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ON THE SOLVABILITY OF EVOLUTION EQUATIONS WITH VARIABLE OPERATOR COEFFICIENTS

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Abstract: In this paper we study nonlinear (linear) evolution equations in a Banach space with linear unbounded nondensely defined operator cofficients, for which the Cauchy problem is well-posed solvable.

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1. In this paper the solvability of the Cauchy problem for the evolution equation

$$\frac{du(t)}{dt} = A(t)u(t) + f(t) \quad (0 < t \leq T), \quad u(0) = u_0 \tag{1}$$

in a Banach space E is studied. Here, $A(t)$ is a linear operator for each fixed $t \in [0,T]$ with the domain of definition $D(A(t))$, and $f(t)$, $t > 0$ is a given

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function from E; u_0 is an element of E. A function $u(t)$ is called the solution of (1) if:

1) $u(t)$ is continious on $[0,T]$ and continiously differentiable on $(0,T]$.

2) $u(t)$ satisfies (1) and the initial condition $u(0) = u_0$.

3) the function $A(t)u(t)$ is continious on $(0,T]$.

The Cauchy problem for evolution equations of the form (1) is studied in various aspects by many authors (see, for example, [1-2] and references therein). The theory of evolution equations with variable operators, which generate an analytic semigroup was developed by Sobolevski [3] and H. Tanabe [4]. The evolution equation with a nonanalytic semigroup in the case of the nonconstant domain $D(A(t))$ was considered by S.Y. Yakubov in [2].

In [5], problem (1) was investigated under the condition that the set $D(A(t))$ is not dense in E. Moreover, it was assumed in [5] that the operator-function $A(t)$ is continously differentiable, and the properties of the resolvent of $A(t)$, are used. In [6], this problem was studied in the case when $A^{-1}(t)$ is not differentiable, and the properties of the resolvent of $A(t)$ are not used. However, it was assumed in [6] that the domain of definition of the operator-function A(t) is constant.

In [7], problem (1) was investigated for a variable domain of definition $D(A(t))$ of the operator $A(t)$ without using properties of the resolvent of this operator and the differentiability of $A^{-1}(t)$. We note that in the above-mentioned papers, it was assumed that the operator-function $A(t)A^{-1}(s)$ is uniformly continious in the sense of Hölder with respect to t in the uniform operator topology.

In [8], a new condition on the continuity of the operator-function $A(t)A^{-1}(s)$ was introduced, which is weaker then previous ones. However, in this paper it is assumed that the domain of definition $D(A(t))$ of the positive operator $A(t)$ is everywhere dense in E and does not depend on t.

In this paper problem (1) is studied in the case of a variable domain of definition $D(A(t))$ of the operator $A(t)$, which is not dense in E. Moreover, we introduce a new weak condition on the continuity of the operator-function $A(t)A^{-1}(s)$, $s \leq t$ and of the function $f(t)$.

In order to study problem (1) we assume that the operator $A(t)$ satisfies the following conditions:

1^0. The linear operator $A(t)$, $t \in [0,T]$ with the domain of definition $D(A(t))$ satisfies the condition

$$\| R(\lambda, A(t)) \| \leq c \mid \lambda \mid^{-\beta} \quad , \quad \mid \arg \lambda \mid \leq \frac{\pi}{2} + \eta \ ,$$

for a certain

$$\beta \in (\frac{1}{2} \ , \quad 1] \ , \ \eta > 0.$$

2^0. The embedding $D(A(\tau)) \subset D(A(t))$ holds for $0 \leq \tau \leq t \leq T$ and the inequality

$$\| [A(t) - A(\tau)A^{-1}(s)] \| \leq cw(\mid t - \tau \mid) \ ,$$

holds for $0 \leq s \leq \tau, t \leq T$, where the function $w(t)$ is pozitive, monotonicially increasing on the interval $(0, \infty)$, and

$$\int_0^T \frac{w^p(r)}{r^p} dr \ < \ \infty \, , \tag{2}$$

for a certain $p \in \left[\frac{1}{2\beta - 1}, \infty\right)$ (for $\beta = 1$, $p = 1$, see [8]).

For example, condition (2) is satisfied for the functions:

1) $w(r) = c \mid \log r \mid^\alpha$, for $\alpha < -2$, $p = 1$.

2) $w(r) = c\dfrac{\log(1+r)}{r^{\frac{1}{p}} \mid \log r \mid^\alpha}$, for $\alpha > \frac{p+1}{p}$, $p \geq 1$.

We note that these functions are not uniformly continious in the sense of Hölder in a neighbourhood of $r = 0$.

