International Journal of Pure and Applied Mathematics
Volume 4 No. 3 2003, 281-304

EXISTENCES, ASYMPTOTIC BEHAVIORS AND SYMMETRIC PROPERTIES OF SOLUTIONS OF SEMILINEAR ELLIPTIC EQUATIONS IN FLAT FLASK DOMAINS

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Abstract: We assert that there exists a ground state solution of equation (1) in a flat flask domain: the Esteban-Lions domain $\mathbb{S}_0^r$ by adding an arbitrary small width but sufficient long corridor. We also establish the asymptotic behavior and the symmetry of solutions of equation (1) in flat flask domains.

AMS Subject Classification: 35J20, 35J25
Key Words: semilinear elliptic equation, asymptotic behavior, symmetry, flat flask domain

1. Introduction

Consider the semilinear elliptic equation

$$\begin{cases} -\Delta u + u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is an unbounded domain in $\mathbb{R}^N$, $N \geq 2$, and the nonlinear function $f : \mathbb{R}^+ \to \mathbb{R}$ is of class C^1 and satisfies the following conditions:

$(f1)$ $f(\xi) = o(\xi)$ near $\xi = 0$.

Received: December 5, 2002

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$(f2)$

$$\lim_{\xi\to\infty}\frac{f(\xi)}{\xi^s}=0$$

for some $1<s<2^*-1$.

$(f3)$ For some $2<\theta\leq s+1$ we have

$$0\leq\theta F(\xi)<f(\xi)\xi \quad \text{for all } \ \xi>0, \text{ where } F(\xi)=\int_0^\xi f(\tau)d\tau\,.$$

$(f4)$ The function

$$\xi\to\frac{f(\xi)}{\xi}$$

is strictly increasing.

We extend f to be 0 for $\xi<0$. A typical example of the nonlinearity is $f(\xi)=\sum_{i=1}^k a_i\xi^{p_i}$, where $a_i>0$ and $1<p_i<2^*-1$ for each i. Associated with the equation (1), we consider the Sobolev space $H_0^1(\Omega)$, the potential operators a, $b_f : H_0^1(\Omega)\to\mathbb{R}$, and the energy functional $I : H_0^1(\Omega)\to\mathbb{R}$, which are given by

$$a(u)=\int_\Omega\left(|\nabla u|^2+u^2\right), \quad b_f(u)=\int_\Omega f(u)u\,,$$
$$I(u)=\frac{1}{2}a(u)-\int_\Omega F(u).$$

It is known that I is of $C^{1,1}$ and I satisfies the mountain pass hypothesis (see Rabinowitz [10], Proposition B.10 and Lemma 6 (i)). Note that every nonzero critical point of I is a nonzero solution of equation (1).

Sobolev spaces $H_0^1(\Omega)$ provide the proper functional setting for the study of the partial differential equations and Sobolev imbedding theorems (Sobolev operators) provide the connection between Sobolev spaces and Lebesgue spaces.

(i) Standard books describe that the Sobolev critical operator $I : H_0^1(\Omega)\to L^{2^*}(\Omega)$ satisfies

$$\|u\|_{L^{2^*}}\leq c||\nabla u||_{L^2}.$$

Let $S^c(\Omega)$ be the best constant of the Sobolev critical operator defined as

$$S^c(\Omega)=\sup\left\{\frac{\|u\|_{L^{2^*}}}{||\nabla u||_{L^2}}\mid u\in H_0^1(\Omega)\backslash\{0\}\right\}.$$

Then $S^c(\Omega)$ is independent of Ω, $S^c(\Omega)$ is achieved if and only if $\Omega=\mathbb{R}^N$.

(ii) Standard books also describe that the Sobolev subcritical operator $I : H_0^1(\Omega) \to L^p(\Omega)$ satisfies

$$\|u\|_{L^p(\Omega)} \leq c||u||_{H^1(\Omega)}.$$

Let $S(\Omega)$ be the best constant of the Sobolev subcritical operator defined as

$$S(\Omega) = \sup\left\{\frac{\|u\|_{L^p(\Omega)}}{||u||_{H^1(\Omega)}} \mid u \in H_0^1(\Omega)\backslash\{0\}\right\}.$$

Open Question. For which domain Ω, $S(\Omega)$ is achieved?

For the convenience, consider the following definition:

Definition 1. We call that a domain Ω in $\mathbb{R}^N$ is an achieved domain if there is $u \in H_0^1(\Omega)$ such that $\dfrac{\|u\|_{L^p(\Omega)}}{||u||_{H^1(\Omega)}} = S(\Omega)$. Otherwise, we call that Ω is a nonachieved domain.

A bounded domain is achieved. For the unbounded domains, due to the lack of compactness, to solve the open question is very difficult. For convenience, we give the following definition.

Definition 2. A proper unbounded domain Ω in $\mathbb{R}^N$ is called an Esteban-Lions domain if there is $\chi \in \mathbb{R}^N$, $||\chi|| = 1$ such that $n(z) \cdot \chi \gneqq 0$ on $\partial\Omega$, where $n(z)$ denotes the unit outward normal to $\partial\Omega$ at the point z.

