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# VISCOUS DAMPING EXPONENTIAL AND STRONG STABILIZABILITIES OF A PERIODIC SYSTEM OF HYPERBOLIC EQUATIONS

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**Abstract:** This paper studies a periodic system of hyperbolic equations in a finite number of bounded domains in $\mathbb{R}^n$, $n \in N$, which are joined serially at interfaces such that the wave travelling within each of them can be transmitted to the others. It is shown that by applying the appropriate internal controllers in each of m domains, $m \in \mathbb{N}$, the energy of the system decays uniformly exponentially (exponential stability). Withdrawing the damping mechanism from any of these domains may results in losing exponential stability in general. However, in this case, we are able to establish strong stability for the system. Simulations axe presented to illustrate the analytical results.

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## 1. Notation

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|---|---|
| $\Gamma_i$ | the boundary of the bounded domain $\Omega_i \subset \mathbb{R}^n$, $i \in \{1, 2, \ldots, m\}$, $m \in N$; |
| $\Gamma_i^*$ | the common boundary of the domains $\Omega_i$ and $\Omega_{i+1}$, $i \in \{1, 2, \ldots, m-1\}$; |
| $\eta$ | the outward unit normal to the boundary $\Gamma$; |
| $\frac{\partial}{\partial \eta}$ | normal derivative; |
| $\Delta$ | Laplace operator; |
| $L^2(\Omega)$ | the collection of measurable functions $f$ for which $\int_\Omega |f|^2\,dx$ is finite; |
| $L^\infty(\Omega)$ | the class of all functions that are essentially bounded on $\Omega$. |

## 2. Introduction

The exponential stability of the local energy for solutions of the coupled wave equations in a series of bounded domains by means of boundary controllers has been investigated in [2] for one dimensional and in [10] for $n$-dimensional spaces. In this paper, we study the exponential and strong stabilization of a similar system, where the controllers of velocity type are located internally as the following system of hyperbolic equations:

$$(u_1)_{tt} - c_1^2 \Delta(u_1) + \alpha_1 (u_1)_t = 0\,, \quad x \in \Omega_1,\ t > 0, \tag{1.1.1}$$

$$\vdots$$

$$(u_m)_{tt} - c_m^2 \Delta(u_m) + \alpha_m (u_m)_t = 0\,, \quad x \in \Omega_m,\ t > 0. \tag{1.1.m}$$

The initial conditions are

$$u_1(x,0) = u_{10}(x) \in H^l(\Omega_1),\ (u_1)_t(x,0) = u_{11}(x) \in L^2(\Omega_1), \tag{1.2.1}$$

$$\vdots$$

$$u_m(x,0) = u_{m0}(x) \in H^1(\Omega_m),\ (u_m)_t(x,0) = u_{m1}(x) \in L^2(\Omega_m), \tag{1.2.m}$$

where $t$, $x = (x_1, \ldots, x_n)$ are the time and space variables, respectively; $u_i$, $i \in 1, 2, \ldots, m$ are the deflections of the membrane from the equilibrium position and $m \in \mathbb{N}$ is a fixed number. The wave speeds $c_i$, $i \in 1, 2, \ldots, m$ and viscous damping coefficients $\alpha_i(x)$, $i \in 1, 2, \ldots, m$ are the system parameters. $\Omega_i$, $i \in 1, 2, \ldots, m$ are bounded, connected, open sets in $\mathbb{R}^n$ with piecewise smooth boundaries. Suppose $\Gamma_i$ and $\Gamma_{i+1}$ are the boundaries of two domains $\Omega_i$ and $\Omega_{i+1}$, respectively. We join $\Omega_i$ and $\Omega_{i+1}$ on the joined surface $\Gamma_i^*$ as follows:

$$u_i(x,t) = u_{i+1}(x,t), \tag{1.3.a}$$

$$Y_i \frac{\partial u_i}{\partial \eta_i} + Y_{i+1} \frac{\partial u_{i+1}}{\partial \eta_{i+1}} = 0 \quad \text{on} \quad \Gamma_i^* \times (0, \infty), \tag{1.3.b}$$

where $Y_i = c_i^2 \rho_i$, and $\rho_i > 0$ is constant, $i \in \{1, \ldots, m-1\}$. We note that on the common boundary, external normals with respect to each domain are different from each other. In fact, $\eta_i = -\eta_i + 1$ on $\Gamma_i^*$. The indices of the external normals are dropped if there is no confusion. We study the decay of the system in space $\mathbb{R}^n$, $n > 1$, where $\alpha_i \in L^\infty(\Omega_i)$, $\alpha_i(x) \geq \alpha_i > 0$, $i \in \{1, \ldots, m\}$, and we let the deflection around the boundaries be zero, i.e.

$$u_1(x,t) = 0 \qquad \text{on} \quad \bar{\Gamma}_1 := \Gamma_1 - \Gamma_1^*, \tag{1.4.1}$$

$$u_i(x,t) = 0 \qquad \text{on} \quad \bar{\Gamma}_i := \Gamma_i - (\Gamma_{i-1}^* \cup \Gamma_i^*), \tag{1.4.i}$$

$$u_m(x,t) = 0 \qquad \text{on} \quad \bar{\Gamma}_m := \Gamma_m - \Gamma_{m-1}^*, \tag{1.4.m}$$

