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A NOTE ON THE TAFT'S PROBLEM

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Abstract: Let F be a field of characteristic zero. In this paper we work out the linearly recursive relation on Lie multiplication $[f, g]$ in Witt algebras $(W_1^{(i)})^o$(resp. $(W^{(i)})^o$). This is an open problem proposed by Earl J. Taft. We show that if the characteric polynomial $p(x)$(resp. $q(x)$) of f(resp. g)$\in (W^{(i)})^o$ or $(W_1^{(i)})^o$ satisfy $p(x)|(x^i - a^i)$ and $q(x)|(x^i - a^i)$ for a in the algebraically closure of F, then $[f, g]$ satisfies $LCM(p(x), q(x))$,the least common multiple of $p(x)$ and $q(x)$. Some examples illustrate the results.

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1. Introduction

Let F be a field of characteristic zero. In this paper we describe some special recursive relations on Lie multiplications in Lie duals of the (one-sided) Witt algebra in one variable $W_1 = \operatorname{Der} F[x]$and of the 2-sided Witt algebra(or Vi-

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rasoro algebra without central charge) in one variable $W = \mathrm{Der}\, F[x, x^{-1}]$. For the general case, see [2] for more details. The aim of this paper is to consider the case, in which the characteric polynomial of f, g have only singular root, and for any root a of characteric polynomial of f, b of characteric polynomial of g satisfy $a^i = b^i$. The Lie dual structure of W and W_1 have been studyed by Michaelias [3, 4, 6], Nichols [7, 8] and Taft [13, 14, 15, 16].

Recall that a *Lie algebra* L over F has a skew-symmetric multiplication [,] satisfying the Jacobi identity. Reversing the arrows, a *Lie coalgebra* M over F has a comultiplication δ from M to $M \wedge M$, the skew-symmetric tensors in $M \otimes M$, which satisfies the co-Jacobi identity $(1 + \sigma + \sigma^2)(1 \otimes \delta)\delta = 0$, where σ is the permutation (123) in S_3 acting in the usual way on $M \otimes M \otimes M$. A Lie algebra L, which is simultaneously a Lie coalgebra is called a *Lie bilagebra* if $\delta \in Z^1(L, L \wedge L)$. If $\delta = \delta_r \in B^1(L, L \wedge L)$ for some $r \in L \wedge L$, L is called a *coboundary Lie bialgebra*. The condition is that $\delta_r(x) = [r, x]$ for all $x \in L$.

Every Lie algebra L over F has a dual Lie coalgebra L^o, which is the sum of the good subspaces of L^*. A subspace V of L^* is *good* if the map $L^* \to (L \otimes L)^*$ dual to the Lie multiplication of L takes V to $V \otimes V$(see [3, 15] for moe details).

Let Witt algebras $W_1 = \mathrm{Der}\, F[x]$, the Lie algebra of derivations of the polynomial algebra $F[x]$. W_1 has a basis $\{e_i\}$ for $i \geq -1$, where $e_i = x^{i+1}d/dx$ and $[e_i, e_j] = (j - i)e_{i+j}$. We identity W_1^* with sequences $(f_i)_{i \geq -1}$, where $f \leftrightarrow (f_i)$ means $f_i = f(e_i)$, then W_1^o has been identified as the space of linearly recursive sequences (see [7] for details). Similarly, for (full) Witt algebras $W = \mathrm{Der}\, F[x, x^{-1}]$, W has a basis $\{e_i\}$ for $i \in \mathbf{Z}$, where $e_i = x^{i+1}d/dx$ and $[e_i, e_j] = (j - i)e_{i+j}$. We identity W^o as the space of back-solving linearly recursive sequences. The sequence $(f_i)_{i \in \mathbf{Z}}$ is $back - solving$ linearly recursive sequences if f_i satisfying the recursive relation over F.