Let us introduce the functional space

$$F^\alpha_{p,\mu}((0,T);E) = \{f(.) : f \in C((0,T);E) \ ,$$

$$\| f \|_{F^\alpha_{p\mu}((0,T);E)} = \sup_{t \in (0,T]} \| t^\mu f(t) \| + \sup_{0 \leq \tau < t \leq T} \int_\tau^t \frac{\| f(t) - f(s) \|^p}{\mid t - s \mid^{p\alpha}} s^{\mu p} ds \, \} \, , \quad \mu \geq 0 \, .$$

Theorem 1. *Let assumptions 1^0 and 2^0 hold. Then there exisits an operator-function $U(t,\tau) \in B(E)$, defined for $0 \leq \tau \leq t \leq T$, with the following properties:*

1) $U(t,s)$ is continious for $0 \leq \tau \leq \ t \leq T$.

2) The equality $U(t,\tau) = U(t,s)U(s,\tau)$ *is valid for* $0 \le \tau < s < t \le T$.

3) The range of $U(t,\tau)$ *belongs to* $D(A(t))$, *and the operator-function* $A(t)U(t,\tau)$ *is continious for* $0 \le \tau \le \ t \le T$.

4) For any $0 \le \tau \le \ t \le T$

$$\frac{\partial U(t,\tau)}{\partial t} = A(t)U(t,\tau).$$

5) For any $0 \le \tau \le \ t < t + \Delta t \le \xi \le T$, *the following estimates hold*

$$\| A^{\alpha}(t)U(t,\tau)A^{-\delta}(\tau) \| \le c(\alpha,\beta,\delta) \mid t-\tau \mid^{\delta+\beta-\alpha-1} \qquad (0 \le 1-\beta \le \delta \le \alpha < 1+\epsilon)\,,$$

$$\| A^{\alpha}(\xi)[U(t+\delta t,\tau) - U(t,\tau)]A^{-\delta}(\tau) \| \le c(\alpha,\beta,\delta,\nu)\Delta t^{2(\beta-1)+\nu-\alpha} \mid t-\tau \mid^{\delta+\beta-\nu-1} w(\Delta t)$$

$(0 \le \alpha \le 1)$, $0 \le 1-\beta \le \delta \le \nu < 1+\epsilon$, $0 \le \nu - \alpha < 1$.

If either $\xi = t + \Delta t$ *or* $\delta < \gamma$, *then in the last estimate, one can take* $\gamma - \alpha = 1$.

Theorem 2. *Let assumptions* 1^0 *and* 2^0 *hold, and let*

3^0. $\| f(t+\Delta t) - f(t) \| \le ct^{-\mu} w(\mid \Delta t \mid)$, $\mu \in [0,\beta)$.

4^0. $u_0 \in D(A^{\delta}(0))$ *for a certain* $\delta \in (1-\beta, 1]$.

Then problem (1) has the unique solution $u(t)$ *defined by the formula*

$$u(t) = U(t,0)u_0 + \int_0^t U(t,s)f(s)ds\,. \tag{3}$$

In the case of $w(t) = t^{\alpha}$ Theorems 1 and 2 partially intersect with results by S.Y. Yakubov [2], p. 134, and Y.T. Silchenko [7].

Theorem 3. *Let assumptions* 1^0 *and* 2^0 *hold. Suppose that:*

5^0. $f \in F^{\alpha}_{p,\mu}((0,T);E)$ *for certain* $\mu \in [0, \frac{p-1}{p})$, $\alpha \in (\frac{p+1}{p} - \beta, 1]$ *and* $p \in [\frac{1}{2\beta-1}, \infty)$.

6^0. *$u_0 \in D(A^\delta(0))$ if $\overline{D(A(0))} \neq E$, $\delta \in (1-\beta, 1]$, and $u_0 \in E$, if $\overline{D(A(0))} = E, \beta = 1$, $p = 1$.*

Then problem (1) has the unique solution $u(t)$ defined by (3).

We note that if $D(A(t)) = D(A)$, $\overline{D(A)} = E$ and $\beta = 1$, Theorem 3 intersects with results by S. Kawatsu [8].

2. Let us consider the Cauchy problem for the nonlinear equation

$$\frac{du(t)}{dt} = A(t, u(t))u(t) + f(t, u(t)) \qquad (0 < t \leq t_0),\ t_0 \in (0, T]\,, \tag{4}$$

$$u(0) = u_0\,, \tag{5}$$

in a Banach space E.

