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A GENERALIZATION OF DATKO-PAZY THEOREM FOR STRONGLY CONTINUOUS SEMIGROUPS

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Abstract: We give another proof for a result of van Neerven in [3].

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1. Introduction

Let $\mathbf{T} = \{T(t)\}_{t\geq 0}$ be a C_0-semigroup on a Banach space X and $\omega_0(T) := \lim\limits_{t\to\infty} \frac{\ln ||T(t)||}{t}$ be its growth bound. It is well known theorem of Datko [1], that if for each $x \in X$, the map $t \mapsto ||T(t)x||$ belongs to $L^2(\mathbf{R}_+)$, then $\omega_0(\mathbf{T}) < 0$. This result was generalized by Pazy [4], who showed that the exponent $p = 2$ may be replaced by $1 \leq p < \infty$. Recently Neerven proved in [3] the following improvements of Datko-Pazy theorem:

Theorem A. *Suppose E is a complete normed function space over $\mathbf{R}_+$, satisfying $\lim\limits_{t\to\infty} \Psi_E(t) = \infty$. If $\mathbf{T} = \{T(t)\}_{t\geq 0}$ is a C_0-semigroup on a Banach space X, such that for all $x \in X$ the function $t \to ||T(t)x||$ defines an element in E, then the growth bound of $\mathbf{T}$ is negative.*

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Theorem B. *Let* $\mathbf{T}$ *be a* C_0*-semigroup on a Banach space* X *and* $1 \leq p < \infty$. *If*

$$\int_0^\infty \beta(t)||T(t)x||^p dt < \infty, \quad \forall x \in X\,,$$

where $0 \leq \beta \in L^1_{loc}(\mathbf{R}_+)$ *satisfies* $\int_0^\infty \beta(t)dt = \infty$, *then* $\omega_0(\mathbf{T}) < 0$.

In this paper we present a result which contain Theorem A and Theorem B. In fact our proof of the result is an easy extension of the proof of Theorem B in [3]. We mention here that the proof of Theorem A in [3] is based on the Pazy's proof of boundedness of $\mathbf{T}$ which also used the *invariant translation*-property of the space E.

2. Normed Function Space

Let $(\mathbf{R}_+, \mathcal{L}, m)$ be a Lebesgue measure space and let $\mathcal{M}$ be the linear space of all measurable functions $f : \mathbf{R}_+ \to \mathbf{C}$, identifying functions which are equal a.e.. Suppose that there exists a function $\rho : \mathcal{M} \to [0, \infty]$ with the following properties:

- $(\mathbf{n_1})$ $\rho(f) = 0$ if and only if $f = 0$;
- $(\mathbf{n_2})$ $\rho(af) = |a|\rho(f)$ for all scalar $a \in \mathbf{C}$ and all $f \in \mathcal{M}$ with $\rho(f) < \infty$;
- $(\mathbf{n_3})$ $\rho(f + g) \leq \rho(f) + \rho(g)$ for all $f, g \in \mathcal{M}$.

Let $F = F_\rho$ be the set of all $f \in \mathcal{M}$ such that $|f|_F := \rho(f) < \infty$. It is clear that $(F, |\cdot|_F)$ is a normed linear space. The normed linear subspace E of F is called *normed function space* if the following two conditions hold:

- $(\mathbf{n_4})$ if $f \in \mathcal{M}$, $g \in E$ and $|f| \leq |g|$ a.e., then $f \in E$ and $|f|_E \leq |g|_E$;
- $(\mathbf{n_5})$ $\chi_{[0,t]} \in E$ for all $t > 0$ ($\chi_{[0,t]}$ is the characteristic function of $[0, t]$).

Let E be a normed function space. For all $t > 0$ we define $\Psi_E(t) = |\chi_{[0,t)}|_E$ and $\Psi_E(\infty) = \lim\limits_{t \to \infty} \Psi_E(t)$. For further details about normed function spaces we refer to [5] and [6].

Theorem 1. *Let* $\mathbf{T}$ *be a* C_0*-semigroup on a Banach space* X *and let* E *be a normed function space over* $\mathbf{R}_+ = [0, \infty)$ *with* $\Psi_E(\infty) = \infty$. *If for each* $x \in X$ *the map* $t \to ||T(t)x||$ *belongs to* E, *then* $\omega_0(\mathbf{T}) < 0$.

3. Proof of Theorem 1.

We begin with the following lemma (see [3, Lemma 4.6]).

Lemma 2. *Let* $\mathbf{T}$ *be a* C_0*-semigroup on* X *with* $\omega_0(\mathbf{T}) \geq 0$. *Then there is a constant* $C > 0$ *with the following property. For all* $\gamma \in C_0(\mathbf{R}_+, \mathbf{C})$ *of norm one there exists a norm one vector* $x \in X$ *such that*

$$||T(t)x|| \geq C|\gamma(t)|, \quad \forall t \geq 0.$$

Proof. The proof of Lemma 2 uses a result of Müller (see [2]).