A typical example of an Esteban-Lions domain is the upper semi-strip domain $\mathbb{S}_0^r$. Esteban-Lions [5] asserted that there does not exist any solutions in $H_0^1(\Omega)$ for the equation (1) in an Esteban-Lions domain Ω. In particular, there does not exist any solutions in $H_0^1(\mathbb{S}_0^r)$ for equation (1) in $\mathbb{S}_0^r$: $\mathbb{S}_0^r$ is a nonachieved domain. Berestycki conjectured that there exists a solution of equation (1) in $\mathbb{S}_0^r$ with a hole. Lien-Tzeng-Wang [8] asserted that there exists a ground state solution of equation (1) in $\mathbb{S}^r$ and Chen-Chen-Wang [4] asserted that every solution $u(x, y)$ of equation (1) is radially symmetric in x and axially symmetric in y; that is, $u(x, y - \sigma) = u(|x|, |y - \sigma|)$ for some σ. Using these results, Wang [12] asserted the Berestycki conjecture.

To describe the existence of solutions in equation (1) in a general domain is extremely difficult. We start with equation (1) in the domains, which are the perturbations of the Esteban-Lions domain $\mathbb{S}_0^r$ by adding a domain to it.

For $r > 0$, $R > 0$, denote by $\mathbf{B}^N(z_0; R)$ the $N-$ball, $\mathbb{S}^r$ the strip domain, $\mathbb{S}_0^r$ the upper semi-strip domain, and the flat flask domain Ω_ε with a small width but sufficiently long corridor, where $\delta > 0$ and $\varepsilon > 0$ are two small numbers, as follows:

$$\begin{aligned}
&\mathbf{B}^N(z_0; R) = \{z \in \mathbb{R}^N \mid |z - z_0| < R\}\,,\\
&\mathbf{B}_+(0; r+\delta) = \{(x,y) \in \mathbf{B}^N(0; r+\delta) | y > 0\}\,,\\
&\mathbf{B}_-(0; r+\delta) = \{(x,y) \in \mathbf{B}^N(0; r+\delta) | y < 0\}\,,\\
&\mathbb{S}^r = \{(x,y) \in \mathbb{R}^{N-1} \times \mathbb{R} \mid |x| < r\}\,,\\
&\mathbb{S}_0^r = \{(x,y) \in \mathbb{S}^r \mid 0 < y\}\,,\\
&E_\varepsilon = \mathbf{B}_+(0; r+\delta) \cup \{(x,y) \in \mathbb{R}^N \mid (x, \varepsilon y) \in \mathbf{B}_-(0; r+\delta)\}\,,\\
&\Omega_\varepsilon = \mathbb{S}_0^r \cup E_\varepsilon.
\end{aligned}$$

In this article we assert that there exists an $\varepsilon_0 > 0$, such that if $\varepsilon \leq \varepsilon_0$, then the flat flask domain Ω_ε is achieved. Furthermore, we assert that the asymptotic behavior of the solutions of equation (1) in Ω_ε and the symmetry of solutions of equation (1) in Ω_ε.

2. Palais-Smale Values

In this section, we study systematically the $(PS)-$values and the index of a domain Ω. We obtain the index comparison criterion (see Theorem 13.) We use this criterion to prove our existence results in next section.

The nonlinearity f satisfies the followings properties.

Lemma 3. *Assume that f satisfies $(f1)-$ $(f3)$, then:*
$(f5)$ $|f(\xi)| \leq \varepsilon |\xi| + c_1 |\xi|^s$ for any ξ and $1 < s < 2^* - 1$,
$(f6)$ $F(\xi) \geq c(r)\xi^\theta$ for $\xi \geq r > 0$, where $2 < \theta \leq s + 1$,
$(f7)$ $F(\xi) \geq c_2\xi^\theta - c_2\xi^2$ for any $\xi > 0$, where $c_2 = F(1) > 0$.

Proof. $(f5)$ Given $\varepsilon > 0$. By $(f1)$, there exists $\delta > 0$ such that $|\xi| < \delta$ implies $|f(\xi)| \leq \varepsilon |\xi|$. By $(f2)$, there exists $N > 0$ such that $|\xi| > N$ implies $|f(\xi)| \leq \varepsilon |\xi|^s$. For $\xi \in [\delta, N]$, since $f(\xi)$ is continuous on $[\delta, N]$, then there exist $c' > 0$, $c'' > 0$ such that for $\xi \in [\delta, N]$

$$|f(\xi)| \leq c' = c''\delta^s \leq c'' |\xi|^s\,.$$

Thus, $|f(\xi)| \le \varepsilon |\xi| + c_1 |\xi|^s$ for any ξ.

$(f6)$ By $(f3)$, we have

$$\left(\frac{F(\xi)}{\xi^\theta}\right)' = \frac{f(\xi)\xi^\theta - F(\xi)\theta\xi^{\theta-1}}{\xi^{2\theta}} = \frac{\xi^{\theta-1}\left[f(\xi)\xi - \theta F(\xi)\right]}{\xi^{2\theta}} > 0.$$

Therefore $\frac{F(\xi)}{\xi^\theta}$ is a strictly increasing function: there exists $r > 0$ such that $\xi \ge r > 0$ implies $\frac{F(\xi)}{\xi^\theta} \ge \frac{F(r)}{r^\theta} = c(r)$. Hence $F(\xi) \ge c(r)\xi^\theta$ for $\xi \ge r > 0$, where $2 < \theta \le s+1$.

$(f7)$ We know $\left(\frac{F(\xi)}{\xi^\theta}\right)' > 0$ for any $\xi > 0$. For $\xi \ge 1$, we have

$$\frac{F(\xi)}{\xi^\theta} \ge \frac{F(1)}{1^\theta} = F(1) = c_2 > 0.$$

Therefore $F(\xi) \ge c_2\xi^\theta \ge c_2\xi^\theta - c_2\xi^2$. For $0 < \xi < 1$, we have

$$c_2\xi^\theta - c_2\xi^2 = c_2\xi^2\left(\xi^{\theta-2} - 1\right) \le 0 \le F(\xi).$$

Hence $F(\xi) \ge c_2\xi^\theta - c_2\xi^2$ for any $\xi > 0$, where $c_2 = F(1) > 0$. □

In the following definitions, we simply denote Palais-Smale by (PS).