**Example 1.** We may construct such a system as a model used for scattering and inverse problems:

Assume that two bars represented as two regions $\Omega_1$ and $\Omega_2$ are composed of two homogeneous materials joined at an interface $x = 0$. One material has constant density $\gamma_1$ and stiffness (Young's modulus) $E_1$ and the other has constant density $\gamma_2$ and stiffness $E_2$. The sound velocities in domains $\Omega_1$ and $\Omega_2$ are $c_1$ and $c_2$, respectively, where $c_i^2 = \dfrac{E_i}{\gamma_i}$, $i = 1, 2$. Suppose an incident right travelling wave $e^{i(x-c_1t)}$ of unit amplitude and unit wave number is sent from the far left. When the wave impinges upon the interface, a wave is reflected back into the domain $\Omega_1$ and another wave is transmitted into the other domain. Let $\alpha_1 u_t$ and $\alpha_2 v_t$ be dissipative perturbations to the system. This dissipation may be due to viscous effects, medium impurities or artificially imposed dampers. Then the displacements of travelling waves $u$ and $v$ in domains $\Omega_1$ and $\Omega_2$ satisfy the one dimensional wave equations:

$$u_{tt} - c_1^2 u_{xx} + \alpha_1 u_t = 0, \tag{1.5.a}$$

$$v_{tt} - c_2^2 v_{xx} + \alpha_2 v_t = 0. \tag{1.5.b}$$

At the interface $x = 0$ the displacements must be continuous, since we assume that the domains do not separate. Therefore,

$$u(0,t) = v(0,t), \qquad t > 0. \tag{1.6}$$

Further, we require that the force across the interface to be continuous:

$$E_1 u_x(0,t) = E_2 v_x(0,t), \qquad t > 0. \tag{1.7}$$

Equations (1.6) and (1.7), which are the special cases of (1.3) in one-dimensional space, are called the interface conditions.

In Section 2, we set out the notations and reformulate the system (1.1) - (1.4) into an evolution equation. Next we present a brief study of the existence and uniqueness of the problem with the aid of semigroup theory and obtain some consequences to be used in the latter sections. Section 3 is devoted to the study of the exponential stability of the system. The energy method is applied to achieve the result. We observe that, for a desired result in $n$-dimensional space, $n \geq 2$, we need all of $\alpha_i$, $i \in \{1, \ldots, m\}$ to be nonzeros, i.e. we cannot ignore any of the dampers for exponential stabilizability of our system in general. However, as it is shown in Section 4, we are able to show that the system is strongly stable if some of the local dampers, but not all, are removed from their domains. Section 5 is devoted to some numerical results for system in $\mathbb{R}^2$.

## 3. Existence and Uniqueness of Solution and Preliminaries

We reformulate (1.1) - (1.4) in the operational form

$$U_t = \tilde{A}U \qquad \text{in} \quad \bigcup_{i=1}^{m} \Omega_i \times (o, \infty)\,, \tag{2.1.a}$$

$$U(0) = (u_{10}, u_{11}, \ldots, u_{m0}, u_{m1})^T \qquad \text{in} \quad \bigcup_{i=1}^{m} \Omega_i\,, \tag{2.1.b}$$

where $U = (u_1, u_1, \ldots, u_m, z_m)^T$,

$$\tilde{A} = \begin{pmatrix} 0 & I & 0 & \ldots & 0 & 0 \\ c_1^2\Delta & -\alpha_1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 0 & I \\ 0 & 0 & \ldots & 0 & c_m^2\Delta & -\alpha_m \end{pmatrix}$$

is an operator in the Hilbert space,

$$\mathcal{H} = H_1^1 \times L^2(\Omega_1) \times \cdots \times H_m^1 \times L^2(\Omega_m)\,,$$

$$H_j^i = \{\omega \in H^i(\Omega_j)\,|\,\omega = 0 \text{ on } \bar{\Gamma}_j\}\,,$$

$$H^i(\Omega_j) = \{\omega : \Omega_j \longrightarrow \mathbb{R}\,|\,D^k\omega \in L^2(\Omega_j)\,,\ k = 0, 1, \dots, i\} \qquad i \in \{1, 2\} \text{ and } j \in \{1, 2, \dots, m\}\,,$$

where $D^k\omega$ denotes the derivative of $\omega$ in the sense of distributions. The domain of $\tilde{A}$ is defined by

$$D(\tilde{A}) = \left\{ U \in \tilde{\mathcal{H}}| \begin{array}{c} u_i = 0 \text{ on } \bar{\Gamma}_i\,, \quad u_i(x,t) = u_{i+1}(x,t) \text{ on } \Gamma_i^*\,, \\ Y_i \dfrac{\partial u_i}{\partial \eta_i} + Y_{i+1} \dfrac{\partial u_{i+1}}{\partial \eta_{i+1}} \text{ on } \Gamma_i^* \end{array} \right\}, \tag{2.2}$$

where $\mathcal{H} = H_1^2 \times H_1^1 \times \cdots \times H_m^2 \times H_m^1$, is dense in $\mathcal{H}$. The energy of the system is defined as

$$E = \frac{1}{2} \sum_{i=1}^{m} \int_{\Omega_i} \left[ Y_i |\nabla u_i|^2 + \rho_i (u_i)_t^2 \right] dx\,, \quad t \geq 0\,. \tag{2.3}$$

Then the proper norm associated with the energy will be:

$$\|U\|^2 = \sum_{i=1}^{m} \int_{\Omega_i} \left[ Y_i |\nabla u_i|^2 + \rho_i z_i^2 \right] dx\,, \quad t \geq 0\,. \tag{2.4}$$

We observe that $E$ is a non-increasing function of time:

$$\begin{aligned} \frac{dE}{dt} &= \sum_{i=1}^{m} \int_{\Omega_i} \left[ Y_i \nabla u_i \cdot \nabla u_{it} + \rho_i u_{it} u_{itt} \right] dx \\ &= \sum_{i=1}^{m} Y_i \int_{\Omega_i} \left[ \nabla u_i \cdot \nabla u_{it} + (\Delta u_i) u_{it} \right] - \sum_{i=1}^{m} \rho_i \int_{\Omega_i} \alpha_i(x) u_{it}^2 \, dx \\ &= - \sum_{i=1}^{m} \rho_i \int_{\Omega_i} \alpha_i(x) u_{it}^2 \, dx \leq 0\,. \end{aligned}$$

