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HOLOMORPHIC VECTOR BUNDLES AND THEIR SECTIONS WITH PRESCRIBED POLES ON RIEMANN SURFACES

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Abstract: Let X be a smooth connected compact Riemann Surface and $K \subsetneq X$ a proper closed subset. Here we study holomorphic vector bundles on X and meromorphic maps between them with poles contained in K. We also study the case in which X has a real structure, i.e. an anti-holomorphic involution $\sigma : X \to X$ and $\sigma(K) = K$.

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1. Holomorphic Vector Bundles

Let X be a smooth connected compact Riemann Surface and $K \subsetneq X$ – a proper closed subset with $K \neq \emptyset$. Set $Y := X \backslash K$. Thus, Y is a one-dimensional smooth Stein space. If K is finite, then Y is also a smooth affine algebraic curve. Here we study the vector bundles on X and their sections, which are holomorphic on Y and extend to rational sections on X with poles contained in K. Let E be a vector bundles on X. Let $H(X, K, E)$ be the set of all meromorphic sections of E on X, whose poles are contained in K. By the compactness of X

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every element of the **C**-vector space $H(X,K,E)$ is holomorphic outside finitely many points of X, all of them contained in K. We will say that E is rationally trivial or rationally trivial with poles on K if there are $s_1,\dots,s_r \in H(X,K,E)$, $r = \mathrm{rank}(E)$, such that for easch $Q \in Y$ $s_1(Q),\dots,s_r(Q)$ give a basis of the r-dimensional vector space $E|\{Q\}$. We will say that a family $s_i \in H(X,K,E)$, $i \in I$, spans rationally E or spans rationally E with poles in K if for every $Q \in Y$ the vectors $s_i(Q)$, $i \in I$, spans the fiber $E|\{Q\}$ of E over Q. For all integers $d \geq 1$ and any set A let $A^{\times d}$ be the product of d copies of A. For any integer $d \geq 1$ there is a morphism $m_d : X^{\times d} \to \mathrm{Pic}^d(X)$ defined by $m_d((Q_1,\dots,Q_d)) = \mathcal{O}_X(Q_1+\cdots+Q_d)$. If X has genus g, then m_d is surjective for every integer $d \geq g$. We will say that K is additively open if there is a positive integer d, such that $m_d(K^{\times d})$ contains a non-empty open subset of $\mathrm{Pic}^d(X)$.

Remark 1. The case $g = 0$ is trivial, because for any $P \in \mathbf{CP}^1$ any degree d line bundles on $\mathbf{CP}^1$ is isomorphic to $\mathcal{O}_{\mathbf{CP}^1}(dP)$. Thus, if $g = 0$, then $\mathrm{Pic}^d(X)$ is a point for every d and hence every non-empty subset of $\mathbf{CP}^1$ is additively open.

Remark 2. If K contains a non-empty open subset of X, then it is additively open, but the converse is not true even if $g > 0$. Here is a very easy example, which may be generalized to higher genera. Take $g = 1$ and identify X with $S^1 \times S^1$ as topological group. Set $K := S^1 \times \{O\} \cup \{O\} \times S^1$ for some $O \in S^1$. We have $m_2(K^{\times 2}) = \mathrm{Pic}^2(X)$.

Remark 3. By [1], Theorem 30.3, every holomorphic vector bundle on a smooth open Riemann Surface Y is holomorphically trivial. This is not true in the category of smooth affine algebraic curves, but the proof of [1], Theorem 30.3, shows that for every smooth affine algebraic curve Y and every rank r algebraic vector bundle on Y, $r \geq 2$, E is isomorphic (in the category of algebraic schemes) to $\det(E) \oplus \mathcal{O}_Y^{\oplus(r-1)}$. Hence in the category of smooth affine curves the study of vector bundles is reduced (at least in principle) to the study of line bundles.

Remark 4. Let X be a smooth connected complex Riemann Surface and $\emptyset \neq K \subsetneq X$ a proper closed subset. Set $Y := X \backslash K$. A line bundle L on X is rationally trivial with poles in K if there are finitely many points $P_1,\dots,P_s \in K$ and integers $m_1,\dots,m_s$ such that $L \cong \mathcal{O}_X(m_1P_1 + \cdots + m - sP_s)$. Hence if

K is countable, then there are only countably many isomorphism classes of rationally trivial line bundles with poles in K.

Remark 5. A line bundle is rationally trivial with poles in K if and only if its dual is rationally trivial with poles in K.