Definition 4. (i) For $\beta \in \mathbb{R}$, a sequence $\{u_n\}$ in $H_0^1(\Omega)$ is a $(PS)_\beta$-sequence for I if $I(u_n) = \beta + o(1)$ and $I'(u_n) = o(1)$ strongly in $H^{-1}(\Omega)$ as $n \to \infty$;

(ii) $\beta \in \mathbb{R}$ is a $(PS)_\beta$-value for I if there is a $(PS)_\beta$-sequence for I,

(iii) I satisfies the $(PS)_\beta$-condition if every $(PS)_\beta$-sequence for I contains a convergent subsequence,

(iv) I satisfies the (PS)-condition if, for every $\beta \in \mathbb{R}$, every $(PS)_\beta$-sequence for I contains a convergent subsequence.

A $(PS)_\beta$-sequence for I admits a weak limit.

Lemma 5. *Let $\{u_n\}$ in $H_0^1(\Omega)$ be a $(PS)_\beta$-sequence for I, then we have the following results,*

(i) *There exists a $c > 0$ such that $\|u_n\|_{H^1} \le c$ for each n,*

(ii) $a(u_n) = b_f(u_n) + o(1)$ *and* $\beta \ge 0$, *(iii) If $\beta > 0$, then there exist a subsequence $\{u_n\}$ and $c' > 0$ such that $\|u_n\|_{H^1} \geq c'$ for each n.*

Proof. (i) Let $2 < \theta \leq s+1$, $1 < s < 2^* - 1$. By $(f3)$, for large n, $\varepsilon_n = ||I'(u_n)||$,

$$|\beta| + 1 + \frac{\varepsilon_n ||u_n||}{\theta} \geq I(u_n) - \frac{1}{\theta}\langle I'(u_n), u_n\rangle = (\frac{1}{2} - \frac{1}{\theta})\|u_n\|^2_{H^1(\Omega)} - \int_\Omega [F(u_n) - \frac{1}{\theta} f(u_n)u_n] \geq (\frac{1}{2} - \frac{1}{\theta})\|u_n\|^2_{H^1(\Omega)}. \quad (2)$$

Thus, $||u_n||_{H^1} \leq c$ for each n.

(ii) Since $\{u_n\}$ is bounded in $H_0^1(\Omega)$, we have $o(1) = \langle I'(u_n), u_n\rangle = a(u_n) - b_f(u_n)$. By $(f3)$, we have

$$\beta = \frac{1}{2}a(u_n) - \int_\Omega F(u_n) + o(1) \geq \left(\frac{1}{2} - \frac{1}{\theta}\right)\int_\Omega f(u_n)\, u_n + o(1) \geq 0.$$

(iii) Suppose that $\beta > 0$. If $\lim_{n\to\infty} \|u_n\|_{H^1} = 0$, by $(f3)$ and (ii), then $I(u_n) = o(1)$, which is a contradiction. Thus, there exist a subsequence $\{u_n\}$ and $c' > 0$ such that $\|u_n\|_{H^1} \geq c'$ for each n. □

A $(PS)_\beta$-sequence for I admits a weak limit, which is a critical point for I.

Consider the Nehari minimization problem α_M, and the minimax problem α_Γ, where

$$\begin{aligned} \Gamma &= \left\{\gamma \in C\left([0,1], H_0^1(\Omega)\right) \mid I(\gamma(0)) = 0, I(\gamma(1)) < 0\right\}, \\ \alpha_\Gamma &= \inf_{\gamma\in\Gamma} \sup_{\tau\in[0,1]} I(\gamma(\tau)), \\ M &= \left\{u \in H_0^1(\Omega) \setminus \{0\} \mid a(u) = b_f(u)\right\}, \\ \alpha_M &= \inf_{u\in M} I(u). \end{aligned}$$

Let $K(u) = \langle I'(u), u\rangle = a(u) - b_f(u)$, then $K(u)$ enjoys several properties.

Lemma 6. *(i) There exist $c, \delta > 0$ such that for $\|u\|_{H^1} = \delta$, we have $K(u) \geq c\delta^2$, and $I(u) \geq \frac{1}{2}K(u) \geq \frac{c\delta^2}{2}$,*

(ii) For $\gamma \in \Gamma$, there exists a $\tau_0 \in (0,1]$ such that $\|\gamma(\tau_0)\|_{H^1} = \delta$ and $K(\gamma(\tau_0)) > 0$,

(iii) If $\gamma(\tau) \notin M$ for each $\tau \in (0,1]$, then $K(\gamma(\tau)) > 0$ for each $\tau \in (0,1]$.

