International Journal of Pure and Applied Mathematics
Volume 2 No. 3 2002, 311-321

HOMOGENIZATION OF HEAT CONDUCTION EQUATION IN DOMAIN CONTAINING SMALL SOURCES

Sanja Marušić

Dept. of Mathematics
Faculty of Transport and Traffic Engineering
University of Zagreb
Vukelićeva 4, 10 000 Zagreb, CROATIA
e-mail: marusics@mafpz.fpz.hr

Abstract: In this paper we consider the heat equation in a domain containing large number of small heat sources. We study the asymptotic behaviour as the number of sources tends to ∞ and their size tends to 0, using the method of homogenization. The homogenized equation is the same but it has a source term on the right-hand side. The result was proved under weak geometric assumptions (a)-(d).

AMS Subject Classification: 35B27, 35B40
Key Words: homogenization, heat equation, sources, weak geometric conditions

1. Introduction

1.1. The Geometry

The physical situation that we want to study can be described as a heat flow in a domain containing a large number of small heat sources. We neglect the process that goes on it those small grains and assume that they radiate heat in some prescribed manner. The problem reduces to the mixed problem for (linear) parabolic equation with non homogeneous Neumann boundary condition on grains. The same model could be used for some other kind of sources (e.g. radioactive).

Received: April 2, 2002

At the beginning, we do not assume that the distribution of sources is periodic but we impose the following conditions on Ω_ε:

- Let $\Omega \subset \mathbf{R}^n$, $n \geq 2$ be a bounded locally Lipschitz domain.
- Let $\varepsilon \ll 1$ be a small parameter. We denote by

$$Y_\varepsilon(x) = \prod_{i=1}^{n} \left] x_i - \frac{\varepsilon}{2}, x_i + \frac{\varepsilon}{2} \right[$$

Let $A_\varepsilon \subset \Omega$ be a smooth set presenting an array of small sources satisfying the following assumptions:

(a) The set A_ε is a union of bounded smooth domains T_ε^k, $k \in I^\varepsilon \subset \mathbf{Z}$. Its complement Ω_ε is connected.

(b) Each of the sets T_ε^k is assumed to be star-shaped with respect to some point x_ε^k, i.e. it can be described as $T_\varepsilon^k = \{x \in \mathbf{R}^n \; ; \; |x - x_\varepsilon^k| < h_k^\varepsilon(\frac{x-x_\varepsilon^k}{|x-x_\varepsilon^k|}) \}$. In order to have the nontrivial limit we suppose that $h_k^\varepsilon = \varepsilon^{\frac{n}{n-1}} h_k$, with h_k independent from ε. The functions h_k are of class C^1 on the unit sphere S^1 and, for ε small enough, satisfy the inequality[1]

$$|h_k^\varepsilon|_{C^1(S^1)} \leq M \tag{1}$$

with the same constant $M > 0$ for all k. We also suppose that

$$h_k(\omega) \geq c_0, \;\; \omega \in S^1 \tag{2}$$

again with same $c_0 > 0$ for all k.

(c) There exist $m \in \mathbf{N} \cup \{0\}$, $\varepsilon_0 > 0$ and $\sigma > 0$ such that for any $\varepsilon \leq \varepsilon_0$ and any $k \in I^\varepsilon$ each ball $B(x_\varepsilon^k, \sigma \cdot \varepsilon) = \{x \in \mathbf{R}^n \; ; \; |x - x_\varepsilon^k| < \sigma \cdot \varepsilon \}$ has nonempty intersection with at most m sets from the family $\{T_\varepsilon^k\}$.[2]

Those assumptions will be sufficient to pass to the limit, but only up to a subsequence. The limit is, in general, not unique and cannot be completely identified. We are looking for minimal assumptions that will guarantee the complete identification of the limit problem and, consequently, the uniqueness of the limit.

[1]Such condition is needed for the definition of the extension operator.
[2]The meaning of that assumption is that the distribution of T_ε^k's in Ω is not too "dense".