Theorem 4. *Let the following conditions be satisfied:*

7^0. *For any $t \in [0, T]$, $A_0(t) = A(t, u_0)$ is a linear closed operator with the domain of definition $D(A_0(t))$, which is not necessarily dense in E, and in E, the operator $A_0(t)$ has the compact inverse operator $A_0^{-1}(t)$.*

8^0. *For certain $\eta > 0$, $\beta \in (\frac{1}{2}, 1]$, $p \in [\frac{1}{2\beta-1}, \infty)$ and any $t \in [0, T]$ the operator $A(t, u_0) + \lambda I$ has the bounded inverse operator, and*

$$\| R(\lambda, A(t, u_0)) \| \leq c \mid \lambda \mid^{-\beta}, \quad \mid \arg\lambda \mid \leq \frac{\pi}{2} + \eta, \quad \mid \lambda \mid \to \infty.$$

9^0. *For $\alpha \in [1-\beta, 1)$, $p \in (0, 1]$ and any $0 \leq \tau \leq t \leq T$ and $u, v \in E$ with $\| u \| \leq R$, $\| v \| \leq R$ the following relation holds: $D(A(\tau, A_0^{-\alpha}v)) \subset D(A(t, A_0^{-\alpha}v))$. Moreover, for any $0 \leq s < \tau$, $t \leq T$,*

$$\| [A(t, A_0^{-\alpha}u) - A(t, A_0^{-\alpha}v)]A_0^{-1}(s) \| \leq c(R)\,[w(\mid t-\tau \mid) + \| u - v \|^\rho]\,,$$

where $A_0 = A(0, u_0)$ and the function $w(t)$ satisfies (2).

10^0. *For any $\tau, t \in [0, T]$, $\| u \| \leq R$, $\| v \| \leq R$,*

$$\| f(t, A_0^{-\alpha}u) - f(t, A_0^{-\alpha}v) \| \leq c(R)\,[w(\mid t-\tau \mid) + \| u - v \|^\rho]\,,$$

11^0. *$u_0 \in D(A_0^{-\rho})$, for a certain $\delta > \alpha$ and $\| A_0^\alpha u_0 \| \leq R$.*

Then in some interval $[0, t_0]$, there exists at least one solution of problem (4)-(5), which is continuous for $t \in [0, t_0]$ and continiously differentiable for $t > 0$.

If $\rho = 1$, then even without the suggestion of compactness of the operator $A_0^{-1}(s)$, problem (4)-(5) has the unique solution, which can be found by the method of successive approximations.

We note that if $D(A(t)) = D(A)$, $\overline{D(A)} = E$ and $\beta = 1$, $w = t^\alpha$, then Theorem 4 is due to P.E. Sobolevski [3].

3. There are numbers of examples of boundary value problems in various (with respect to t) domains linear and nonlinear differential equations considered.

For example, let Ω_t be a domain in R^n with a smooth boundary Γ_t, and let Q_t be a noncylindered variable domain in R^{n+1} of the form $Q_t = \bigcup\limits_{0<t<T} \{t\}$ x Ω_t .

Denote by $\sum_T = \bigcup_{0<t<T} \{t\}$ x Γ_t the lateral bound of Q_t (see [9]).

In Q_t we consider the boundary value problem

$$\frac{\partial u}{\partial t} = \sum_{i,k=1}^{n} a_{ik}(t,x,u,\frac{\partial u}{\partial x_1},...,\frac{\partial u}{\partial x_n})\frac{\partial^2 u}{\partial x_i \partial x_k} + f(t,x,u,\frac{\partial u}{\partial x_1},...,\frac{\partial u}{\partial x_n}), \qquad \text{in } Q_{t_0}, \quad (6)$$

$$u(t,x) = 0 \qquad \text{in} \quad \sum\nolimits_{t_0}, \quad (7)$$

$$u(0,x) = 0 \qquad \text{in} \quad \Omega_0, \quad (8)$$

Let the following conditions be satisfied:

1) For any $t_0 \in [0,T]$ and $u_0(x)$ the operator

$$\sum_{i,k=1}^{n} a_{ik}(t,x,u,\frac{\partial u}{\partial x_1},...,\frac{\partial u}{\partial x_n})\frac{\partial^2 u}{\partial x_i \partial x_k}$$

is strongly elliptic.

2) $\mid a_{ik}(t,x,u_0(t,x),...) - a_{ik}(\tau,x,v_0(t,x),...) \mid$
$$\leq c\Big[\mid \log \mid t-\tau \mid\mid^\alpha + \parallel u_0 - v_0 \parallel^\rho_{W_2^1(\Omega_0)}\Big].$$

3) $\mid f(t,x,u_0(t,x),...) - f(\tau,x,v_0(t,x),...) \mid$
$$\leq c\Big[\mid \log \mid t-\tau \mid\mid^\alpha + \parallel u_0 - v_0 \parallel^\rho_{W_2^1(\Omega_0)}\Big]$$

Using Theorem 4 one can prove a theorem on existence and uniqueness of a generalized solution of problem (4)-(6) without any restrictions on the growth of nonlinearities.

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