Proof. (i) By $(f5)$ we have

$$f(\xi) \leq \varepsilon |\xi| + c_1 |\xi|^s, \text{ for any } \xi > 0.$$

Then by the Sobolev continuous imbedding, we obtain

$$\begin{aligned} K(u) &\geq \|u\|_{H^1}^2 - \varepsilon \|u\|_{H^1}^2 - c_2 \|u\|_{H^1}^{s+1} \\ &= \|u\|_{H^1}^2 - \left(\varepsilon + c_2 \|u\|_{H^1}^{s-1}\right) \|u\|_{H^1}^2 \\ &= \left[1 - \left(\varepsilon + c_2 \|u\|_{H^1}^{s-1}\right)\right] \|u\|_{H^1}^2 \\ &\geq c\delta^2, \end{aligned}$$

where $\|u\|_{H^1} = \delta$ such that $1 - \left(\varepsilon + c_2 \|u\|_{H^1}^{s-1}\right) \geq c$ for some constant $c > 0$. By $(f3)$ we have

$$\begin{aligned} I(u) &= \frac{1}{2} \|u\|_{H^1}^2 - \int_\Omega F(u) \\ &\geq \frac{1}{2} \|u\|_{H^1}^2 - \frac{1}{\theta} \int_\Omega f(u) u \\ &\geq \frac{1}{2} K(u) \geq \frac{c}{2}\delta^2. \end{aligned}$$

(ii) For $\gamma \in \Gamma$, let

$$U_1 = \{\tau \in [0,1] \mid \|\gamma(\tau)\|_{H^1} < \delta \},$$

and

$$U_2 = \{\tau \in [0,1] \mid \|\gamma(\tau)\|_{H^1} > \delta \}.$$

Then U_1, U_2 are open in $[0,1]$. Since $\gamma(0) = 0$ and $I(\gamma(1)) < 0$, then by part (i), U_1, U_2 are nonempty. Thus, $[0,1]$ is disconnected, a contradiction. Hence there exists a $\tau_0 \in (0,1]$ such that $\|\gamma(\tau_0)\|_{H^1} = \delta$ and $K(\gamma(\tau_0)) > 0$.

(iii) Since $\gamma \in C\left([0,1], H_0^1(\Omega)\right)$ and $\gamma(0) = 0$, by part (ii) there exists a $\tau_0 \in (0,1]$ such that $0 < \|\gamma(\tau_0)\|_{H^1} = \delta$ and $K(\gamma(\tau_0)) > 0$. Since K is continuous, then $K \circ \gamma$ is still continuous. Assume, by contradiction, there exists a $\tau_1 \in (0,1]$ such that $K(\gamma(\tau_1)) < 0$, then we get a $\tau_2 \in [\tau_0, \tau_1] \subset (0,1]$ such that $K(\gamma(\tau_2)) = 0$, that is $\gamma(\tau_2) \in M$. This contradicts the fact that $\gamma(\tau) \notin M$ for all $\tau \in (0,1]$. Hence, $K(\gamma(\tau)) > 0$ for each $\tau \in (0,1]$. □

To examine that several important $(PS)-$values are the same, we should understand the growth condition of the energy functional I. Let $u \in H_0^1(\Omega) \setminus \{0\}$

and $E_+ = \{z \in \Omega \mid u(z) > 0\}$. If $|E_+| = 0$, then clearly we have $I(tu) = \frac{t^2}{2}a(u) \to \infty$ as $t \to \infty$. Note that

$$I(tu) = \frac{t^2}{2}a(u) - \int_\Omega F(tu) \,,$$
$$\frac{d}{dt}I(tu) = ta(u) - \int_\Omega f(tu)\,u\,,$$
$$\frac{d^2}{dt^2}I(tu) = a(u) - \int_\Omega f'(tu)\,u^2.$$

Lemma 7. *For $u \in H_0^1(\Omega)\backslash\{0\}$ such that $|E_+| > 0$. Let*

$$\varphi(u) = \min\left\{t > 0 \mid \frac{d}{dt}I(tu) = 0\right\},$$

then we have the following results:

(i) $I(tu) \to -\infty$ *as* $t \to \infty$,

(ii) $0 < \varphi(u) < \infty$,

(iii) $\frac{d^2}{dt^2}I(tu) < 0$ *for* $t > \varphi(u)$,

(iv) I *has exact one critical point* $\varphi(u)u$,

(v) $\frac{d}{dt}I(tu) \geq 0$ *for* $t \leq \varphi(u)$.

Proof. (i) Since $|E_+| > 0$, there exists $n_0 > 0$ such that $|A| > 0$, where $A = \left\{z \in E_+ \mid u(z) > \frac{1}{n_0}\right\}$. By $(f6)$, for $z \in A$ and $\frac{t}{n_0} \geq r$, we have $F(tu) \geq c(r)(tu)^\theta$. Now

$$\begin{aligned} I(tu) &\leq \frac{t^2}{2}a(u) - \int_A F(tu) \\ &\leq \frac{t^2}{2}a(u) - c(r)t^\theta \int_A u^\theta \\ &\leq t^2\left\{\frac{1}{2}a(u) - c(r)t^{\theta-2}\int_A u^\theta\right\} \to -\infty \text{ as } t \to \infty. \end{aligned}$$

(ii) By the definition of $\varphi(u)$ and part (i).

(iii) By $(f4)$, for $u^+ = \max\{u, 0\}$, we have $\left(\frac{f(tu^+)}{tu^+}\right)' > 0$, that is,

$f'(tu^+)u^{+2} > \dfrac{f(tu^+)\,u^+}{t}$, then

$$\begin{aligned}\frac{d^2}{dt^2}I(tu) &= a(u) - \textstyle\int_\Omega f'(tu)u^2 = \int_\Omega \frac{f(\varphi(u)u^+)}{\varphi(u)} - \int_\Omega f'(tu^+)u^{+2} \\ &< \textstyle\int_\Omega \frac{f(\varphi(u)u^+)\,u^+}{\varphi(u)} - \frac{f(tu^+)\,u^+}{t} \\ &= \textstyle\int_\Omega u^{+2}\left(\frac{f(\varphi(u)u^+)}{\varphi(u)u^+} - \frac{f(tu^+)}{tu^+}\right) \\ &< 0 \text{ for } t > \varphi(u).\end{aligned}$$

(iv) By the definition of $\varphi(u)$, $I(0) = 0$, part (ii) and (iii).