1.2. The Problem

We denote by $\Gamma_\varepsilon = \partial\Omega_\varepsilon \backslash \partial\Omega = \partial A_\varepsilon$. For given $T > 0$, we define $\Gamma_\varepsilon^T = \Gamma_\varepsilon \times]0, T[$, $\Gamma = \partial\Omega$, $\Gamma^T = \Gamma \times]0, T[$ and $\Omega_\varepsilon^T = \Omega_\varepsilon \times]0, T[$. Let $u_0 \in L^2(\Omega)$, $\Phi \in C^1(\Omega^T)$ and $f \in L^2(\Omega^T)$. We assume, in addition, that $A \in L^\infty(\Omega^T)^{n^2}$. Our problem now reads

$$\frac{\partial u_\varepsilon}{\partial t} - \operatorname{div} A\,\nabla u_\varepsilon = f \quad \text{in } \Omega_\varepsilon^T\,, \tag{3}$$

$$\frac{\partial u_\varepsilon}{\partial \nu_A} \equiv A\nabla u_\varepsilon \cdot \nu = \Phi \quad \text{on } \Gamma_\varepsilon^T\,, \tag{4}$$

$$u_\varepsilon = 0 \quad \text{on } \Gamma^T\,, \tag{5}$$

$$u_\varepsilon(\,\cdot\,,0) = u_0 \quad \text{on } \Omega_\varepsilon\,, \tag{6}$$

where ν stands for the exterior unit normal on Γ_ε. We assume that $A(x,t)$ is positive for (a.e.) $(x,t) \in \Omega^T$, i.e. there exist $M, m > 0$ such that

$$m|\xi|^2 \le A(x,t)\xi \cdot \xi \le M|\xi|^2, \;\; \xi \in \mathbf{R}^n, \;(a.e)\; (x,t) \in \Omega^T. \tag{7}$$

We say that u_ε is a weak solution of (3)-(6) if

$$\int_{\Omega_\varepsilon^T} \{-u_\varepsilon \frac{\partial \psi}{\partial t} + A\,\nabla u_\varepsilon\,\nabla\psi - f\,\psi\} = \int_{\Gamma_\varepsilon^T} \Phi\,\psi + \int_{\Omega_\varepsilon} u_0\,\psi(\,\cdot\,,0), \quad \text{for all } \psi \in V, \tag{8}$$

$V = \{v \in H^1(0,T;H^1(\Omega_\varepsilon))\;;\; \psi(\,\cdot\,,T) = 0,\; \psi = 0 \text{ on } \Gamma^T\,\}$. It is well known that such problem admits a unique weak solution $u_\varepsilon \in L^2(0,T;H^1(\Omega_\varepsilon)) \cap L^\infty(0,T;L^2(\Omega_\varepsilon))$ (see e.g. [3]).

2. Homogenization

Using the assumptions (a)-(c) describing the geometric distribution of sources, we study the limit of u_ε as $\varepsilon \to 0$.

2.1. Convergence in the General Case

We begin by noticing that the assumption (b) permits to extend u_ε on the whole Ω in the way that such extension, denoted for simplicity by the same symbol, satisfies

$$\|u_\varepsilon\|_{L^2(0,T;H^1(\Omega_\varepsilon))} \le C\|u_\varepsilon\|_{L^2(0,T;H^1(\Omega))},$$

with some $C > 0$ independent from ε (see [4]).

Our first step is to derive the a priori estimate. To do so we need the following estimate:

Lemma 1. *There exists a constant $C > 0$ such that for any $\psi \in H^1(\Omega)$*

$$\|\psi\|^2_{L^2(\Gamma_\varepsilon)} \leq C(\|\psi\|^2_{L^2(\Omega)} + \lambda(\varepsilon, n)\,\|\nabla\psi\|^2_{L^2(\Omega)}),$$

where

$$\lambda(\varepsilon, n) = \begin{cases} \varepsilon^2 \log(1/\varepsilon), & \text{if } n = 2\,, \\ \varepsilon^2, & \text{if } n > 2\,. \end{cases}$$