The wave operator $\tilde{A}$ can be decomposed as $A+B$, where

$$A = \begin{pmatrix} 0 & I & 0 & \dots & 0 & 0 \\ c_1^2\Delta & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & I \\ 0 & 0 & \dots & 0 & c_m^2\Delta & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & I & 0 & \dots & 0 & 0 \\ 0 & -\alpha_1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & I \\ 0 & 0 & \dots & 0 & 0 & -\alpha_m \end{pmatrix}.$$

$B$ is a bounded operator in $\mathcal{H}$. The following lemma gives some properties of $A$.

**Lemma 2.1.** *The following holds for the operator $A$.*

*(I) $A$ is skew-adjoint, i.e. $A^* = -A$, where $A^*$ is the adjoint operator of $A$.*

*(II) $A^{-1}$ exists and is compact.*

*(III) $A$ is the infinitesimal generator of a $C_0$-semigroup.*

*Proof.* (I) Let $\tilde{U} = (\tilde{u}_1, \tilde{z}_1, \dots, \tilde{u}_m, \tilde{z}_m)^T \in D(A)$ and $\hat{U} = (\hat{u}_1, \hat{z}_1, \dots, \hat{u}_m, \hat{z}_m)^T \in D(-A)$ be arbitrary. We note first that $D(-A) = D(A)$. Moreover,

$$\begin{aligned} \langle AU_1, U_2\rangle_{\mathcal{H}} &= \sum_{i=1}^{m} Y_i \int\limits_{\Omega_i} \left[\nabla \hat{u}_i \cdot \nabla \tilde{z}_i + (\Delta \tilde{u}_i)\hat{z}_i\right] dx \\ &= \sum_{i=1}^{m-1} \left( Y_i \int\limits_{\Gamma_i^*} \frac{\partial \hat{u}_i}{\partial \eta} \tilde{z}_i \, d\Gamma + Y_{i+1} \int\limits_{\Gamma_i^*} \frac{\partial \hat{u}_{i+1}}{\partial \eta} \tilde{z}_{i+1} \, d\Gamma \right) \\ &+ \sum_{i=1}^{m-1} \left( Y_i \int\limits_{\Gamma_i^*} \frac{\partial \tilde{u}_i}{\partial \eta} \hat{z}_i \, d\Gamma + Y_{i+1} \int\limits_{\Gamma_i^*} \frac{\partial \tilde{u}_{i+1}}{\partial \eta} \hat{z}_{i+1} \, d\Gamma \right) \\ &- \sum_{i=1}^{m} Y_i \int\limits_{\Omega_i} \left[\nabla \tilde{u}_i \cdot \nabla \hat{z}_i + (\Delta \hat{u}_i)\tilde{z}_i\right] dx \,. \end{aligned}$$

Since $\tilde{z}_i = \tilde{z}_{i+1}$ and $\hat{z}_i = \hat{z}_{i+1}$ on $\Gamma_i^*$, the expression in each parentheses of above equation vanishes and we obtain

$$\langle A\tilde{U}, \hat{U}\rangle_{\mathcal{H}} = \langle \tilde{U}, -A\hat{U}\rangle_{\mathcal{H}}\,.$$

Therefore, $D(-A) \subseteq D(A^*)$ and $A^*$ is identical to $-A$ in $D(-A)$. To complete the proof we need to verify that $D(A^*) \subseteq D(-A)$ which can be done by an argument similar to that in [6].

(II) Consider the operator $A$ in the variational sense:

$$\begin{aligned} U_t &= AU\,, \\ U(0) &= (u_{10}, u_{11}, \dots, u_{m0}, u_{m1})^T\,, \end{aligned} \tag{2.7}$$

$$\begin{aligned} D(A) = \Big\{ & U \in H_1^1 \times H_1^1 \times \cdots \times H_m^1 \times H_m^1 \mid \\ & \Delta(u_1, \dots, u_m)^T \in L^2(\Omega_1) \times \cdots \times L^2(\Omega_m)\,, \\ & \sum_{i=1}^m Y_i \int_{\Omega_i} [(\Delta u_i)\phi_i + \nabla u_i \cdot \nabla \phi_i]\; dx = 0\,,\ \forall (\phi_1, \dots, \phi_m) \in H^* \Big\}\,, \end{aligned}$$

where

$$\begin{aligned} H^* = \Big\{ & (u_1, \dots, u_m)^T \in H_1^1 \times \cdots \times H_m^1 \mid \\ & u_i = 0 \text{ on } \bar{\Gamma}_i\,,\quad u_i(x,t) = u_{i+1}(x,t) \text{ on } \Gamma_i^*\,, \\ & Y_i \frac{\partial u_i}{\partial \eta} + Y_{i+1} \frac{\partial u_{i+1}}{\partial \eta} = 0 \text{ on } \Gamma_i^* \Big\} \end{aligned}$$

and

$$\|(u_1, \dots, u_m)^T\|_{H^*}^2 = \sum_{i=1}^m Y_i \|\nabla u_i\|^2\,.$$

Define the bilinear form $\mathcal{B}$ in Hilbert space $H^*$ as follows:

$$\mathcal{B}((u_1, \dots, u_m)^T, (\phi_1, \dots, \phi_m)^T) := \sum_{i=1}^m \int_{\Omega_i} [Y_i \nabla u_i \cdot \nabla \phi_i]\; dx\,. \tag{2.8}$$