(v) By the definition of $\varphi(u)$ and part (iv). □

Remark 1. By Lemma 7, we have:

(i) $M = \left\{v = \varphi(u)u \mid u \in H_0^1(\Omega),\ |E_+| > 0\right\}$,

(ii) $\alpha_M = \inf\limits_{\substack{u\in H_0^1\\ u\neq 0}} \sup\limits_{t\geq 0} I(tu) = \inf\limits_{\substack{u\in H_0^1\\ |E_+|>0}} \sup\limits_{t\geq 0} I(tu)$.

Brezis-Nirenberg [3] asserted that α_Γ is a $(PS)-$value and Stuart [11] asserted that α_M is a $(PS)-$value. These two values are the same.

Theorem 8. *If* $(f1)-(f4)$ *hold, then*

$$\alpha_\Gamma = \alpha_M.$$

Proof. For $u \in H_0^1(\Omega)\setminus\{0\}$, by Lemma 7, I has exact one critical point $\varphi(u)u$ and $\frac{d^2}{dt^2}I(tu) < 0$ for $t > \varphi(u)$. We claim that $\alpha_M \geq \alpha_\Gamma$. In fact, for $u \in H_0^1(\Omega)\setminus\{0\}$ and $|E_+| > 0$, there exists $t_1 > 0$ such that $I(t_1u) < 0$. For $\varepsilon > 0$, there exists $u \in H_0^1(\Omega)\setminus\{0\}$ and $|E_+| > 0$ such that

$$\alpha_M + \varepsilon > \sup_{t\geq 0} I(tu) = I(\varphi(u)u) = \sup_{\tau\in[0,1]} I(\gamma(\tau)) \geq \alpha_\Gamma,$$

where $\gamma : [0,1] \to H_0^1(\Omega)$ is continuous with $\gamma(\tau) = (\tau t_1)u$. Thus, $\alpha_M \geq \alpha_\Gamma$.

On the other hand, for $\gamma \in \Gamma$ there exists $\tau \in (0,1]$ such that $\gamma(\tau) \in M$. Otherwise, then $\gamma(\tau) \notin M$ for all $\tau \in (0,1]$. Therefore, by Lemma 6 (iii), we have $K(\gamma(\tau)) > 0$. Then, using $(f3)$ to obtain

$$\begin{aligned} I(\gamma(\tau)) &= \frac{1}{2}\int_\Omega |\nabla\gamma(\tau)|^2 + \gamma(\tau)^2 - \int_\Omega F(\gamma(\tau)) \\ &> \frac{1}{2}\int_\Omega f(\gamma(\tau))\gamma(\tau) - \int_\Omega F(\gamma(\tau)) \\ &\geq \int_\Omega F(\gamma(\tau)) - \int_\Omega F(\gamma(\tau)) \\ &\geq 0 \quad \text{for all } \tau \in [0,1]. \end{aligned}$$

This contradicts the definition of Γ. For $\gamma \in \Gamma$ there exists $\tau_0 \in (0,1]$ such that $\gamma(\tau_0) \in M$, then $\alpha_\Gamma + \varepsilon > \sup_{\tau\in[0,1]} I(\gamma(\tau)) \geq I(\gamma(\tau_0)) \geq \alpha_M$, thus $\alpha_\Gamma \geq \alpha_M$. Then we conclude that $\alpha_\Gamma = \alpha_M$. □

Definition 9. By Theorem 8, we conclude that the numbers α_Γ and α_M are the same. Any one of them is called the index of I in Ω and denoted by $\alpha(\Omega)$. We call that a solution u of equation (1) is a ground state solution if $I(u) = \alpha(\Omega)$, and is a higher energy solution if $I(u) > \alpha(\Omega)$.

Remark 2. It is easy to see that a domain Ω is achieved if and only if there is a ground state solution of equation (1) in Ω.

The energy of each nonzero solution of equation (1) admits a lower bound.

Theorem 10. *(i) Let $\{u_n\}$ in $H_0^1(\Omega)$ be a $(PS)_{\alpha(\Omega)}$-sequence for I and u in $H_0^1(\Omega)$ satisfying $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$. Then, u is a weak solution of equation (1).*

(ii) Let $\{u_n\}$ in $H_0^1(\Omega)$ be a $(PS)_{\alpha(\Omega)}$-sequence for I and u in $H_0^1(\Omega)$ satisfying $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$ and u is nonzero. Then u is a positive ground solution of equation (1).

Proof. (i) Since $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$, then for each $\phi \in C_c^\infty(\Omega)$

$$\int_\Omega \nabla u_n \nabla\phi + u_n\phi \to \int_\Omega \nabla u\nabla\phi + u\phi.$$

By $(f5)$, $u_n \to u$ strongly in $L^q_{loc}(\Omega)$ and a.e. for $1 \leq q < 2^*$, we obtain

$$\int_\Omega f(u_n)\phi \to \int_\Omega f(u)\phi.$$

Thus,

$$\begin{aligned} \langle I'(u), \phi\rangle &= \textstyle\int_\Omega \nabla u\nabla\phi + u\phi - \int_\Omega f(u)\phi \\ &= \lim_{n\to\infty}\left(\textstyle\int_\Omega \nabla u_n\nabla\phi + u_n\phi - \int_\Omega f(u_n)\phi\right) \\ &= \lim_{n\to\infty}\langle I'(u_n), \phi\rangle = 0. \end{aligned}$$