Proof. We give the proof in case $n = 2$ but the case $n > 2$ is completely analogous. Let $\psi \in C^1(\overline{\Omega})$. We denote by $\Sigma^k_\varepsilon = \partial T^k_\varepsilon$ so that

$$\Sigma^k_\varepsilon = \{x = (r\,\cos\,\varphi, r\,\sin\,\varphi) \in \mathbf{R}^2\ ;\ \varphi \in [0, 2\,\pi\,[,\ r = \varepsilon^2\, h(\varphi)\}.$$

Then

$$\psi(\varepsilon^2\, h(\varphi), \varphi) = \int_r^{\varepsilon^2 h} \frac{\partial\psi}{\partial r}(\rho, \varphi)\, d\rho + \psi(r, \varphi),$$

where (r, φ) are the polar coordinates and we denote shorter $\psi(r, \varphi) = \psi(r\,\cos\,\varphi, r\,\sin\,\varphi)$. Thus the Cauchy inequality gives

$$\psi(\varepsilon^2\, h(\varphi), \varphi)^2 = 2\int_{\varepsilon^2 h}^{\sigma\,\varepsilon} \left(\frac{\partial\psi}{\partial r}(\rho, \varphi)\right)^2 \rho\, d\rho\, \int_{\varepsilon^2 h}^{\sigma\varepsilon} \frac{d\rho}{\rho} + 2\,\psi(r, \varphi)^2.$$

We now multiply the above inequality by $\varepsilon^2\, \sqrt{h^2 + (h')^2}$ and integrate with respect to φ over $[0, 2\pi]$. After noticing that

$$\int_{\Sigma^k_\varepsilon} \psi^2 = \varepsilon^2\, \int_0^{2\pi} \psi(\varepsilon^2\, h(\varphi), \varphi)^2\, \sqrt{h^2 + (h')^2}\, d\varphi$$

we get

$$\int_{\Sigma^k_\varepsilon} \psi^2 \leq C\varepsilon^2 (\int_0^{2\pi} \psi(r, \varphi)^2\, d\varphi + \log\,(1/\varepsilon)\,|\nabla\psi|^2_{L^2(B(x_\varepsilon, \sigma\varepsilon))}).$$

Multiplying by r and integrating with respect to r from 0 to $\sigma\,\varepsilon$ we get

$$\int_{\Sigma^k_\varepsilon} \psi^2 \leq C(|\psi|^2_{L^2(B(x^k_\varepsilon, \sigma\varepsilon))} + \lambda(\varepsilon, 2)\,|\nabla\psi|^2_{L^2(B(x_\varepsilon, \sigma\varepsilon))}).$$

Now the summation with respect to k and the assumption (c) imply the claim. □

We are now ready to prove:

Lemma 2. *There exists a constant $C > 0$ independent on ε such that*

$$|u_\varepsilon|_{L^2(0,T;H^1(\Omega_\varepsilon))} \leq C\,, \tag{9}$$

$$|u_\varepsilon|_{L^\infty(0,T;L^2(\Omega_\varepsilon))} \leq C\,. \tag{10}$$

Proof. We start from the variational equality

$$\begin{aligned}
&\frac{1}{2}\,\frac{d}{dt}\|u_\varepsilon(\,\cdot\,,t)\|^2_{L^2(\Omega_\varepsilon)} + (A\nabla u_\varepsilon(\,\cdot\,,t))|\nabla u_\varepsilon(\,\cdot\,,t))_{L^2(\Omega_\varepsilon)}\\
&= (f(\,\cdot\,,t)|u_\varepsilon(\,\cdot\,,t))_{L^2(\Omega_\varepsilon)} + \int_{\Gamma_\varepsilon} \Phi(\,\cdot\,,t)\,u_\varepsilon(\,\cdot\,,t)
\end{aligned}$$

and integrate over $[0,s]$ with respect to t. Then, using the positivity of A and Lemma 1, we get the claim. □

2.2. The Limit

Our next step is to pass to the limit as $\varepsilon \to 0$. One should notice that, without additional assumptions on geometry, we do not have a unique limit and we can prove the convergence only up to a subsequence. We have:

Theorem 1. *There exists a subsequence, denoted for simplicity by the same symbol $\{u_\varepsilon\}$ and a functional $\ell \in L^2(0,T;H^{-1}(\Omega))$, such that $u_\varepsilon \rightharpoonup u$ weakly in $L^2(0,T;H^1(\Omega))$ and weak* in $L^\infty(0,T;L^2(\Omega))$, where u is a weak solution to the homogenized problem*

$$\frac{\partial u}{\partial t} - div\, A(x,t)\,\nabla u = f + \ell \quad in\ \ \Omega^T\,, \tag{11}$$

$$u = 0 \quad on\ \ \Gamma^T\,, \tag{12}$$

$$u(\,\cdot\,,0) = u_0 \quad on\ \ \Omega\,. \tag{13}$$

Proof. We first notice that A_ε is vanishing and therefore

$$\lim_{\varepsilon\to 0} \|u_\varepsilon\|_{L^2(0,T;H^1(A_\varepsilon)} = 0. \tag{14}$$

It remains to pass to the limit in the variational equation (8). Passage to the limit through all terms except the last one is straightforward. For instance

$$\int_{\Omega_\varepsilon} A\,\nabla u_\varepsilon \nabla\psi = \int_\Omega A\nabla u_\varepsilon \nabla\psi + o(\varepsilon) \to \int_\Omega A\nabla u\nabla\psi.$$

The last term can be written as

$$\int_0^T \int_{\Gamma_\varepsilon} \Phi\,\psi = \int_0^T \quad {}_{H^{-1}(\Omega)}\langle\, \delta_{\Gamma_\varepsilon} \mid \Phi\,\psi\rangle_{H_0^1(\Omega)}.$$

The functional $\delta_{\Gamma_\varepsilon} \in L^2(0,T;H^{-1}(\Omega))$ is bounded due to the Lemma 1. Therefore, there exists some $\ell \in L^2(0,T;H^{-1}(\Omega))$ such that, up to a subsequence,

$$\delta_{\Gamma_\varepsilon} \rightharpoonup \ell \quad \text{weakly in } \; L^2(0,T;H^{-1}(\Omega))$$

Furthermore $\delta_{\Gamma_\varepsilon} \geq 0$ so that $\delta_{\Gamma_\varepsilon} \to \ell$ strongly in $(W^{1,q}(\Omega^T))'$, $q > 2$ (see [5]). □

3. Homogenization under Additional Geometric Assumptions

The main goal of this section is to see under which additional assumption on geometry of Ω_ε we can identify the limit functional ℓ and find the unique limit u. We start with some classical examples.

3.1. Examples

Example 1. First example of Ω_ε is the periodic one. We suppose that the array A_ε is ε-periodic and consists of small inclusions with surface that has (surface) measure of order ε^n. More precisely

$$A_\varepsilon = \bigcup_{\alpha\in J_\varepsilon} \varepsilon\, B_\varepsilon^\alpha, \quad B_\varepsilon^\alpha = \alpha + B_\varepsilon\,,$$
$$J_\varepsilon = \{k \in \mathbf{Z}^n\ ;\ \Omega \cap Y_\varepsilon(\varepsilon\, k) \neq \emptyset\}.$$

where $B_\varepsilon = \varepsilon^{\frac{1}{n-1}}\, B$ and $B \subset]-1/2, 1/2[^n$ is C^1 domain, star-shaped with respect to the origin with the strictly positive shape function h. Now B_ε has a (volume) measure $O(\varepsilon^{\frac{n}{n-1}})$ and ∂B_ε has a (surface) measure $O(\varepsilon)$. In that case

$$\int_{\Gamma_\varepsilon^T} \Phi\,\psi = \int_0^T \sum_{k\in J_\varepsilon} \int_{\Sigma_\varepsilon^k} \Phi\,\psi\ ,$$

where $\Sigma_\varepsilon^k = \partial(\varepsilon\, B_\varepsilon^k\,)$. Then we have, for $x_\varepsilon^k = \varepsilon\, k$,

$$\int_{\Sigma_k^\varepsilon} \psi(\,\cdot\,,t) = \kappa^\varepsilon\,\left[\,\psi(x_\varepsilon^k,t) + o(\varepsilon)\,\right],$$