It can be shown that $\mathcal{B}$ is bounded and coercive. Therefore, by the Lax-Milgram theorem [8], given any bounded linear functional

$\mathcal{L}((\phi_1,\dots,\phi_m)^T)$ on $H^*$, there exists a $(u_1,\dots,u_m)^T \in H^*$ such that

$$\mathcal{B}((u_1,\dots,u_m)^T,(\phi_1,\dots,\phi_m)^T) = \mathcal{L}((\phi_1,\dots,\phi_m)^T)\,, \qquad \forall (\phi_1,\dots,\phi_m)^T \in H^*. \quad (2.9)$$

Now, let $F = (f_1, g_1, \dots, f_m, g_m)^T \in \mathcal{H}$ be arbitrary, and define the bounded linear function $\mathcal{L}$ on $H^*$ as follows:

$$\mathcal{L}((\phi_1,\dots,\phi_m)^T) := -\sum_{i=1}^{m}\int_{\Omega} \rho_i g_i \phi_i \, dx\,. \quad (2.10)$$

From (2.8), (2.9) and (2.10) it follows that

$$\sum_{i=1}^{m}\int_{\Omega_i} \left[Y_i \nabla u_i \cdot \nabla \phi_i + \rho_i g_i \phi_i\right] \, dx = 0 \quad (2.11)$$

has a unique solution $(u_1,\dots,u_m)^T \in H^*$. Applying Green's formula we obtain

$$c_i^2 \Delta u = g_i \in L^2(\Omega_i)\,, \quad i \in \{1,2,\dots,m\}.$$

Hence, (2.11) can be written as

$$\sum_{i=1}^{m} Y_i \int_{\Omega_i} \left[(\Delta u_i)\phi_i + \nabla u_i \cdot \nabla \phi_i\right] \, dx = 0\,.$$

Let $z_i = f_i \in H_i^1$, $i \in \{1,2,\dots,m\}$, then we observe

$$U = (u_1, z_1, \dots, u_m, z_m) \in D(A)\,,$$

and

$$AU = \begin{pmatrix} 0 & I & 0 & \dots & 0 & 0 \\ c_1^2\Delta & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & I \\ 0 & 0 & \dots & 0 & c_m^2\Delta & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ z_1 \\ \vdots \\ u_m \\ z_m \end{pmatrix} = \begin{pmatrix} z_1 \\ c_1^2 \Delta u_1 \\ \vdots \\ z_m \\ c_m^2 \Delta u_m \end{pmatrix} = \begin{pmatrix} f_1 \\ g_1 \\ \vdots \\ f_m \\ g_m \end{pmatrix}.$$

Furthermore,

$$\|(u_1, z_1, \dots, u_m, z_m)^T\|_{\tilde{\mathcal{H}}} \le K \|(f_1, g_1, \dots, f_m g_m)^T\|_{\mathcal{H}}\,,$$

where $K$ is a constant. Therefore, $A^{-1}$ exists and is bounded on $\mathcal{H} \longrightarrow \widetilde{\mathcal{H}}$. The compactness of $A^{-1}$ is a result of the Sobolev embedding theorem.

(III) This part can be followed by (I). In fact, $A$ and $A^* = -A$ are both dissipative:

$$\begin{aligned}\langle AU, U\rangle_{\mathcal{H}} &= \sum_{i=1}^{m} Y_i \int\limits_{\Omega_i} [\nabla u_i \cdot \nabla z_i + (\Delta u_i) z_i]\, dx \\ &= \sum_{i=1}^{m-1} \left( Y_i \int\limits_{\Gamma_i^*} \frac{\partial u_i}{\partial \eta} z_i \, d\Gamma + Y_{i+1} \int\limits_{\Gamma_i^*} \frac{\partial u_{i+1}}{\partial \eta} z_{i+1} \, d\Gamma \right) = 0 = \langle U, A^* U\rangle_{\mathcal{H}}\,,\end{aligned}$$

and $A$ is a closed operator. Therefore, by Corollary 4.3.1 of [1] we obtain the desired result. □

An immediate conclusion of the above lemma is that, according to the Perturbation Theorem [8], $\tilde{A}$ generates a $C_0$-semigroup, $\tilde{S}(t)$, satisfying $\|\tilde{S}(t)\| \leq Me^{(\omega + M\|B\|)t}$, where $M$ and $\omega$ are constants satisfying $\|S(t)\| \leq Me^{\omega t}$ for the semigroup $S(t)$ generated by $A$. This guarantees the existence and uniqueness of the solution of the system (1.1) - (1.4).

## 4. Systems with Internal Controllers in $n$-Dimensional Space

Consider the system (1.1)-(1.4), where $\alpha_i(x) \in L^\infty(\Omega_i)$ are positive functions and

$$\operatorname*{Essup}_{s\in\Omega_i} \alpha_i(x) = \alpha_i \,, \qquad i \in 1, 2, \dots, m \,.$$

We would like to establish the following theorem:

**Theorem 3.1.** *The system (1.1) along with initial and boundary conditions (1.2) - (1.4) is exponentially stabilizable, i.e.*

$$E(t) < Me^{-\omega t}, \quad M > 0,\ w > 0,\ \forall t > 0\,, \tag{EI}$$

*provided none of the velocity feedback coefficients $\alpha_i(x)$ vanish in their corresponding domains.*

We apply energy method technique as in [9] to achieve the goal. We note that as shown in Section 2, the energy of the system is non-increasing

$$\frac{DE}{dt} = -\sum_{i=1}^{m} \rho_i \int\limits_{\Omega_i} \alpha_i(x) u_{it}^2 \, dx \le 0\,. \tag{3.1}$$