Let E and Ψ_E as in Theorem 1. First we prove that there exists a strictly increasing sequence $(t_n)_{n\geq 0}$ such that $|\chi_{[t_n,t_{n+1})}|_E \geq (n+1)^2$ for all $n \in \{0, 1, 2, \ldots\}$. Put $t_0 = 0$ and let $t_1 > 0$ be so large that $|\chi_{[t_0,t_1)}|_E \geq 1$. Inductively, suppose $t_1 < t_2 < \ldots < t_{n-1}$ have been chosen such that

$$|\chi_{[t_{k-1},t_k)}|_E \geq k^2, \quad k = 1, 2, \ldots, n-1.$$

Then

$$\begin{aligned} |\chi_{[t_{n-1},t)}|_E &= |\chi_{[0,t)} - \chi_{[0,t_{n-1})}|_E \\ &\geq |\chi_{[0,t)}|_E - |\chi_{[0,t_{n-1})}|_E, \end{aligned}$$

hence $\lim\limits_{t\to\infty} |\chi_{[t_{n-1},t)}|_E = \infty$. It follows that there exists $t_n > t_{n-1}$ such that $|\chi_{[t_{n-1},t_n)}|_E \geq n^2$. This completes the induction step.

Suppose, for a contradiction, that $\omega_0(\mathbf{T}) \geq 0$. Let $\gamma \in C_0(\mathbf{R}_+)$ such that $\gamma(t) \geq \frac{1}{n}$ for every $t \in [t_{n-1}, t_n)$. Let C and x as in Lemma 2.

Then for $n \in \{1, 2, \ldots\}$, we have

$$\begin{aligned} |||T(\cdot)x|||_E &\geq |\chi_{[t_{n-1},t_n)}||T(\cdot)x|||_E \\ &\geq C|\chi_{[t_{n-1},t_n)}\gamma|_E \\ &\geq \tfrac{C}{n}|\chi_{[t_{n-1},t_n)}|_E \\ &\geq Cn. \end{aligned}$$

It follows $|||T(\cdot)x|||_E = \infty$. This contradiction concludes the proof. □

Theorem 1 can be applied to various "weighted" normed function spaces. It is not hard to see that the condition $(\mathbf{n_1})$ is not used in the proof of Theorem 1.

Corollary 3. *Let $0 \leq \phi \in L^1_{loc}(\mathbf{R}_+)$ such that $\sup\limits_{t\geq 0} \int_t^{t+1} \phi(s)ds = \infty$ and let* $\mathbf{T}$ *be a C_0-semigroup on X. If*

$$\sup_{t\geq 0} \int_t^{t+1} \phi(s)||T(s)x||ds < \infty, \quad \forall x \in X$$

then $\omega_0(\mathbf{T}) < 0$.

Proof. It results by Theorem 1 for

$$E =: \{f \in L^1_{loc}(\mathbf{R}_+, \mathbf{C}) : \rho(f) = \sup_{t\geq 0} \int_t^{t+1} \phi(t)|f(t)|dt < \infty\}.$$

The conditions $(\mathbf{n_2}) - (\mathbf{n_5})$ are easy verified. Moreover, $\Psi_E(\infty) = \infty$. Indeed, if $t \geq 2$ then

$$\Psi_E(t) = \sup_{u\geq 0} \int_u^{u+1} \chi_{[0,t)}\phi(s)ds \geq \int_{t-1}^{t} \phi(s)ds\,,$$

hence

$$\Psi_E(\infty) \geq \lim_{t\to\infty} \int_{t-1}^{t} \phi(s)ds = \sup_{t\geq 2} \int_{t-1}^{t} \phi(s)ds = \infty. \qquad \square$$

Remark. Corollary 3 can be deduced directly from Theorem B. In fact, if ϕ is as in Corollary 3, then there exists a sequence $(t_n)_{n\geq 1}$ such that $\int\limits_{t_n}^{t_n+1} \phi(s)ds \geq n^2$. Define $\Psi \in L^1_{loc}(\mathbf{R}_+)$, by

$$\Psi = \sum_n \frac{1}{n^2}\chi_{[t_n,t_{n+1}]}\phi.$$

Then $\int\limits_{t_n}^{t_n+1} \phi(s)ds \geq 1$, and hence $\int\limits_0^\infty \psi(s)ds = \infty$. But also

$$\int\limits_0^\infty \psi(s)||T(s)x||ds = \sum_n \frac{1}{n^2}\int\limits_{t_n}^{t_n+1} \phi(s)||T(s)x||ds < \infty,$$

since, by assumption, the supremum over the integrals on the right hand side is finite. Now Theorem B can be applied to ψ.

Corollary 4. *Let $a > 0$ be a constant and $\mathbf{T}$ be a C_0-semigroup on X. If*

$$\sup_{t\geq 0}\int\limits_t^{t+1} s^a||T(s)x||ds < \infty, \quad \forall x \in X$$

then $\omega_0(\mathbf{T}) < 0$.

Proof. It results by Corollary 3 for $\phi(s) = s^a$. ☐

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