(ii) By *(i)*, u is a nonzero solution of equation (1). Thus, $u \in M$ and $I(u) \geq \alpha(\Omega)$. Since I is weakly lower semicontinuous, we have

$$\alpha(\Omega) \leq I(u) \leq \liminf_{n\to\infty} I(u_n) = \alpha(\Omega).$$

Let $u^- = \max\{-u, 0\}$. Multiply equation (1) by u^- and integrate to obtain

$$\int_\Omega \nabla u \nabla u^- + uu^- = \int_\Omega f(u)u^-.$$

Consequently, $\int_\Omega |\nabla u^-|^2 + |u^-|^2 = 0$, that is, $u^- = 0$. By the maximum principle, u is a positive ground state solution of equation (1). □

Remark 3. If the Nehari minimization problem α_M or the minimax problem α_Γ admits a nonzero solution u, then u must be a ground state solution. If I satisfies the $(PS)_{\alpha(\Omega)}$-condition, then the Nehari minimization problem α_M and the minimax problem α_Γ admit a ground state solution.

Let $\{u_n\}$ be a $(PS)_\beta$-sequence for I, where $\beta > 0$, and $t_n u_n \in M$ for $t_n = \varphi(u_n) > 0$, then $\{||t_n u_n||_{H^1}\}$ is bounded above and bounded away from zero.

Lemma 11. *Let $\{u_n\}$ in $H_0^1(\Omega)$ be a $(PS)_\beta$-sequence for I, where $\beta > 0$ and $t_n u_n \in M$ for $t_n > 0$, then there exist a subsequence $\{t_n\}$ and a subsequence $\{u_n\}$ such that $c' \leq t_n \leq c''$ and $C' \leq ||t_n u_n||_{H^1} \leq C''$ for each $n \in \mathbb{N}$, where c', c'', C' and $C'' > 0$.*

Proof. *(i)* Claim $t_n \leq c''$. By Lemma 5, $\{u_n\}$ is bounded, say $\|u_n\|_{H^1} \leq c_1$.

$$\begin{aligned} &I(u_n) = \tfrac{1}{2}a(u_n) - \textstyle\int_\Omega F(u_n) = \beta + o(1), \\ &< I'(u_n), u_n >= a(u_n) - \textstyle\int_\Omega f(u_n)u_n = o(1). \end{aligned}$$

Suppose $\int_\Omega f(u_n)u_n = o(1)$, then $a(u_n) = o(1)$ and

$$\int_\Omega F(u_n) \leq \frac{1}{\theta}\int_\Omega f(u_n)u_n = o(1).$$

Thus, $I(u_n) = o(1)$, a contradiction. Therefore there exist a subsequence $\{u_n\}$ and $l > 0$ such that

$$l \leq \int_\Omega f(u_n)u_n \leq \varepsilon\int_\Omega |u_n|^2 + c\int_\Omega |u_n|^{s+1} \leq \varepsilon c_1^2 + c\int_\Omega |u_n|^{s+1}.$$

Let $\varepsilon c_1^2 \leq \frac{l}{2}$ to obtain $\int_\Omega |u_n|^{s+1} \geq c_2 > 0$. Since $\theta \leq s+1 < 2^*$, let $s+1 = (1-\xi)\,\theta + \xi 2^*$, $0 \leq \xi < 1$, then

$$\begin{aligned} c_2 &\leq \int_\Omega |u_n|^{s+1} = \int_\Omega |u_n|^{(1-\xi)\theta} |u_n|^{\xi 2^*} \\ &\leq \left(\int_\Omega |u_n|^\theta\right)^{1-\xi} \left(\int_\Omega |u_n|^{2^*}\right)^{\xi} \leq c_3 \left(\int_\Omega |u_n|^\theta\right)^{1-\xi}. \end{aligned}$$

Thus, $\int_\Omega |u_n|^\theta \geq c_4 > 0$. Hence by Lemma 3, $(f7)$ and Lemma 5, we have

$$\begin{aligned} 0 &\leq I\left(t_n u_n\right) \leq \frac{t_n^2}{2} a\left(u_n\right) + c_5 \int_\Omega |t_n u_n|^2 - c_5 \int_\Omega |t_n u_n|^\theta \\ &\leq c_6 t_n^2 - c_7 t_n^\theta, \text{ where } c_5, c_6, c_7 > 0. \end{aligned}$$

Thus, there is a $c'' > 0$ such that

$$t_n \leq c'' \text{ for each } n \in \mathbb{N}.$$

(ii) Claim $c' \leq t_n$. By Lemma 6 (i), there exist $c, \delta > 0$ such that for $\|u\|_{H^1} = \delta$, we have $I(u) \geq \frac{c\delta^2}{2}$. Since $I\,(0) = 0$ and I is continuous, we have $I(u) < \frac{c\delta^2}{2}$ for $\|u\|_{H^1} < \delta_1 < \delta$. By Lemma 5, we have $0 < c_9 \leq \|u_n\|_{H^1} \leq c_{10}$ for each $n \in \mathbb{N}$. Since I is achieved the maximum at $t_n u_n$ for each $n \in \mathbb{N}$, then

$$t_n c_{10} \geq t_n \left\|u_n\right\|_{H^1} = \left\|t_n u_n\right\|_{H^1} \geq \delta_1 \text{ for each } n \in \mathbb{N}.$$

Therefore there is $c' > 0$ such that $t_n \geq c'$. Hence, by part (i) and (ii), we have $c' \leq t_n \leq c''$ for each $n \in \mathbb{N}$. □

