $\mathrm{Ker}(g)$ such that $\mathrm{Ker}(g) \cong M \oplus \mathbf{K}$ and $\mathrm{Ker}(f) \cong M \oplus \mathbf{K}$. Hence, even in this case $\mathbf{P}(\mathrm{Ker}(f))$

and $\mathbf{P}(\mathrm{Ker}(g))$ are projectively equivalent. Thus $\mathbf{P}(P^1(V))$ is homogeneous for the action of $PGL(V)$.

Example 2. Fix an integer $d \geq 2$ and a closed hyperplane $H = \{f = 0\} \subset \mathbf{P}(V)$. Thus $dH = \{f^d = 0\}$ is a degree d hypersurface. By Example 1, the orbit of dH is the set of all dM with M any closed hyperplane of $\mathbf{P}(V)$. Notice that this orbit is closed.

Lemma 1. *Let $\mathbf{C}^n \to \mathbf{C}$ be a homogeneous degree d polynomial, $f \neq 0$, and $X = \{f = 0\} \subset \mathbf{P}^{n-1}$ the associated degree d hypersurface. Let $f = f_1^{a_1} \dots f_s^{a_s}$, $s \geq 1$, be a decomposition of f (unique up to non-zero multiplicative constants) in positive degree polynomials such that $a_1 > \dots > a_s$ and each f_j, $i \leq j \leq s$, has no multiple factor. Set $d_j := \deg(f_j)$. Then for a general line $D \subset \mathbf{P}^{n-1}$ the scheme $D \cap X$ is the union of d_1 points with multiplicities a_1, d_2 points with multiplicity a_2, ..., and d_s points with multiplicities a_s.*

Proof. Set $X_j := \{f_j = 0\}$. By a very elementary form of a classical theorem of Bertini for a general line $D \subset \mathbf{P}^{n-1}$ the scheme $X_j \cup D$ is formed by d_j points with multiplicity one, while $D \cap X_i \cap X_j = \emptyset$ if $i \neq j$. □

Proof of Theorem 1. Fix a closed hyperplane $H = \{w = 0\}$ of V. Since H has a closed supplement, for any $P \in H$ we have $H + \mathbf{K}P = V$ as topological vector spaces. For any $\lambda \in \mathbf{K}\backslash\{0\}$ and $x \in H$, set $g_\lambda(x, mP) = (x, (m/\lambda)P)$. Thus $g_\lambda \in GL(V)$. Fix a degree d form $f \neq 0$ representing a closed orbit A of $PGL(V)$ in $\mathbf{P}(P^d(V))$. For a suitable choice of H and P we may assume $f(P) \neq 0$. Write $f = \sum_{i=0}^d f_i w^i$ with $f_i \in P^{d-i}(H)$. Seeing H as a quotient $V = H + \mathbf{K}P \to H$ of V we see each f_i as an element of $P^{d-i}(V)$. We have $g_\lambda(f) = \sum_{i=0}^d \lambda^{-i} f_i w^i$. Hence $f_d \in \mathbf{K}$. By the polynomial case of Weierstrass preparation theorem (see [3], Th. 5.2, but it holds even if $\mathbf{K} = \mathbf{R}$) we may find H and P such that $f_d \neq 0$. Since $f_d \neq 0$, the net associated to $g_\lambda(f)$, $\lambda \in \mathbf{K}\backslash\{0\}$, tends to the hypersurface $X = \{f_d w^d = 0\}$ when λ goes to zero. Thus X is in the closure of the orbit A. Since A is closed, A is the orbit of X, proving the only if part of the theorem. The if part of the theorem is explained in Example 2. □

Proof of Theorem 2. The closure of an orbit is $PGL(V)$-invariant and hence it is a union of orbits. Obviously, the only orbit B adherent to A must be closed. By Theorem 1 B is the orbit formed by all d-powers of continuos linear forms. Fix a closed hyperplane M of $\mathbf{P}(V)$ such that $Q \notin M$, say $M = \{w = 0\}$ with $w \in V'$. Let z be an equation of the hyperplane $T_Q X$ tangent to X at Q and f be an equation of X. Write $f = \sum_{i=0}^d f_i z^i$ with $f_i \in P^{d-i}(T_Q X)$. Since

$f(Q) = 0$, we have $f_0 = 0$.

Claim. *The hypersurface* $Y = \{zw^{d-1} = 0\}$ *is the limit of the hypersurfaces* $g_\lambda(X)$ *when* λ *goes to zero.*

Proof of the Claim: Fix a finite-dimensional projective space W. By enlarging W we may assume $Q \in W$, $Q \subseteq X$, and $W \subseteq T_Q X$. Hence $g_\lambda(W) = W$ for every $\lambda \in \mathbf{K}\backslash\{0\}$, and we are reduced to prove the Claim in the finite-dimensional case. Since W has finite-dimension, every connected component of the Hilbert scheme $\mathrm{Hilb}(W)$ of W is projective ([4]) and, in particular, it is compact and Hausdorff. Hence to show that $Y \cap W$ is the limit of the family $g_\lambda(X \cap W)$ it is sufficient to take a general line D through Q and see that $Y \cap D$ is the limit of the family $g_\lambda(X \cap D)$; notice that this family is well-defined because $g_\lambda(D) = D$ since $Q \in D$. In particular we assume that D is not contained in $W \cap T_Q X$. Thus the zero-dimensional degree d scheme $X \cap D$ contains Q with multiplicity one (Lemma 1). The other $d-1$ roots of $f|D$ goes the point of $H \cap D$ when λ goes to zero. Since the scheme $Y \cap D$ is the union of Q with multiplicity one and $H \cap D$ with multiplicity $d-1$, we are done.

Since $Y \notin B$ and $A \cup B$ is the closure of A, the Claim implies that A is the orbit of Y, proving the only if part of the theorem. The if part is easy and left to the reader. □

Remark 1. In the proofs of Theorems 1 and 2 we used that V is locally convex only to have some weak form of Hahn - Banach (e.g. just enough to have the weak form of Weierstrass preparation theorem we used in those proofs). Any vector space topology with many continuos linear forms (e.g. the finite topology) will do.

Proof of Theorem 3. Since to be smooth is an open condition, we may find Y, X and Q such that Y is smooth at Q and $Q \notin X$. We fix a closed hyperplane H of $\mathbf{P}(V)$ with $Q \notin H$ and consider the family g_λ, $\lambda \in \mathbf{K}\backslash\{0\}$, of projective linear transformations constructed in the proof of Theorem 2 using H and Q. The proof of Theorem 2 showes that when λ goes to zero, X has dH as limit, while Y has $(d-1)H \cup T_Q Y$ as limit. Since $Q \in T_Q Y$, while $Q \notin H$, we have $T_Q Y \neq H$. Thus $dH \neq (d-1)H \cup T_Q Y$ and hence the pair $(dH, (d-1)H \cup T_Q Y)$ define $T \in \mathrm{Flag}(1, 2, P^d(V))$ contained in the closure of the orbit A. Since A is closed, we have $A = PGL(V)T$, as wanted. □

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