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SOLVABILITY OF BOUNDARY VALUE PROBLEMS WITH OPERATORS IN BOUNDARY CONDITIONS FOR THE ARBITRARY ORDER-DIFFERANTIAL OPERATORS EQUATIONS

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Abstract: Correct solvability of boundary value problems is established for the arbitrary order differential operator equations in a Banach space, whose boundary conditions contain linear unbounded operators.

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Correct solvability of boundary value problems is established for the arbitrary order differential operator equations in a Banach space, whose boundary conditions contain linear unbounded operators. In this paper, at first, the operators in boundary conditions may the unbounded, secondly, for some classes of boundary problems these operators may have higher order than the operators contained in the equations.

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As is known [1, 7, 9] varies boundary (non-local and irregular) problems for differantial equations may be reduced to these problems.

In a complex Banach space E, boundary problems are considered for linear arbitrary order equations

$$L(u) \equiv u^{(n)}(t) + \sum_{k=1}^{n} A_k u^{(n-k)}(t) = f(t), \qquad 0 < t \le T\,, \tag{1}$$

$$L_i(u) \equiv \sum_{k=1}^{n} \left[P_{ik} u^{(n-k)}(0) + B_{ik} u^{(n-k)}(T) \right] = g_i, \quad i = 1, 2, \ldots, n\,, \tag{2}$$

here A_k, B_{ik}, P_{ik} $(i, k = 1, 2, \ldots, n)$ are linear, generally speaking, unbounded operators acting in E, $f(t)$ is a continuous for $t \geq 0$, $g_i \in E$,$(i = 1, 2, \ldots, n)$. The forms $L_i(u)$ are assumed to be linear independently.

The boundary value problems for evaluation equations, when the coefficients in boundary conditions are complex numbers are investigated in papers [1, 5, 7, 9, 10, 11] .

The construction of Green's function investigation of a spectrum, and solvability of a boundary value problem for evalution equations in the case, when the coefficients of boundary conditions are linear bounded operators, are carried out in papers [7, 14, 15].

In paper [2] A.A.Dezin firstly applied the method of spectral theory for abstract operators (see also [3]) to investigate the correct solvability (Hörmander sense) of boundary-value problem (1)-(2) for $n = 1$. Lately, by the same method in papers [9, 10] scalar boundary coditions (2), for which problem (1)-(2) is correctly solvable were found by V.K. Romanko for some class of arbitrary order differential-operator equations with unbounded operator coefficients.

In Ju.A. Dubinskii's paper [12], general boundary value problems, Cauchy problem, the classification problems of equations (1), the statement of problems for these equations and thesis solvability in Sobolev-Slobedetski spaces are studied.

As it is known, the boundary-value problem on the whole axis (see for intance [4, 7]) have been studied in detail for a higher order general coercive differential-operator equations, and at the limit segment the boundary-problem has not been studied.

The noncoercive arbitrary order differential-operator equations, the boundary-value problems, whose operators in boundary conditions have not almost been investigated.

As A.A. Dezin notes in his book [1], p. 155, the passage in studying of the general differantial-operator equation of the form (1), to the case $n > 2$ in connected with arising new difficulties.

Introduce the denotetions. E is a Banach space with the norm $\|.\|$, A is a closed operator in E with a definition domain $D(A)$,

$$E(A) = \left\{ u : u \in D(A), \ \ \|u\|_{E(A)} = \|Au\| + \|u\| \right\} ,$$

$$u^{(k)}(t) = d^k u(t)/dt^k,$$

$$C^n([0,T] : E(A_n), \ldots, E(A_1), E) = \left\{ f(.) : A_n f, \ldots, A_1 f^{(n-1)}, f^{(n)} \in C([0,T]\,; E \right\}.$$

Definition. We call the function

$$u \in C^n((0,T]; E(A_n) \cap E(A_{n-1}, \ldots, E(A_1), E) \cap C^{n-1}([0,T]\,; E(A_{n-1}), \ldots, E(A_1), E)$$

satisfying equation (1) on $[0,T]$ and boundary conditions (2) the solution of problem (1)-(2).

1. First Consider a Boundary-Value Problem for the First Order Evaluation Equation

$$u'(t) + Au(t) = f(t) \quad (0 < t \le T)\,, \tag{3}$$

$$\mu P u(0) + Bu(t) = g, \tag{4}$$

in a complex Banach space $E; A, B$ and P are linear, generaly speaking unbounded operators in E, μ is a complex parameter.

Theorem 1. *Let the following conditions be fulfilled:*

1) The linear operator A for some $n > 0$ and $B \in (0,1]$ fulfilled condition

$$\|R(\lambda, -A)\| \le C\,|\lambda|^{-\beta}\,, \quad |\arg \lambda| \le \frac{\pi}{2} + \eta;$$

2) The operator A has a completely continuous inverse operator A^{-1}.

3) The operator $BA^{-(1+\alpha)}$ is bounded in some $\alpha \in [1-\beta, 1)$.

4) The linear operator P has a bounded inverse operator P^{-1} and for some $\alpha \geq 1-\beta$ the operator $A^{\alpha}P^{-1}$ is bounded.

5) For some $\gamma \in (1+\alpha-\beta, 1]$, $f \in C^{\alpha}([0,T];E)$.

6) $g \in E$, $\mu \neq 0$.