The various Lie coalgebra structure $W_1^{(i)}$ for $i \geq -1$, which are maturally non-isomorphic, are construct in [15] as follows. Let $r_i = e_0 \wedge e_i$. r_i satisfy the classical Yang-Baxter equation(CYBE) for W_1, i. e. , $r_i \in W_1 \otimes W_1$ is a solution of the triple tensor product condition

$$(CYBE) \quad [r_i^{12}, r_i^{13}] + [r_i^{12}, r_i^{23}] + [r_i^{13}, r_i^{23}] = 0\,,$$

in $W_1 \otimes W_1 \otimes W_1$. The notation is that if $r_i = \sum a_j \otimes b_j$, then $r_i^{12} = \sum a_j \otimes b_j \otimes 1$, $r_i^{13} = \sum a_j \otimes 1 \otimes b_j$ and $r_i^{23} = \sum 1 \otimes a_j \otimes b_j$. Thus, $W_1^{(i)} = (W_1, \delta_{r_i})$ is a triangular coboundary Lie bialgebra (see [1] and [15] Proposition 1). In [9, 10], Ng and Taft show that Taft's Lie bialgebra structure $W_1^{(i)}$ on W_1 are all of the Lie bialgebra structures on W_1 up to isomorphism when F is algebraically closed of characteristic zero. We assume $i \in \mathbf{Z}^+$, as $\delta_0 = 0$ gives $(W_1^{(0)})^o$ the structure of an abelian Lie algebra. Let $\delta_i = \delta_{r_i}$, i. e, $\delta_i(x) = [e_0 \wedge e_i, x]$. Then $\delta_i(e_n) = n(e_n \wedge e_i) + (n-i)(e_0 \wedge e_{n+i})$ be the Lie cobracket in $W_1^{(i)}$. The Lie multiplication in $(W_1^{(i)})^o$ is described by

$$[e_0^*, e_n^*] = (n-2i)e_{n-i}^*, \qquad for \;\; n \neq 0\,,$$

$$[e_n^*, e_i^*] = ne_n^*, \qquad for \;\; n \neq 0, i\,,$$

with all other Lie multiplication of the e_n^* being zero. Thus let $f = \sum a_n e_n^*, g = \sum b_m e_m^* \in (W_1^{(i)})^o$,

$$[f,g] = \sum c_p e_p^*,$$

where

$$c_p = p(a_0 b_{p+i} - b_0 a_{p+i} + b_i a_p - a_i b_p) + i(a_{p+i} b_0 - a_0 b_{p+i}) \tag{1}$$

The formulas (1) is very important for obtain the algorithm on recursive relations of $[f,g] \in (W_1^{(i)})^o$. For $i \in \mathbf{Z}$, a similar discussion is possible for the 2-sided Witt algebra W. Let $W^{(i)}$ is the Lie bialgebra (W, δ_i) with $\delta_i(x) = [e_0 \wedge e_i, x]$ for $i \in \mathbf{Z}$. The formulas (1) is also held for $(W^{(i)})^o$. Note that $e_k* \notin (W^{(i)})^o$, we will use the basis $(a^i i^n)_{i \geq -1}$ for $a \in F^\times$ and $n \in \mathbf{N}$.

In [13, 14, 15], Taft proposes an open problem on finding an algorithm for multiplying two given linearly recursive sequences under the above Lie multiplication. Let F be an algebraically closed field of characteristic zero, we have consider the Taft's problem in [2]. We show that for $f, g \in (W_1^{(i)})^o$(resp. $(W^{(i)})^o$) with $i \neq 0$, $[f,g]$ satisfies

$$x^{max\{r_0,s_0\}+1}(x-a_1)^{max\{r_1,s_1\}+1} \cdots (x-a_k)^{max\{r_k,s_k\}+1}(x-c_1)^{r_{k+1}+1} \cdot$$
$$\cdots (x-c_n)^{r_{k+n}+1} \cdot (x-d_1)^{s_{k+1}+1} \cdots (x-d_m)^{s_{k+m}+1}$$
$$(resp.\;\; (x-a_1)^{max\{r_1,s_1\}+1} \cdots (x-a_k)^{max\{r_k,s_k\}+1}(x-c_1)^{r_{k+1}+1} \cdot$$
$$\cdots (x-c_n)^{r_{k+n}+1} \cdot (x-d_1)^{s_{k+1}+1} \cdots (x-d_m)^{s_{k+m}+1})$$