We define $\mathcal{E}(t)$ and $J$ as follows:

$$\mathcal{E}(t) := E(t) + \varepsilon J, \tag{3.2}$$

$$J := \sum_{i=1}^{m} \int\limits_{\Omega_i} \rho_i u_i u_{it} \, dx\,. \tag{3.3}$$

From (3.3) we obtain

$$J' := \sum_{i=1}^{m} \int\limits_{\Omega_i} \rho_i \left[ u_{it}^2 + \left( c_i^2 \Delta u_i - \alpha_i u_{it} \right) u_i \right] \, dx\,. \tag{3.4}$$

Applying Green's formula and then utilizing boundary conditions (1.4) and joined conditions (1.3) in (3.4) leads us to the following equation:

$$J' = \sum_{i=1}^{m} \left( \int\limits_{\Omega_i} \rho_i u_{it}^2 \, dx - Y_i \int\limits_{\Omega_i} |\nabla u_i|^2 \, dx - \int\limits_{\Omega_i} \rho_i \alpha_i(x) u_{it} u_i \, dx \right)\,. \tag{3.5}$$

On the other hand, applying the Cauchy-Schwarz inequality we obtain:

$$\left|\int\limits_{\Omega_i} \rho_i\alpha_i(x)u_{it}u_i\,dx\right| \le \frac{\rho_i\alpha_i}{2\delta_i}\int\limits_{\Omega_i}|u_{it}|^2\,dx + \frac{\rho_i\mu_i\delta_i\alpha_i}{2}\int\limits_{\Omega_i}|\nabla u_i|^2\,dx\,, \tag{3.6}$$

where $\delta_i$ is any arbitrary positive number and $\mu_i$ is the constant of Poincare's inequality (P) introduced in domain $\Omega_i$:

$$\int\limits_{\Omega_i} u_i^2\,dx \le \mu_i\int\limits_{\Omega_i}|\nabla u_i|^2\,dx\,, \quad \mu_i>0,\ u\in H_i^1\,. \tag{$P$}$$

Then (3.5), (3.6) give the following estimate:

$$J' \le \sum_{i=1}^{m}\Bigg(\int\limits_{\Omega_i}\rho_i u_{it}^2\,dx - Y_i\int\limits_{\Omega_i}|\nabla u_i|^2\,dx + \frac{\rho_i\alpha_i}{2\delta_i}\int\limits_{\Omega_i}u_{it}^2\,dx + \frac{\rho_i\mu_i\delta_i\alpha_i}{2}\int\limits_{\Omega_i}|\nabla u_i|^2\,dx\Bigg)\,. \tag{3.7}$$

Finally, (3.1), (3.2) and (3.7) give

$$\mathcal{E}'(t) \le -\sum_{i=1}^{m}\Bigg(\rho_i\left[\alpha_i - \varepsilon\left(1+\frac{a_i}{2\delta_i}\right)\right]\int\limits_{\Omega_i}u_{it}^2\,dx + \varepsilon\left[Y_i - \delta_i\frac{\rho_i\mu_i a_i}{2}\right]\int\limits_{\Omega_i}|\nabla u_i|^2\,dx\Bigg)\,. \tag{3.8}$$

Choose $\delta_i = \dfrac{c_i^2}{\mu_i a_i}$, $i\in\{1,2,\dots,m\}$, then we have

$$\sum_{i=1}^{m}\varepsilon\left[Y_i - \delta_i\frac{\rho_i\mu_i a_i}{2}\right]\int\limits_{\Omega_i}|\nabla u_i|^2\,dx = \sum_{i=1}^{m}\frac{\varepsilon}{2}Y_i\int\limits_{\Omega_i}|\nabla u_i|^2\,dx\,. \tag{3.9}$$

Let $\varepsilon$ be small enough so that

$$\left[a_i - \varepsilon\left(1+\frac{a_i}{2\delta_i}\right)\right] \ge \frac{1}{2}\varepsilon\,.$$

It is seen that in order to obtain the above estimate, $\varepsilon$ should lie in the interval $\left[0, \dfrac{2c_i^2 a_i}{3c_i^2 + a_i^2\mu_i}\right]$. Now, we choose

$$\varepsilon_0 = \min\left\{\frac{2c_i^2 a_i}{3c_i^2 + a_i^2\mu_i}\,,\ i \in \{1, 2, \dots, m\}\right\}.$$

Then for $\varepsilon \leq \varepsilon_0$ we have:

$$-\sum_{i=1}^{m} \rho_i \left[a_i - \varepsilon\left(1 + \frac{a_i}{2\delta_i}\right)\right] \int_{\Omega_i} u_{it}^2 \leq -\sum_{i=1}^{m} \frac{\varepsilon}{2} \int_{\Omega_i} \rho_i u_{it}^2\, dx\,. \tag{3.10}$$

Relations (3.8) - (3.10) finally yield

$$\mathcal{E}'(t) \leq -\varepsilon E(t)\,, \quad \forall t \geq 0\,. \tag{3.11}$$

*Proof of Theorem 3.1.* Applying the Cauchy-Schwarz and Poincare's inequalities to (3.3) yields:

$$|J| \leq \kappa_1 E(t)\,, \quad \forall t \geq 0\,, \tag{3.12}$$

where $\kappa_1 = \max\left\{\left(1 + \dfrac{\mu_i}{c_i^2}\right)\,,\ i \in \{1, 2, \dots, m\}\right\}$. From (3.2) and (3.12) we have the following estimate:

$$(1 - \varepsilon\kappa_1)E \leq \mathcal{E} \leq (1 + \varepsilon\kappa_1)E\,. \tag{3.13}$$