By Theorem 10, a $(PS)_{\alpha(\Omega)}$-sequence for I admits a weak limit u, which is a critical point for I. However, we should do more effort to assert that u is nonzero, that is, u is a positive solution of equation (1). From now on $\alpha\,(\Omega)$ is simply denoted by α. Let

$$\begin{aligned} &\Omega_n = \Omega \cap \mathbf{B}^N(0;n)\,, \\ &\widetilde{\Omega}_n = \Omega \setminus \overline{\mathbf{B}^N(0;n)}\,, \\ &M_n = \left\{u \in H_0^1(\Omega_n) \backslash \{0\} \mid a\,(u) = b_f\,(u)\right\}, \\ &\widetilde{M}_n = \left\{u \in H_0^1(\widetilde{\Omega}_n) \backslash \{0\} \mid a\,(u) = b_f\,(u)\right\}, \\ &\alpha_n = \alpha(\Omega_n) = \inf_{u \in M_n} I\,(u)\,, \\ &\widetilde{\alpha}_n = \alpha(\widetilde{\Omega}_n) = \inf_{u \in \widetilde{M}_n} I\,(u)\,. \end{aligned}$$

We shall see what will happen whenever u is zero.

Theorem 12. *Let $\{u_n\}$ in $H_0^1(\Omega)$ be a $(PS)_\alpha$-sequence for I.*

(i) Suppose that $u_n \rightharpoonup 0$ weakly in $H_0^1(\Omega)$, then there is a subsequence $\{u_n\}$ such that

$$\int_{\Omega_n} f(u_n)\, u_n = o(1) \text{ and } \int_{\Omega_n} F(u_n) = o(1);$$

(ii) Suppose there is a subsequence $\{u_n\}$ such that

$$\int_{\Omega_n} f(u_n)\, u_n = o(1),$$

then we have $\alpha = \widetilde{\alpha}_n$ for each $n \in \mathbb{N}$.

(iii) Suppose that $u_n \rightharpoonup 0$ weakly in $H_0^1(\Omega)$, then $\alpha = \widetilde{\alpha}_n$ for each $n \in \mathbb{N}$.

Proof. (i) Since $u_n \rightharpoonup 0$ weakly in $H_0^1(\Omega)$, then $u_n \to 0$ strongly $L^q_{loc}(\Omega)$, where $1 \leq q < 2^*$. Thus, for each $m \in \mathbb{N}$, $\lim_{n\to\infty} \int_{\Omega_m} |u_n|^p = 0$. We can take a subsequence $\{u_{n_m}\}$ such that $\int_{\Omega_m} |u_{n_m}|^p < \dfrac{1}{m}$. Therefore there is a subsequence $\{u_n\}$ such that $\int_{\Omega_n} |u_n|^p = o(1)$. By $(f3)$, and Lemma 3 $(f5)$, we have

$$\int_{\Omega_n} f(u_n)\, u_n = o(1) \quad \text{and} \quad \int_{\Omega_n} F(u_n) = o(1).$$

(ii) Let $\{u_n\}$ in $H_0^1(\Omega)$ be a $(PS)_\alpha$-sequence for I such that

$$\begin{aligned} I(u_n) &= \frac{1}{2}a(u_n) - \int_\Omega F(u_n) = \alpha + o(1), \\ a(u_n) &= b_f(u_n) + o(1). \end{aligned}$$

Suppose there is a subsequence $\{u_n\}$ such that

$$\int_{\Omega_n} f(u_n)\, u_n = o(1). \tag{3}$$

Let $\xi \in C^\infty([0,\infty))$ such that

$$0 \leq \xi \leq 1,\ \xi(t) = \begin{cases} 0 & \text{for } t \in [0,1], \\ 1 & \text{for } t \in [2,\infty). \end{cases}$$

Let $\xi_n(z) = \xi(\frac{2|z|}{n})$. Since $\{\xi_n^2 u_n\}$ is bounded in $H_0^1(\Omega)$,

$$\begin{aligned} o(1) &= \left\langle I'(u_n), \xi_n^2 u_n \right\rangle \\ &= \textstyle\int_\Omega \left(\xi_n^2 |\nabla u_n|^2 + 2\xi_n u_n \nabla \xi_n \cdot \nabla u_n + \xi_n^2 u_n^2\right) - \int_\Omega \xi_n^2 f(u_n)\, u_n\ . \end{aligned}$$

Note that $|\nabla \xi_n(z)| \le \frac{c}{n}$ and by (3), we have

$$\int_\Omega \xi_n u_n \nabla \xi_n \cdot \nabla u_n = o(1), \tag{4}$$

$$\int_\Omega \xi_n^2 f(u_n) u_n = \int_\Omega f(u_n) u_n + o(1) = a(u_n) + o(1), \tag{5}$$

and

$$\int_\Omega F(\xi_n u_n) = \int_\Omega F(u_n) + o(1). \tag{6}$$

We conclude that

$$\int_\Omega \xi_n^2 (|\nabla u_n|^2 + u_n^2) = a(u_n) + o(1). \tag{7}$$

Let $v_n = \xi_n u_n$. By (3) $-$ (7), we have

$$\begin{aligned} a(v_n) &= \int_\Omega (|\nabla v_n|^2 + v_n^2) \\ &= \int_\Omega \left[|\nabla \xi_n|^2 u_n^2 + \xi_n^2 (|\nabla u_n|^2 + u_n^2) + 2\xi_n u_n \nabla \xi_n \cdot \nabla u_n \right] \\ &= a(u_n) + o(1), \end{aligned}$$

$$\begin{aligned} b_f(v_n) &= \int_\Omega \xi_n f(\xi_n u_n) u_n \\ &= b_f(u_n) + o(1), \end{aligned}$$

and

$$\begin{aligned} I(v_n) &= \frac{1}{2} \int_\Omega (|\nabla v_n|^2 + v_n^2) - \int_\Omega F(v_n) \\ &= \frac{1}{2} \int_\Omega \left[|\nabla \xi_n|^2 u_n^2 + \xi_n^2 (|\nabla u_n|^2 + u_n^2) + 2\xi_n u_n \nabla \xi_n \cdot \nabla u_n \right] \\ &\quad - \int_\Omega F(v_n) \\ &= \frac{1}{2} a(u_n) - \int_\Omega F(u_n) + o(1) \\ &= \alpha + o(1). \end{aligned}$$