Theorem 1. *Let X be a smooth connected compact Riemann Surface and $K \subsetneq X$ a proper closed subset with $K \neq \emptyset$. Every line bundle on X is rationally trivial with poles in K if and only if K is additively open.*

Proof. The case $g = 0$ is trivial by Remark 1. Assume $g > 0$. Thus $\mathrm{Pic}^0(X)$ is a g-dimensional complex torus and for every integer d the complex space $\mathrm{Pic}^d(X)$ is isomorphic to $\mathrm{Pic}^0(X)$. Fix $P \in K$. A line bundle L is rationally trivial (with poles in K) if and only if $L(P)$ is rationally trivial. Hence we see that if every degree zero line bundle on X is rationally trivial, then for any integer $d > 0$ every degree d line bundle on X is trivial. By Remark 5 the same is true for $d < 0$. Hence if every degree zero line bundle on X is rationally trivial, then every line bundle on X is rationally trivial. For all integer $d \geq 1$ set $K(d) := \{L \in \mathrm{Pic}^0(X)$: there are $Q_1, \ldots, Q_d \in K$ such that $L \cong \mathcal{O}_X(Q_1 + \cdots + Q_d - dP)$. Hence $K(d)$ is a compact subset of $\mathrm{Pic}^0(X)$. K is additively open if and only if there is an integer $d \geq 1$ such that $K(d)$ contains a non-empty open subset of $\mathrm{Pic}^0(X)$. Hence the thesis easily follows from Baire Category Theorem and the obvious relation $K(d) + K(d) \subseteq K(2d)$. □

Theorem 2. *Let X be a smooth connected compact Riemann Surface of genus g, $K \subsetneq X$ a proper closed subset with $K \neq \emptyset$ and E a holomorphic vector bundle on X such that $r := rank(E) \geq 2$. Then there exists a rationally trivial line bundle L with poles in K such that E fits in an exact sequence*

$$0 \to L^{\oplus(r-1)} \to E \to det(E) \otimes (L^*)^{\otimes(r-1)} \to 0 \tag{1}$$

Proof. Fix $P \in K$. For all integers $t \gg 0$ the vector bundle $E(tP)$ is spanned by its global sections. Since X is one-dimensional, a standard and well-known dimensional count shows that we may find $r-1$ sections of $E(tP)$ such that at each point Q of X they generate an $(r-1)$-dimensional linear subspace of the fiber $E(tP)|\{Q\}$, i.e. such that they induce an injective morphism $f : \mathcal{O}_X^{\oplus(r-1)} \to E(tP)$ with $\mathrm{Coker}\,(f)$ locally free and with rank one. Take $L = \mathcal{O}_X(-tP)$ and let (1) be the exact sequence induced by f. L is rationally trivial with poles in K, because $P \in K$. □

Theorem 3. *Let X be a smooth connected compact Riemann Surface of genus g and $K \subsetneqq X$ a proper closed subset with $K \neq \emptyset$. Then for every integer $r \geq 2$ and every rank r holomorphic vector bundle E on X there are rationally trivial line bundles L, M and morphisms $u : L^{\oplus r} \to E$ and $v : E \to M^{\oplus r}$ such that u and v are injective as morphisms of sheaves, surjective outside finitely many points and with rank at least $r-1$ at each point of X, i.e. Coker(u) and Coker(v) have dimension one at each point of their support. The line bundle det(L) is rationally trivial with poles in K if and only if we may find L and u as above and such that Coker(u) is supported by points of K. The line bundle det(L) is rationally trivial with poles in K if and only if we may find M and v as above and such that Coker(v) is supported by points of K.*

Proof. Fix $P \in K$. For all integers $t \gg 0$ the vector bundle $E(tP)$ is spanned by its global sections. Since X is one-dimensional, a standard and well-known dimensional count shows that we may find r sections of $E(tP)$ such that at each point Q of X they generate at least an $(r-1)$-dimensional linear subspace of the fiber $E(tP)|\{Q\}$ and they generate the fiber $E(tP)|\{Q\}$ except at finitely many points of X, i.e. such that they induce an injective morphism $f : \mathcal{O}_X^{\oplus r} \to E(tP)$ with Coker(f) a skyscraper sheaf with length one at each point of its support. Take $L = \mathcal{O}_X(-tP)$ and as u the morphism induced by f. L is rationally trivial with poles in K, because $P \in K$. Apply the same construction to E^* to obtain the rationally trivial line bundle $M = \mathcal{O}_X(tP)$, $t \gg 0$, and the morphism v. From $r-1$ of our sections of $E(tP)$ we construct an exact sequence (1). The remaining section gives a morphism $u' : \mathcal{O}_X(-tP) \to \det(E)$ such that Coker(u) = Coker(u'). Since $\mathcal{O}_X(-tP)$ is rationally trivial with poles in K, we obtain Supp(Coker(u)) = Supp(Coker(u')) if and only if $\det(E)$ is rationally trivial with poles in K. taking duals we obtain the last assertion concerning Coker(v). □

The proof of Theorem 3 gives the following two corollaries.