where $\kappa^\varepsilon = meas(\varepsilon\,\partial B_\varepsilon) = \varepsilon^n\; meas(\partial B)$ and $meas(\,.\,)$ denotes the $n-1$ dimensional measure of the surface in brackets. Now, denoting $\kappa^0 = meas(\partial B)$ and noticing that $\varepsilon^n = meas(Y_\varepsilon(x_\varepsilon^k)\,)$, we get

$$\int_{\Gamma_\varepsilon^T} \Phi\;\psi = \int_0^T \sum_{k\in J_\varepsilon} \kappa^0\;\Phi(x_\varepsilon^k,t)\,\psi(x_\varepsilon^k,t)\;meas(Y_\varepsilon(x_\varepsilon^k)\,) + o(\varepsilon) \quad ,$$

but the first term on the right-hand side is exactly the Riemmannian integral sum for $\int_\Omega \Phi\;\kappa^0\;\psi$ so that $\langle \ell|\psi\rangle = meas(\partial B)\int_{\Omega^T}\;\Phi\,\psi$.

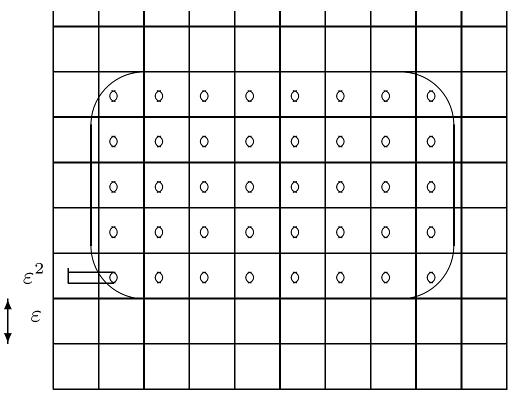

Figure 1: Ω_ε in the periodic case

Since the problem

$$\frac{\partial u}{\partial t} - \operatorname{div} A(x,t)\,\nabla u = f + \kappa^0\,\Phi \quad \text{in}\;\; \Omega^T\,, \tag{15}$$

$$u = 0 \quad \text{on}\;\; \Gamma^T\,, \tag{16}$$

$$u(\,\cdot\,,0) = u_0 \quad \text{on}\;\; \Omega. \tag{17}$$

obviously has a unique solution, Theorem 1 now implies that:

Lemma 3. *The whole sequence $\{u_\varepsilon\}$ converges to u, the unique solution of the homogenized problem (15)-(17), weakly in $L^2(0,T;H^1(\Omega))$ and weak* in $L^\infty(0,T;L^2(\Omega))$.*

Example 2. Second example is the quasi-periodic case that can be described using the ideas from [1] (see also [4]). Obviously, it is a generalization of the first example. Here the surface Γ_ε is a quasi-periodic. More precisely, let $\phi_\varepsilon : \Omega\times]-1/2,1/2[^n\to \mathbf{R}$ be 1-periodic in second variable, smooth and $\phi_\varepsilon(x,0) = -1$, while $\phi_\varepsilon \geq c > 0$ on $\Omega \times \partial([-1/2,1/2]^n)$. Then $A_\varepsilon = \{x \in \Omega \ ; \ \phi_\varepsilon(x,\frac{x}{\varepsilon}) < 0 \ \}$ and $\Gamma_\varepsilon = \{x \in \Omega \ ; \ \phi_\varepsilon(x,\frac{x}{\varepsilon}) = 0 \ \}$. To satisfy the conditions (a)-(c) we need the perforations smaller then the (quasi-) period ε (their diameter needs to be of order $O(\varepsilon^{\frac{n}{n-1}})$). More precisely, to describe the dependence of ϕ_ε on the parameter ε, we assume that the surface $S_\varepsilon(x) = \{y \in]-1/2,1/2[^n \ ; \ \phi_\varepsilon(x,y) = 0 \ \}$ has the form $S_\varepsilon(x) = \varepsilon^{\frac{1}{n-1}}\ \Gamma(x)$, where $\Gamma(x)$, $x \in \Omega$ is a family of smooth star-shaped surfaces satisfying (1), (2) and not depending on ε. In such case, denoting $\kappa^\varepsilon(x) = \varepsilon^n\ meas(\Gamma(x))$, and $\kappa^0(x) = meas(\Gamma(x))$, we get again Lemma 3.