Then for problem (3)-(4) in the space

$$C([0,T];E) \cap C^{1}([0,T];E(A),E)$$

the following alternative holds: the corresponding homogeneus problem has a finite number of linearly independent solutions, or problem (3)-(4) has a unique solution $u \in C([0,T];E) \cap C^{1}((0,T];E(A),E)$ and for the solution of $u(t)$ the following estimates hold:

$$\|u\|_{C([0,T];E)} \leq C(\mu)(\|g\| + \|f\|_{C^{\gamma}([0,T];E)}), \tag{5}$$

$$\|u\|_{C^{1}([\delta,T];E(A),E)} \leq C(\mu,\delta)(\|g\| + \|f\|_{C^{\gamma}([0,T];E)}), \tag{5'}$$

for any $0 < \delta < T$.

2. A Boundary-Value Problem with a Perturbed Equation

Apply some results of abstract parabolic equations theory to the following boundary-value problem:

$$u'(t) + (A + A_1)u(t) = f(t) \quad (0 < t \leq T), \tag{6}$$

$$\mu P u(0) + B u(T) = g, \tag{7}$$

where A, A_1, B and P are generally speaking, unbounded, not necessarily densely given operators acting in the Banach space. E, $\mu \neq 0$ is a complex parameter.

Theorem 2. *Let conditions (1)-(6) of Theorem1 be fulfilled. Later, let*

7) At any $\varepsilon > 0$ and $u \in D(A) \subset D(A_1)$ it holds the estimate

$$\|A_1 u\| \leq \varepsilon \|Au\| + C(\varepsilon)\|u\|, \qquad u \in D(A).$$

Then for problem (6)-(7) in the space

$$C([0,T];E) \cap C^{1}((0,T];E(A),E),$$

the following alternative holds: the corresponding homogenous problem has a finite number of linearly independent solutions, or problem (6)-(7) has a unique solution $u \in C([0,T];E) \cap C^{1}((0,T];E(A),E)$ and for the solution the estimate (5) and (5$'$) holds.

3. Consideration of the Boundary-Value Problem (1)-(2)

Theorem 3. *Let fulfilled the conditions:*

1) Linear operators A_k $(k = 1, 2, \ldots, n)$ *has bounded inverse operators* A_k^{-1} $(k = 1, 2, ..., n)$ *in* E *respectively.*

2) The operators $A_k A_k^{-1}$ $(k = 1, 2, \ldots, n;\ A_0 \equiv I)$ *are closed in* E *and for* $Re\lambda \geq 0$ *the estimates hold*

$$\left\| R(\lambda, A_k A_k^{-1}) \right\| \leq C \left|\lambda\right|^{-1} .$$

3) The operators $A_k A_k^{-1}$ *are completely continuous.*

4) For any $\varepsilon > 0$ *and* $u \in D(A_k A_{k-1}^{-1}) \subset D(A_{k+1} A_k^{-1})$ $(k = 1, 2, \ldots, n)$ *the estimates*

$$\left\| A_{k+1} A_k^{-1} u \right\| \leq \varepsilon \left\| A_k A_{k-1}^{-1} u \right\| + C(\varepsilon) \left\| u \right\|, \quad u \in D(A_k A_{k-1}^{-1}).$$

5) The operators $B_{k_1} A_1^{-1}$ $(k = 1, 2, \ldots, n)$ $B_{ij} A_{j-1}^{-1} A_{j-2} A_{j-1}^{-1}$ $(i, j = 2, \ldots, n)$ *are bounded.*

6) The operator-matrix $\mathbf{Q} = (Q_{ij})$ $(i, j = 1, 2, \ldots, n)$*, where* $Q_{ij} = P_{ij} A_{j-1}^{-1} - P_{i,j+1} A_j^{-1}$; $P_{i,n+1} = 0$ $(i = 1, 2, \ldots, n)$*, has the bounded inverse operator* $\mathbf{Q}^{-1}$ *in the space* $E^n = E \times E \times \ldots \times E$*.*

7) For some $g_i \in E$, $i = 1, 2, \ldots, n$*.*

Then for the problem (1)-(2) the following alternative holds: the corresponding homogenous problem has a finite number of linearly independent solution, or the problem (1)-(2) has a unique solution in this class $u \in C^n((0, T]\,; E(A_n) \cap E(A_{n-1}), \ldots, E(A_1), E) \cap C^{n-1}([0, T] : E(A_{n-1}), \ldots, E(A_1), E)$ *and for the solution* $u(t)$ *the estimates*

$$\left\| u \right\|_{C^{n-1}([0,T]:E(A_{n-1}),\ldots,E(A_1),E)} \leq C(\sum_{i=1}^{n} \left\| g_i \right\| + \left\| f \right\|_{C^{\gamma}([0,T]:E)}),$$

$$\left\| u \right\|_{C^{n}([\delta,T]:E(A_n)\cap E(A_{n-1}),\ldots,E(A_1),E)} \leq C(\delta)(\sum_{i=1}^{n} \left\| g_i \right\| + \left\| f \right\|_{C^{\gamma}([0,T]:E)}),$$

for any $0 < \delta < T$ *are valid.*

For some class of boundary value problems condition (6) of Theorem 3 may be substituted by the conditions through the given boundary operators.

Consider the boundary conditions:

$$L_i'(u) \equiv P_i u^{(n-1)}(0) + \sum_{k=1}^{n} B_{ik} u^{(n-k)}(T) = g_i, i = 1, 2, \ldots, n\,. \quad (8)$$

Let all the conditions of Theorem 3 except the condition (6) be fulfilled. Later let

8) *The operators* P_i, $i = 1, 2, \ldots, n$ *respectively, and the operators* $P_k A_{k-1}^{-1}$ *have the bounded inverse operators* $A_{k-1} P_k^{-1}$ *(*$k = 1, 2, \ldots, n$*;* $A_0 \equiv I$*) respectively.*

Then for the problem (1), (8) the statemed of Theorem 3 is valid.

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