So $\mathcal{E}$ is positive if $\varepsilon$ is sufficiently small, i.e. $0 < \varepsilon < \kappa_1^{-1}$. The inequality (3.11) along with (3.13) for such $\varepsilon$ yield

$$\mathcal{E}'(t) \leq -\kappa_2\varepsilon\mathcal{E}\,, \tag{3.14}$$

where $\kappa_2 = (1 + \varepsilon + \kappa_1)^{-1}$. Integrating (3.14) yields

$$\mathcal{E}(t) \leq e^{-\kappa_2\varepsilon t}\mathcal{E}(0)\,, \quad \forall t \geq 0\,.$$

Now, applying (3.13) and choosing a fixed number $k$ in range

$$\left\{\varepsilon\,|\,0 < \varepsilon < \min\{\kappa_1^{-1}, \varepsilon_0\}\right\}$$

give us the desired inequality:

$$E(t) \leq Me^{-\omega t}\,, \quad \forall t \geq 0\,,$$

where $M = \left(\dfrac{1 + k\kappa_1}{1 - k\kappa_1}\right) E(0)$ and $\omega = k\kappa_2$. □

## 5. Strong Stability of the System with One Locally Distributed Damper

As we noticed in Section 3, in order to obtain exponential stability, we need to have viscous damping in all domains. An interesting question then would be "what will happen if we drop one of the dampers from its domain?" In general, we are able to show that the system is strongly stable. By that, we mean the system comes to rest through time in a rate which is not necessarily exponential. However, as some numerical simulations suggest in the next section, we lose the exponential rate of decay of local energy in $n$-dimensional space if $n > 1$. The main objective of this section is the follqwing theorem:

**Theorem 4.1.** *Let $\alpha_j(x)$, for a fixed $j$, $j \in \{1, 2, \ldots, m\}$ be zero in the system introduced by (1.1)-(1.4). Assume that the interface surfaces $\Gamma_i^*$, $i \in \{1, 2, \ldots, m\}$ have non-empty interiors in space of $\mathbb{R}^{n-1}$. Then the system is strongly asymptotically stable, i.e.*

$$E(u, v, t) \longrightarrow 0 \text{ as } t \to \infty \,. \tag{$E2$}$$

To prove the theorem, the following result is utilized [4]:

**Theorem 4.2.** (Strong Stability Criterion) *Let $S(t)$ be a uniform bounded $C_0$-semigroup on Banach space, and let $Re\lambda < 0$ for all $\lambda \in \sigma(A)$, then $S(t)$ is strongly asymptotically stable; conversely, let $S(t)$ be strongly asymptotically stable, then $S(t)$ is uniformly bounded, $Re\lambda \leq 0$ for all $\lambda \in \sigma(A)$, and there is on imaginary axis neither point nor residual spectrum of $A$.*

*Proof of Theorem 4.1.* Without loss of generality let $j = m$, i.e. $\alpha_m = 0$, and consider the evolution equation (2.1), we want to show that 0 is not an eigenvalue of $\tilde{A} = A + B$. For this, with the same approach as Lemma 2.1, Part (II), we define the bilinear form $\tilde{\mathcal{B}}$ as

$$\mathcal{B}\left((u_1, \ldots, u_m)^T, (\phi_1, \ldots, \phi_m)^T\right) := \sum_{i=1}^{m} \int_{\Omega_i} \left[Y_i \nabla u_i \cdot \nabla \phi_i\right] dx \tag{4.1}$$

and for arbitrary element $F = (f_1, g_1, \ldots, f_m, g_m)^T \in \mathcal{H}$, define the bounded linear function $\mathcal{L}$, on $H^*$ as

$$\mathcal{L}\left((\phi_1,\ldots,\phi_m)\right) := -\left[\sum_{i=1}^{m-1}\int\limits_{\Omega_i}\rho_i(\alpha_i f_i + g_i)\phi_i\,dx + \int\limits_{\Omega_m}\rho_m g_m\phi_m\,dx\right]. \quad (4.2)$$

Then (4.1), (4.2) yield

$$c_i^2\Delta u_i = \alpha_i f_i + g_i \in L^2(\Omega_i)\,, \quad i\in\{1,2,\ldots,m-1\}\,,$$
$$c_m^2\Delta u_m = g_m \in L^2(\Omega_m)\,,$$

and with assumptions $z_i = f_i \in H_i^1$ we finally have

$$U = (u_1, z_1, \ldots, u_m, z_m) \in D(\tilde{A})\,,$$

and

$$\tilde{A}U = \begin{pmatrix} 0 & I & 0 & \ldots & 0 & 0\\ c_1^2\Delta & -\alpha_1 & 0 & \ldots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \ldots & 0 & 0 & I\\ 0 & 0 & \ldots & 0 & c_m^2\Delta & 0 \end{pmatrix}\begin{pmatrix} u_1\\ z_1\\ \vdots\\ u_m\\ z_m\end{pmatrix} = \begin{pmatrix} z_1\\ c_1^2\Delta u_1 - \alpha_1 z_1\\ \vdots\\ z_m\\ c_m^2\Delta u_m\end{pmatrix} = \begin{pmatrix} f_1\\ g_1\\ \vdots\\ f_m\\ g_m\end{pmatrix} = F\,.$$