Example 3. We call the following example the case of periodic surface measure. We assume that the condition (b) holds for the connected components of A_ε and that we can cover A_ε with an ε-net with nods $\varepsilon\, k$, $k \in \mathbf{Z}^n$ such that the grain (or grains) T_ε^j contained in each mash $Y_\varepsilon(k\,\varepsilon)$ of the net has the same surface measure, i.e. $meas(Y_\varepsilon(k\,\varepsilon) \cap A_\varepsilon) = K\,\varepsilon^n$, $K = const.$. That means that the grains in each mash are neither necessarily placed in the same spot, nor they need to have the same shape, but their surfaces must have the same measure. In such case, denoting

$$\kappa^\varepsilon(x) = \varepsilon^n\ K, \quad \kappa^0 = K$$

we again have Lemma 3.

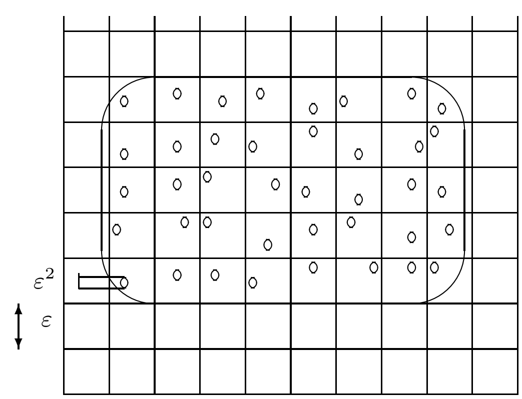

Figure 2: Ω_ε in the periodic surface measure case

3.2. General Case

Obviously, the same existence and uniqueness result, as for the original problem (3)-(6) applies that, for $\kappa^0 \in L^2(\Omega)$, the homogenized problem

$$\frac{\partial u}{\partial t} - \operatorname{div} A(x,t)\,\nabla u = f + \kappa^0\,\Phi \quad \text{in} \ \ \Omega^T\,, \tag{18}$$

$$u = 0 \quad \text{on} \ \ \Gamma^T\,, \tag{19}$$

$$u(\,\cdot\,,0) = u_0 \quad \text{on} \ \ \Omega \tag{20}$$

admits a unique solution.

Closer look to the three previous examples shows that the key role in identification of ℓ was played by the function

$$\kappa^\varepsilon(x) = meas(\Gamma_\varepsilon \cap Y_\varepsilon(x)\ )$$

and its convergence as $\varepsilon \to 0$. Therefore, we set the following abstract assumptions, obviously satisfied in three previous examples:

(d) Let $x_{\varepsilon,\alpha} = \alpha\,\varepsilon$, $\alpha \in \mathbf{Z}^n$. Let $\kappa^\varepsilon(x_{\alpha,\varepsilon}) = meas(\Gamma_\varepsilon \cap Y_\varepsilon(x_{\alpha,\varepsilon})\ )$, where $meas(\,\cdot\,)$ denotes the $n-1$-dimensional Lebesgue measure of the surface in parenthesis. We assume that there exists a function κ^0, Riemman integrable on Ω. Furthermore, for sufficiently small ε and all $\alpha \in J_\varepsilon = \{\alpha \in \mathbf{Z}^n\ ;\ \Omega \cap\ Y_\varepsilon(\varepsilon\,\alpha) \neq \emptyset\}$, there exists $C(\varepsilon) > 0$ such that

$$|\varepsilon^{-n}\,\kappa^\varepsilon(x_{\alpha,\varepsilon}) - \kappa^0(x_{\alpha,\varepsilon})| \le C(\varepsilon) \quad \text{and} \quad \lim_{\varepsilon\to 0} C(\varepsilon) = 0.$$

The essential part of the convergence process is contained in the following generalization of Lemma 3:

Lemma 4. *Suppose that the assumptions (a)-(d) hold. Let $\psi \in C^\infty(\Omega^T)$. Then*

$$\int_{\Gamma_\varepsilon^T} \Phi\ \psi \to \int_{\Omega^T} \Phi\ \kappa^0\ \psi$$

as $\varepsilon \to 0$.