in general, where the characteric polynomial of f is $p(x) = x^{r_0}(x-a_1)^{r_1} \cdots (x-a_k)^{r_k}(x-c_{k+1})^{r_{k+1}} \cdots (x-c_{k+n})^{r_{k+n}}$(resp. $p(x) = (x-a_1)^{r_1} \cdots (x-a_k)^{r_k}(x-c_{k+1})^{r_{k+1}} \cdots (x-c_{k+n})^{r_{k+n}}$) and the characteric polynomial of g is $q(x) = x^{s_0}(x-a_1)^{s_1} \cdots (x-a_k)^{s_k}(x-d_{k+1})^{s_{k+1}} \cdots (x-d_{k+m})^{s_{k+m}}$(resp. $q(x) = (x-a_1)^{s_1} \cdots (x-a_k)^{s_k}(x-d_{k+1})^{s_{k+1}} \cdots (x-d_{k+m})^{s_{k+m}}$), and $\deg(p(x)) > 0, \deg(q(x)) > 0$.

Where $a_1, \cdots, a_k, c_{k+1}, \cdots, c_{k+n}, d_{k+1}, \cdots, d_{k+m}$ are distinct in $F^\times$, $r_0, s_0 \in \mathbf{N}$ and $r_1, \cdots, r_{k+n}, s_1, \cdots, s_{k+m} \in \mathbf{Z}^+$. But for the case, which the root a(resp. b) of the characteric polynomial of f(resp. g)$\in$ $(W_1^{(i)})^o$ (or $(W^{(i)})^o$) satisfy

$a^i = b^i$, it is more complex. In this paper, we will give a more explicit algorithm for the recursive relations on $[f, g] \in (W_1^{(i)})^o$, $(W^{(i)})^o$ in this case.

Throughtout, the set of non-zero elements of F is denoted $F^\times$. We use $\mathbf{Z}$ denote integers, $\mathbf{N}$ for the non-negative integers, $\mathbf{N}_{-1}$ for the integers greater than -1, and $\mathbf{Z}^+$ for the positive integers. In Section 2, 3, 4 and 5(except for the last section, Section 6), we assume that F is an algebraically closed field. See [11, 16] for a development of the Hopf algebraic structure of linearly recursive sequences and see [12] for Hopf algebra and coalgebra backgroud.

Lemma 1. *Let F be an algebraic closure filed. If $f = \{f_j\} \in (W_1^{(i)})^o$ is a linearly recursive sequence with the characteric polynomial $p(x) = (x - a_1)\cdots(x - a_n)$ and $a_1, \cdots, a_n$ are distinct in $F^\times$ then*

$$f_j = t_1 a_1^j + \cdots + t_n a_n^j,$$

for $j \in \mathcal{Z}$.

Lemma 2. *Let $\{a^j\}, \{b^k\} \in (W^{(i)})^o$ for $j, k \in \mathcal{Z}$, where $a, b \in F^\times$ such that $a^i = b^i$. Then $[\{a^j\}, \{b^k\}] = 0$ for $a = b$ and $[\{a^j\}, \{b^k\}]$ satisfies $(x - a)(x - b)$ for $a \neq b$.*

Proof. If $a = b$ then $\{a^j\} = \{b^k\}$. Thus $[\{a^j\}, \{b^k\}] = 0$.

Now let $a \neq b$. By the formulas $c_p = [\{a^j\}, \{b^k\}]_p$ of (1)

$$c_p = p(b^{p+i} - a^{p+i} + b^i a^p - a^i b^p) + i(a^{p+i} - b^{p+i}).$$

Since $a^i = b^i$,

$$\begin{aligned} c_p &= p(b^p a^i - a^p b^i + b^i a^p - a^i b^p) + i(a^{p+i} - b^{p+i}) \\ &= i(a^{p+i} - b^{p+i}). \end{aligned}$$

So $[\{a^j\}, \{b^k\}] = \{c_p\}$ satisfies $(x - a)(x - b)$. This completes the proof of Lemma. □

Lemma 3. *Let f, g be the linearly recursive sequences. Let the characteric polynomial of f be $p(x)$ and the characteric polynomial of g be $q(x)$ with* $\deg(p(x)) > 0, \deg(q(x)) > 0$. *Then $f + g$ satisfies $LCM(p(x), q(x))$, the least common multiple of $p(x)$ and $q(x)$.*

Now we can proof our main result.