This shows that 0 is not an eigenvalue of $A+B$. Utilizing Theorem A.1 in [3] guarantees existence and compactness of operator $\tilde{A}^{-1}$. Hence, by Theorem 6.29 in Chapter 3 of [5], $\sigma(\tilde{A})$ consists of only isolated eigenvalues. Since $\tilde{A}$ is dissipative, we have no eigenvalue in the open right half plane. What's left is to show that there is not any eigenvalue at the imaginary axis. We assume the contrary that there exists $\lambda_0 \in \sigma(A)$ such that $\lambda_0 = i\zeta_0$, where $i^2 = -1$ and $\zeta_0 \in \mathbb{R}$. Then

$$(\tilde{A} - \lambda_0 I)(u_1, z_1, \ldots, u_m, z_m)^T = 0 \quad (4.3)$$

has a nontrivial solution $(\bar{u}_1, \bar{z}_1, \ldots, \bar{u}_m, \bar{z}_m)^T \in D(A)$. Therefore, $(\bar{u}_1, \bar{z}_1, \ldots, \bar{u}_m, \bar{z}_m)^T$ satisfies the following equation:

$$\mathcal{F}(\bar{u}_1, \bar{z}_1, \ldots, \bar{u}_m, \bar{z}_m)^T = 0\,,$$

where

$$\mathcal{F} = \tag{4.4}$$

$$\begin{pmatrix} c_1^2\Delta - \alpha_1\lambda_0 - \lambda_0^2 & 0 & 0 & \dots & 0 & 0 \\ 0 & c_2^2\Delta - \alpha_2\lambda_0 - \lambda_0^2 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & c_{m-1}^2\Delta - \alpha_{m-1}\lambda_0 - \lambda_0^2 & 0 \\ 0 & 0 & \dots & 0 & 0 & c_m^2\Delta - \lambda_0^2 \end{pmatrix}$$

Let $(u_1, \dots, u_m)^T = e^{\lambda_0 t}(\bar{u}_1, \dots, \bar{u}_m)^T$. Then $(u_1, \dots, u_m)^T$ will be a solution to the system (1.1)-(1.4). Furthermore,

$$\begin{aligned} \|(u_1, z_1, \dots, u_m, z_m)^T\| &= \|e^{\lambda_0 t}(\bar{u}_1, \bar{z}_1, \dots, \bar{u}_m, \bar{z}_m)^T\| \\ &= \|(\bar{u}_1, \bar{z}_1, \dots, \bar{u}_m, \bar{z}_m)^T\|\,. \end{aligned} \tag{4.5}$$

Equation (4.5) shows that the energy of the system is a constant function in $t$:

$$\frac{d}{dt}E(t) = -\sum_{i=1}^{m-1} \rho_i \int\limits_{\Omega_i} \alpha_i(x) u_{it}^2 \, d\Gamma = 0\,. \tag{4.6}$$

But this implies $u_{it} = \lambda_0 e^{\lambda_0 t}\bar{u}_i = 0$, $i \in \{1, \dots, m-1\}$. Thus, we immediately conclude that $\bar{u}_i = 0$ in $\Omega_i$, and $u_{itt}$ are zero almost everywhere in $\Omega_i$, and $\bar{u}_i = \dfrac{\partial \bar{u}_i}{\partial \eta} = 0$ on $\Gamma_i^*$, $i \in \{1, \dots, m-1\}$. Applying the joined conditions (1.3), the system (1.1)-(1.4) then reduces to the following:

$$u_{mtt} - c_2^2 \Delta u_m = 0\,, \qquad x \in \Omega_2,\ t > 0\,, \tag{4.7.a}$$

$$u_m(x, 0) = \bar{u}_m(x) \in H^1(\Omega_m)\,, \qquad u_{mt} = \bar{z}_m(x) \in L^2(\Omega_m)\,. \tag{4.7.b}$$

$$u_m(x, t) = 0 \qquad \text{on } \Gamma_m \times (0, \infty)\,, \tag{4.8.a}$$

$$\frac{\partial u_m}{\partial \eta} = 0 \qquad \text{on } \Gamma_{m-1}^* \times (0, \infty)\,, \tag{4.8.b}$$

from which the following system of elliptic equation can be extracted:

$$\zeta_0^2 \bar{u}_m + c_2^2 \Delta \bar{u}_m = 0\,, \qquad x \in \Omega_m,\ t > 0\,, \tag{4.9}$$

$$\bar{u}_m = 0\,, \qquad x \in \Gamma_m\,, \tag{4.10.a}$$

$$\frac{\partial \bar{u}_m}{\partial \eta} = 0\,, \qquad x \in \Gamma_{m-1}^*\,. \tag{4.10.b}$$

Let $x_*$ be a point in the interior of $\Gamma^*_{m-1}$ and $\Omega_*$ be a ball centered at $x_*$. We choose $\Omega_*$ so small that it contains only interior points of $\Gamma^*_{m-1}$. Let $\bar{u}_m = 0$ in $\Omega_* - \bar{\Omega}_m$, then applying boundary conditions (4.10), by uniqueness extension of the elliptic equations we obtain that $\bar{u}_m$ is analytic on the interior of $\bar{\Omega}_* \cup \bar{\Omega}_m$, and since $u_m = 0$ in open set $\Omega_* - \bar{\Omega}_m \subset \text{ int } \left(\bar{\Omega}_* \cup \bar{\Omega}_m\right)$, we conclude that $\bar{u}_m = 0$ everywhere including $\Omega_m$. Thus, $(\bar{u}_1, \bar{z}_1, \ldots, \bar{u}_m, \bar{z}_m)^T = 0$ which is a contradiction. This proves that there is not any eigenvalue in the imaginary axis and, therefore, $\sigma(A)$ is a subset of the open left half plane. The proof of the theorem, then, is an immediate consequence of Theorem 4.2. □