Proof. Let $\Sigma_\alpha^\varepsilon = \Gamma_\varepsilon \cap Y_\varepsilon(\varepsilon\,\alpha)$. We write

$$\int_{\Gamma_\varepsilon^T} \Phi\ \psi = \int_0^T \sum_{\alpha\in J_\varepsilon} \int_{\Sigma_\alpha^\varepsilon} \Phi\,\psi\ .$$

Then we have

$$\int_{\Sigma_\alpha^\varepsilon} \Phi(\,\cdot\,,t)\,\psi(\,\cdot\,,t) = \kappa^\varepsilon(x_{\varepsilon,\alpha})\,[\,\Phi(x_{\varepsilon,\alpha},t)\,\psi(x_{\varepsilon,\alpha},t) + o(\varepsilon)\,].$$

Now we use the hypothesis (d) to obtain

$$\varepsilon^{-n}\kappa^\varepsilon(x_{\varepsilon,\alpha}) = \kappa^0(x_{\varepsilon,\alpha}) + o(\varepsilon),$$

where $o(\varepsilon) \to 0$ uniformly with respect to $\alpha \in J_\varepsilon$. The rest of the proof is the same as in the periodic case solved in Lemma 3. □

We have proved our main result:

Theorem 2. *Let u_ε be a solution of a problem (3)-(6) and u the (unique) solution of the homogenized problem (18)-(20). If the assumptions (a)-(d) hold, then*

$$u_\varepsilon \rightharpoonup u \quad \text{weakly in } \; L^2(0,T;H^1(\Omega_\varepsilon)) \quad \text{and weak* in } \; L^\infty(0,T;L^2(\Omega)).$$

Remark 1. The result is easy to generalize on the case when grains are radiating heat according to the Fourier law [2], which can be described by the boundary condition

$$\frac{\partial u_\varepsilon}{\partial \nu_A} = \lambda(\Phi - u_\varepsilon).$$

The only difference is that, in addition, we have to pass to the limit in integral of the type $\int_{\Gamma_\varepsilon^T} \psi\, u_\varepsilon$. This is easy to handle since we have from our equation that

$$\left|\frac{\partial u_\varepsilon}{\partial t}\right|_{L^2(0,T;H^{-1}(\Omega))} \le C$$

and, due to the Dubinskii theorem $u_\varepsilon \to u$ strongly in $L^2(\Omega^T)$. Thus Lemma 1 implies

$$\left|\int_{\Gamma_\varepsilon^T} \psi\,(u_\varepsilon - u)\right| \le C(|u_\varepsilon - u|_{L^2(\Omega^T)} + o(\varepsilon)\,|\nabla(u_\varepsilon - u)|_{L^2(\Omega^T)}) \to 0.$$

In that case the homogenized equation reads

$$\frac{\partial u}{\partial t} - \operatorname{div} A(x,t)\,\nabla u = f + \kappa^0\,\lambda(\Phi - u) \quad \text{in } \; \Omega^T.$$

References

[1] G.A. Chechkin, A.L. Piatnitski, Homogenization of boundary value problem in a locally periodic perforated domain, *Applicable Anal.*, **71** (1999), 215-235.

[2] J.B. Fourier, *Théorie Analytique de la Chaleur*, Didrot (1822).

[3] J.L. Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites non Linéaires*, Dunod (1969).

[4] M.L. Mascarenhas, D. Poliševski, The warping, the torsion and the Neumann problems in a quasi-periodically perforated domain, *RAIRO Math. Mod. Num. Anal.*, **28** (1994), 37-57.

[5] F. Murat, L'injection du cône positif de H^{-1} dans $W^{-1,q}$ est compacte pour tout $q < 2$, *J. Math. Pures et Appl.*, **60** (1981), 309-322.