Theorem 4. *Let F be an algebraically closed field. Let $f, g \in (W^{(i)})^o$ and f with the characteric polynomial of $p(x) = (x - a_1)\cdots(x - a_k)(x - c_1)\cdots(x - c_n)$, g with the characteric polynomial of $q(x) = (x - a_1)\cdots(x -$

$a_k)(x-d_1)\cdots(x-d_m)$. If $a_1,\cdots,a_k,c_1,\cdots,c_n,\ d_1,\cdots,d_m$ are distinct in $F^\times$ and satisfy $a_\alpha^i = b_\beta^i = c_\gamma^i$ for $1 \le \alpha \le k, 1 \le \beta \le n, 1 \le \gamma \le m$, then $[f,g]$ satisfies

$$(x-a_1)\cdots(x-a_k)(x-c_1)\cdots(x-c_n)(x-d_1)\cdots(x-d_m).$$

Proof. By Lemma 1,

$$f = \{f_j\} = \{t_1 a_1^j + \cdots + t_k a_k^j + t_{k+1} c_1^j + \cdots + t_{k+m} c_n^j\},$$

and

$$g = \{g_{j_1}\} = \{s_1 a_1^{j_1} + \cdots + s_k a_k^{j_1} + s_{k+1} d_1^{j_1} + \cdots + s_{k+m} d_m^{j_1}\}.$$

Thus,

$$\begin{aligned}
[f,g] &= [\{t_1 a_1^j + \cdots + t_k a_k^j + t_{k+1} c_1^j + \cdots + t_{k+n} c_n^j\}, \{s_1 a_1^{j_1} + \cdots \\
&\quad + s_k a_k^{j_1} + s_{k+1} d_1^{j_1} + \cdots + s_{k+m} d_m^{j_1}\}] \\
&= \textstyle\sum_{u=1}^{k}\sum_{v=1}^{k} t_u s_v [\{a_u^j\}, \{a_v^{j_1}\}] \\
&\quad + \textstyle\sum_{u=1}^{k}\sum_{v=1}^{m} t_u s_{k+v} [\{a_u^j\}, \{d_{k+v}^{j_1}\}] \\
&\quad + \textstyle\sum_{u=1}^{n}\sum_{v=1}^{k} t_{k+u} s_v [\{c_{k+u}^j\}, \{a_v^{j_1}\}] \\
&\quad + \textstyle\sum_{u=1}^{n}\sum_{v=1}^{m} t_{k+u} s_{k+v} [\{c_{k+u}^j\}, \{d_{k+v}^{j_1}\}] \\
&= \text{sum } (1) + \text{sum } (2) + \text{sum } (3) + \text{sum } (4).
\end{aligned}$$

By Lemma 2, the sum (1) satisfies

$$(x-a_1)\cdots(x-a_k),$$

the sum (2) satisfies

$$(x-a_1)\cdots(x-a_k)(x-d_1)\cdots(x-d_m),$$

the sum (3) satisfies

$$(x-a_1)\cdots(x-a_k)(x-c_1)\cdots(x-c_n),$$

the sum (4) satisfies

$$(x-c_1)\cdots(x-c_n)(x-d_1)\cdots(x-d_m).$$

So $[f,g]$ satisfies

$$(x-a_1)\cdots(x-a_k)(x-c_1)\cdots(x-c_n)(x-d_1)\cdots(x-d_m),$$

by Lemma 3. This complets the proof of theorem. □

As for the case that F may be not algebraically closed field, we have the followed corollary, which followed from [7] Lemma 2.

Corollary 5. *Let $f, g \in (W^{(i)})^o$ and f with the characteric polynomial of $p(x)$, g with the characteric polynomial of $q(x)$. If $p(x)|(x^i - a^i)$ and $q(x)|(x^i - a^i)$, where a is in the algebraically closure of F, then $[f, g]$ satisfies*

$$LCM(p(x), q(x)).$$

The following lemma is the Lemma 5 in [2]. We omit the proof.