## 6. A Numerical Result for a Coupled Rectangular Domains

So far, as shown in Section 3, it seems reasonable to say in a higher dimensional space, exponential stability requires the presence of viscous damping in all regions. The study of some geometrical domains in two-dimensional space confirms this opinion. For instance, consider coupled unit square domains joined on one side where there is no internal or boundary damper in the system. For simplicity let $c_1 = c_2 = 1$. Then the eigenvalues of the system will be:

$$\lambda_{lm} = i\left(\frac{l^2\pi^2}{2} + m^2\pi^2\right)^{\frac{1}{2}}, \quad l, m \in \mathbb{Z}\,.$$

These eigenvalues do not have a gap. More than that, they are so condensed that a damper of type velocity in only one of the domains may not take all eigenvalues to a uniform distance from the imaginary axis in the left half plane (according to Section 4, however, all the eigenvalues are in the left half plane). By an elaboration of the approach in [7], it may be shown that there exists a sequence of eigenvalues which approaches the imaginary axis asymptotically if the viscous damping acts in only one of the regions.

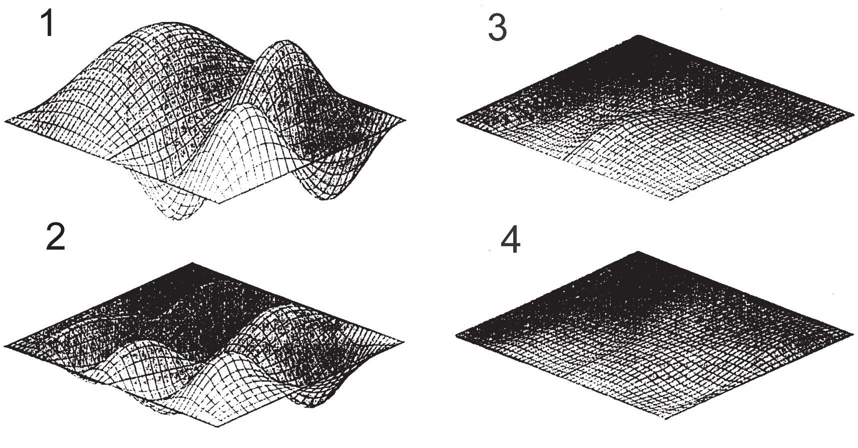



Figure 5.1: The solution of the system studied in Section 3 for time $t = 48K^{*}\Delta t$, $K = 0, \ldots, 3$.

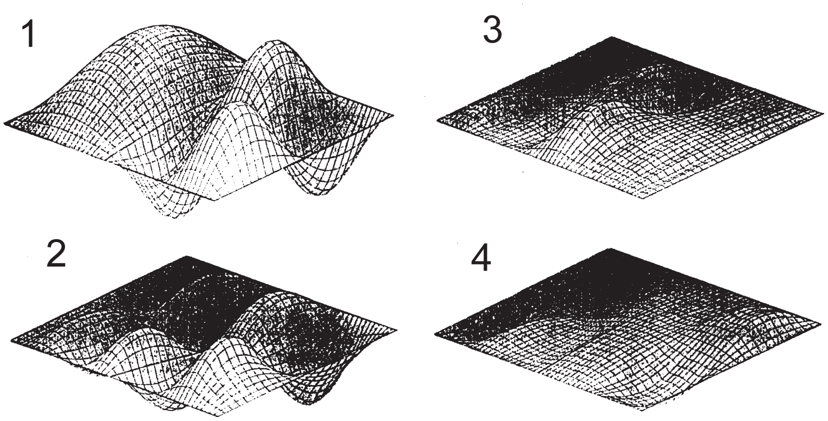
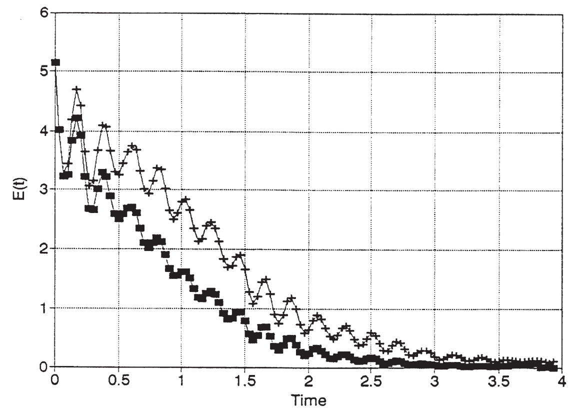



Figure 5.2: The solution of the system studied in Section 4 for time $t = 48K^{*}\Delta t$, $K = 0, \ldots, 3$.

Figures 5.1-5.3 illustrate a numerical comparison between the cases studied in Section 3 and Section 4. These results are based on the finite difference method. The graphs have been plotted in Maple V to manifest the rate of dissipation of waves in three equal-distance periods of time. The Simpson's method was employed to handle the double integrals introduced in the energy equation of the systems.




$$U_{tt} - c_1^2(U_{xx} + U_{yy}) + a_1 U_t = 0\,,$$

$$U_{tt} - c_2^2(U_{xx} + U_{yy}) + a_2 U_t = 0\,.$$

—■— Energy for Figure 5.1
—|— Energy for Figure 5.2

Figure 5.3: Comparison of decay of energy of the systems presented in Figures 5.1-5.2 with respect to time $t$.

## 7. Conclusion

This paper presents what appears to be the first in formulations for well-posedness and exponential stability of periodic hyperbolic equations in $n$-dimensional domains by means of internal controllers. To achieve the goal, the energy method is employed for $n$-dimensional spaces. With fewer controllers, i.e. controllers in some domains, we establish strong stability of the system in $\mathbb{R}^n$, $n > 1$. Later numerical simulations suggest, that for coupled square domains with fewer dampers on the boundaries, the system may not be exponentially stable.

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