Corollary 1. *Let X be a smooth connected compact Riemann Surface of genus g and $K \subsetneqq X$ a proper closed subset with $K \neq \emptyset$. Fix $P \in K$ and a rank r holomorphic vector bundle E on X such that det(E) is rationally trivial with poles in K. Then there exists an integer $t(E)$ such that for all integers $t \geq t(E)$ there are r sections $s_1, \dots, s_r \in H^0(X, E(tP))$, which are independent at each point of $X \backslash K$.*

Corollary 2. *Let X be a smooth connected compact Riemann Surface of genus g and $K \subsetneqq X$ a proper closed subset with $K \neq \emptyset$. Assume that every holomorphic line bundle on X is rationally trivial with poles in K. Fix*

$P \in K$ and a rank r holomorphic vector bundle E on X. Then there exists an integer $t(E)$ such that for all integers $t \geq t(E)$ there are r sections $s_1, \ldots, s_r \in H^0(X, E(tP))$, which are independent at each point of $X \backslash K$.

2. Real Curves

Let X be a smooth connected compact Riemann Surface of genus $g \geq 0$ and $\sigma : X \to X$ an anti-holomorphic involution. The existence of σ is equivalent to require that the algebraic curve X may be defined over Spec($\mathbf{R}$) (see [2]). We will say that the pair (X, σ) is a real curve. To emphasize this equivalence we will often write $X(\mathbf{C})$ for the usual set of points of X with the euclidean topology. Set $X(\mathbf{R}) := \{P \in X(\mathbf{C}) : \sigma(P) = P\}$. It is known that $X(\mathbf{C}) \backslash X(\mathbf{R})$ is either connected ($a(X, \sigma) = 0$ with the notation of [2], §3) or with two connected components ($a(X, \sigma) = 1$ with the notation of [2], §3). Each connected component of $X(\mathbf{R})$ is a smooth circle. Call $c(X, \sigma)$ the number of the connected components of $X(\mathbf{R})$. If $a(X, \sigma) = 0$, then $0 \leq c(X, \sigma) \leq g + 1$ and $c(X, \sigma) \equiv g + 1 \pmod 2$. If $a(X, \sigma) = 1$, then $1 \leq c(X, \sigma) \leq g$. Conversely, for any such pair (a, c) there is a pair (X, σ) with these prescribed invariants and any two such pairs (X, σ) are topologically equivalent. The same Riemann Surface may have several non-isomorphic real structures, but only finitely many ones. For instance $\mathbf{CP}^1$ has two real structures, the usual one with $\mathbf{CP}^1(\mathbf{R}) = \mathbf{RP}^1$ and the one with $\mathbf{CP}^1(\mathbf{R}) = \emptyset$ corresponding to the plane conic $\{x^2 + y^2 + z^2 = 0\} \subset \mathbf{CP}^2$. Take a closed subset K of X with $K \neq X$ and $K \neq \emptyset$. We will say that K is σ-invariant if $\sigma(K) \subseteq K$. Since σ is an involution, we have $\sigma(K) \subseteq K$ if and only if $\sigma(K) = K$. From now on we assume that K is σ-invariant and set $K(\mathbf{R}) := K \cap X(\mathbf{R})$. For any integer $t \geq 1$ and any subset A of $X(\mathbf{C})$, let $S^t(A)$ be its symmetric product. $S^t(X)$ is a smooth t-dimensional projective variety defined over Spec($\mathbf{R}$); let $S^t(X)(\mathbf{R})$ be the set of its real points, i.e. the set of all σ-invariant effective degree t divisors on $X(\mathbf{C})$. Notice that for any $Q \in X(\mathbf{C})$ the divisor $Q + \sigma(Q)$ is an element of $S^2(X)(\mathbf{R})$ and that $S^t(X)(\mathbf{R}) = \emptyset$ if and only if t is odd and $X(\mathbf{R}) = \emptyset$. For any positive integer d the map m_d induces a map $a_d : X(\mathbf{R})^{\times d} \to S^d(X)(\mathbf{R})$. If $X(\mathbf{R}) \neq \emptyset$ we will say that K is odd-real additively open if there is an integer $d \geq 1$ such that the compact set $a_d(K(\mathbf{R})^{\times d})$ contains a non-empty open subset of $S^d(X)(\mathbf{R})$. It is easy to check that in this case for all integers $k \geq 2$ $a_{kd}(K(\mathbf{R})^{\times kd})$ contains a non-empty open subset of $S^{kd}(X)(\mathbf{R})$. If K is odd-relatively open, then $K(\mathbf{R}) \neq \emptyset$. For any integer $d \geq 1$ let $b_{2d} : X(\mathbf{C})^{\times d} \to S^{2d}(X)(\mathbf{R})$ be the real analytic map defined by $b_{2d}((Q_1, \ldots, Q_d)) = Q_1 + \cdots + Q_d + \sigma(Q_1) + \cdots + \sigma(Q_d)$.