Lemma 6. *Let $\{a^j\}, \{b^k\} \in (W^{(i)})^o$ for $j, k \in \mathcal{Z}$, where $a, b \in F^\times$ such that $a^i \neq b^i$. Then $[\{a^j\}, \{b^k\}]$ satisfies $(x-a)^2(x-b)^2$.*

As a consequence of Lemma 6, we have the following corollary. The proof is similar to Theorem 4.

Corollary 7. *Let F be an algebraically closed field. Let $f, g \in (W^{(i)})^o$ and f with the characteric polynomial of $p(x) = (x-a_1)\cdots(x-a_k)(x-c_1)\cdots(x-c_n)(x-b_1^{(1)})\cdots(x-b_s^{(1)})$, g with the characteric polynomial of $q(x) = (x-a_1)\cdots(x-a_k)(x-d_1)\cdots(x-d_m)(x-b_1^{(2)})\cdots(x-b_t^{(2)})$. If $a_1, \cdots, a_k, c_1, \cdots, c_n, d_1, \cdots, d_m, b_1^{(1)}, \cdots, b_s^{(1)}, b_1^{(2)} \cdots, b_t^{(2)}$ are distinct in $F^\times$, they satisfy $a_\alpha^i = c_\beta^i = d_\gamma^i$ for $1 \leq \alpha \leq k, 1 \leq \beta \leq n, 1 \leq \gamma \leq m$ and $a_\alpha^i \neq (b_\eta^{(1)})^i, a_\alpha^i \neq (b_\lambda^{(2)})^i$ for $1 \leq \alpha \leq k, 1 \leq \eta \leq s, 1 \leq \lambda \leq t$, then $[f, g]$ satisfies*

(1) $(x-a_1)\cdots(x-a_k)(x-c_1)\cdots(x-c_n)(x-d_1)\cdots(x-d_m)$ for $s = t = 0$,

(2) $(x-a_1)^2\cdots(x-a_k)^2(x-c_1)^2\cdots(x-c_n)^2(x-b_1^{(1)})\cdots(x-b_s^{(1)})(x-d_1)^2\cdots(x-d_m)^2$(resp. $(x-a_1)^2\cdots(x-a_k)^2(x-c_1)^2\cdots(x-c_n)^2(x-d_1)^2\cdots(x-d_m)^2(x-b_1^{(2)})\cdots(x-b_t^{(2)})$) for $s \neq 0, t = 0$(resp. $s = 0, t \neq 0$),

(3) $(x-a_1)^2\cdots(x-a_k)^2(x-c_1)^2\cdots(x-c_n)^2(x-b_1^{(1)})^2\cdots(x-b_s^{(1)})^2(x-d_1)^2\cdots(x-d_m)^2(x-b_1^{(2)})^2\cdots(x-b_t^{(2)})^2$ for $s \neq 0, t \neq 0$.

We close this paper with the following examples.

Example 8. Denote the linearly recursive sequences $f = \{1\}$ be the sequence $(\cdots, 1, 1, 1, 1, \cdots)$, $g = \{-1\}$ be the sequence $(\cdots, -1, 1, -1, 1, \cdots)$, where the 0-th term is 1 and $h = \{2\}$ be the sequence $(\cdots, \frac{1}{2}, 1, 2, 4, 8, \cdots)$, where the 0-th term is 1. Note that the characteric polynomial of f(resp. g and $f+h$) is $x-1$(resp. $x+1$ and $(x-1)(x-2)$).

For $i = 2$, $[f, g] \in (W^{(2)})^o$ is $(\cdots, 0, 4, 0, 4, 0, 4, \cdots)$. It satisfies $x^2 - 1$. Obviously, $x^2 - 1$ is also the characteric polynomial of $[f, g] \in (W^{(2)})^o$. $[f +$

$h, g] \in (W^{(2)})^o$ is $[f,g]_p = (-3p-2)(-1)^p + (-3p+8)2^p + 2$. It satisfies $x^5-3x^4-x^3+7x^2-4 = (x-1)(x+1)^2(x-2)^2$. Obviously, $x^5-3x^4-x^3+7x^2-4$ is also the characteric polynomial of $[f,g] \in (W^{(2)})^o$.

However, for $i=-1$, $[f,g] \in (W^{(-1)})^o$ is $[f,g]_p = (-2p-1)((-1)^p+1)$. Thus, $x^4-2x^2+1 = (x^2-1)^2$ is the characteric polynomial of $[f,g] \in (W^{(-1)})^o$. $[f+h,g] \in (W^{(-1)})^o$ is $[f,g]_p = (-2p-1) + (-\frac{7}{2}p-2)(-1)^p) + (-\frac{3}{2}p-\frac{1}{2})2^p$. Thus, $x^6-4x^5+2x^4+8x^3-7x^2-4x+4 = (x^2-1)^2(x-2)^2$ is the characteric polynomial of $[f+h,g] \in (W^{(-1)})^o$.

Example 9. Let the linearly recursive sequence $f = (f_t)_{t\in\mathbf{Z}}$ be $f_t = -2f_{t-1} - 4f_{t-2}$, and $f_0 = 1, g_1 = 1$. The characteric polynomial of f is x^2+2x+4. Denote the linearly recursive sequences $g = \{2\}$ be the sequence $(\cdots, \frac{1}{2}, 1, 2, 4, 8, \cdots)$, where the 0-th term is 1 and $h = \{1\}$ be the sequence $(\cdots, 1, 1, 1, 1, \cdots)$. The characteric polynomial of g(resp. $f+h$ and $g+h$) is $x-2$(resp. $x^3+x^2+2x-4 = (x^2+2x+4)(x-1)$ and $(x-1)(x-2)$). Note that $(x^2+2x+4)|(x^3-8)$. So $f_{p+3} = 8f_p$ for $p \in \mathcal{Z}$.

For $i=3$, $[f,g] \in (W^{(3)})^o$ is $[f,g]_p = 24f_p - 24\cdot 2^p$. Thus, the characteric polynomial of $[f,g] \in (W^{(3)})^o$ is x^3-8. $[f+h,g] \in (W^{(3)})^o$ is $[f,g]_p = (7p+3) + (7p-48)2^p + 24f_p$. Thus, the characteric polynomial of $[f+h,g] \in (W^{(3)})^o$ is $x^6-4x^5+5x^4-10x^3+32x^2-40x+16 = (x-1)^2(x+1)^2(x^2+2x+4)$. $[f,g+h] \in (W^{(3)})^o$ is $[f,g]_p = (-7p-3) - 24\cdot 2^p + (-7p+48)f_p$. Thus, the characteric polynomial of $[f,g+h] \in (W^{(3)})^o$ is $x^7+x^5-14x^4+4x^3-8x^2+48x-32 = (x-1)^2(x-2)(x^2+2x+4)^2$. Since $[f+h,g+h] = [f+h,g]+[f,g+h]$, the characteric polynomial of $[f+h,g+h] \in (W^{(3)})^o$ is $x^8-2x^7+x^6-16x^5+32x^4-16x^3+64x^2-128x+64 = (x-1)^2(x^3-8)^2$.

However, for $i=2$, $[f,g] \in (W^{(2)})^o$ is $[f,g]_p = (10p-8)2^p + (-p+2)f_{p+2} + 4pf_p$. Thus, the characteric polynomial of $[f,g] \in (W^{(2)})^o$ is $x^6-16x^3+64 = (x^3-8)^2$. $[f+h,g] \in (W^{(2)})^o$ is $[f,g]_p = (5p+2) + (-3p-16)2^p + 4pf_p + (p+2)f_{p+2}$. $[f,g+h] \in (W^{(2)})^o$ is $[f,g]_p = (7p-2) + (10p-8)2^p + 5pf_p + (-2p+4)f_{p+2}$. $[f+h,g+h] = [f+h,g]+[f,g+h] \in (W^{(2)})^o$. Thus, the characteric polynomial of $[f+h,g], [f,g+h]$ and $[f+h,g+h] \in (W^{(-1)})^o$ are the same. It is $x^8-2x^7+x^6-16x^5+32x^4-16x^3+64x^2-128x+64 = (x-1)^2(x^3-8)^2$.

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