We will say that K is even-real additively open if there is an integer $d \geq 1$ such that $b_{2d}(K)$ contains a non-empty open subset of $S^{2d}(X)(\mathbf{R})$. A holomorphic line bundle L on X is called real if, seeing X as an algebraic curve over Spec($\mathbf{R}$), L can be defined over Spec($\mathbf{R}$). L is called σ-invariant if $\sigma^*(L) \cong L$. If L is real, then it is σ-invariant and the converse is true if and only if $X(\mathbf{R}) \neq \emptyset$ (see [2]). If $X(\mathbf{R}) = \emptyset$, then for every σ-invariant line bundle L the line bundle $L^{\otimes 2}$ is real. The set of all isomorphism classes of σ-invariant line bundles of degree d will be denoted with $\mathrm{Pic}^d(X)(\mathbf{R})$ because it is exactly the set of all real points of the $\mathbf{R}$-algebraic smooth scheme $\mathrm{Pic}^d(X)$. If $X(\mathbf{R}) \neq \emptyset$ the natural map $S^d(X)(\mathbf{R}) \to \mathrm{Pic}^d(\mathbf{R})$ is surjective for every integer $d \geq g$ and it is a submersion for every integer $d \geq 2g-1$. Composing the maps a_d and b_{2t} (if $d = 2t$ is even) with the natural map $S^d(X) \to \mathrm{Pic}^d(X)$ we obtain real-analytic maps $\alpha_d : X(\mathbf{R})^{\times d} \to \mathrm{Pix}^d(\mathbf{R})$ and $\beta_{2t} : X(\mathbf{C})^{\times t} \to \mathrm{Pix}^{2t}(\mathbf{R})$. It is obvious that the notions of odd-real and even-real additive openness may be defined using the maps α_d and β_{2t} instead of the maps a_d and b_{2t}. We will say that a holomorphic vector bundle on X is real (with respect to the real structure σ) if it comes from an algebraic vector bundle defined over Spec($\mathbf{R}$).

Just taking only real points in the proof of Theorem 1 we obtain the following two propositions.

Proposition 1. *Let (X, σ) be a real curve with $X(\mathbf{R}) \neq \emptyset$ and K a σ-invariant closed subset of X with $\emptyset \neq K \neq X$. K is odd-real additively open if and only if for every real line bundle L on (X, σ) there are $P_1, \dots, P_x \in K(\mathbf{R})$ and integers $m_1, \dots, m_x$ such that $L \cong \mathcal{O}_X(m_1P_1 + \cdots + m_xP_x)$.*

Proposition 2. *Let (X, σ) be a real curve and K a σ-invariant closed subset of X with $\emptyset \neq K \neq X$. K is even-real additively open if and only if for every real line bundle L on (X, σ) with even degree there are $P_1, \dots, P_x \in K$ and integers $m_1, \dots, m_x$ such that $L \cong \mathcal{O}_X(m_1P_1 + \cdots + m_xP_x + m_1\sigma(P_1) + \cdots + m_x\sigma(P_x))$.*

The proof of Theorem 3 gives the following two results.

Proposition 3. *Let (X, σ) be a real curve, K a σ-invariant closed subset of X with $K \neq X$ and $K(\mathbf{R}) \neq \emptyset$ and E a real vector bundle on X with $r := \mathrm{rank}(E) \geq 2$. Fix $P \in K(\mathbf{R})$. Then there exists an integer k such that for all integers $m \geq k$, E fits in an exact sequence (1) with $L \cong \mathcal{O}_X(mP)$.*

Proposition 4. *Let (X, σ) be a real curve and K a σ-invariant closed subset of X with $K \neq X$ and $K(\mathbf{R}) \neq \emptyset$. Fix $P \in K(\mathbf{R})$ and a rank r real vector bundle on (X, σ) such that $\det(E)$ is rationally trivial with poles in $K(\mathbf{R})$. Then there exists an integer t_E such that for all integers $t \geq t_E$ there are*

r sections $s_1, \ldots, s_r \in H^0(X, E(tP))$ defined over $\mathbf{R}$ and which are independent at each point of $X \backslash K(\mathbf{R})$.

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