Let $h_n(t) = I(tv_n)$ for each $n \in \mathbb{N}$ and $t \ge 0$, then $h_n'(t) = t \int_\Omega (|\nabla v_n|^2 + v_n^2) - \int_\Omega f(tv_n) v_n$ and $h_n'(1) = \langle I'(v_n), v_n \rangle = o(1)$. Claim $|h_n'(t)| \le |h_n'(1)| + t\varepsilon$ for any $\varepsilon > 0$ and large n. Case 1: for $t_n \le t \le 1$, by Lemma 7, then $|h_n'(t)| \le |h_n'(1)|$. Case 2: for $1 < t \le t_n$, since $a(v_n) = b_f(v_n) + o(1)$, $h_n'(1) = o(1)$ and Lemma 7 (v), then for any $\varepsilon > 0$ we have $0 \le h_n'(1) = a(v_n) - b_f(v_n) < \varepsilon$ for large n and $h_n'(t) \geq 0$. Thus, by $(f4)$,

$$\begin{aligned} h_n'(t) - h_n'(1) &= t\int_\Omega (|\nabla v_n|^2 + v_n^2) - \int_\Omega f(tv_n)v_n - h_n'(1) \\ &< t\int_\Omega \frac{f(v_n)v_n^2}{v_n} + t\varepsilon - t\int_\Omega \frac{f(tv_n)v_n^2}{tv_n} \\ &= t\int_\Omega v_n^{+2}\left[\frac{f(v_n^+)}{v_n^+} - \frac{f(tv_n^+)}{tv_n^+}\right] + t\varepsilon \\ &< t\varepsilon \quad \text{for large } n. \end{aligned}$$

Therefore $|h_n'(t)| \leq |h_n'(1)| + t\varepsilon$ for any $\varepsilon > 0$ and large n. Let $\{t_n\}$ be a sequence such that $t_n v_n \in \widetilde{M}_{\frac{n}{2}}$. Thus, by the mean value theorem, Cases 1 and 2 and from Lemma 11, a subsequence $\{t_n\}$ is bounded, we have

$$\begin{aligned} \lim_{n\to\infty} |I(v_n) - I(t_n v_n)| &= \lim_{n\to\infty} |h_n(1) - h_n(t_n)| \\ &= \lim_{n\to\infty} \left|h_n'(\theta_n)\right| |1 - t_n|, \text{ where } \theta_n \text{ is between 1 and} \\ & \qquad t_n \leq c\lim_{n\to\infty} \left|h_n'(1)\right| = 0. \end{aligned}$$

Hence $I(t_n v_n) = I(v_n) + o(1) = \alpha + o(1)$. Since $t_n v_n \in \widetilde{M}_{\frac{n}{2}}$, we have $\widetilde{\alpha}_n \leq I(t_{2n}v_{2n})$ for each $n \in \mathbb{N}$. Taking limit to obtain $\lim_{n\to\infty} \widetilde{\alpha}_n \leq \alpha$. Since $\Omega \supset \widetilde{\Omega}_n \supset \widetilde{\Omega}_{n+1}$, we have $\alpha \leq \widetilde{\alpha}_n \leq \widetilde{\alpha}_{n+1}$ for each $n \in \mathbb{N}$. Then we conclude that $\alpha = \widetilde{\alpha}_n$ for each $n \in \mathbb{N}$.

(iii) By part (i) and part (ii). □

We have the index comparison criterion .

Theorem 13. *If $\alpha < \widetilde{\alpha}_n$ for some $n \in \mathbb{N}$, then there is a positive ground state solution of equation (1) in Ω.*

Proof. Let $\{u_n\}$ in $H_0^1(\Omega)$ be a $(PS)_{\alpha(\Omega)}$-sequence for I and u in $H_0^1(\Omega)$ satisfying $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$. By Theorem 12, u is nonzero. By Theorem 10 (ii), u is a positive ground state solution of equation (1). □

3. Existence of Solutions

In this section, we apply Theorem 13 to prove the following existence results:

Theorem 14. *There exists an $\varepsilon_0 > 0$, such that if $\varepsilon \leq \varepsilon_0$, then there exists a positive ground state solution of equation (1) in Ω_ε.*

Proof. The decomposition Lemma 4.1 in Lien-Tzeng-Wang [8] can be modified to hold for the equation (1). As a consequence, we have that $\alpha(\mathbb{S}^r)$ admits a minimizer. Since $\mathbb{S}^r \subsetneq \mathbb{S}^{r+\delta}$ and $\alpha(\mathbb{S}^r)$ admits a minimizer, by Lien-Tzeng-Wang [8], Lemma 2.7, we have $\alpha(\mathbb{S}^{r+\delta}) < \alpha(\mathbb{S}^r)$. Since $E_\varepsilon \subset \mathbb{S}^{r+\delta}$ and $\lim_{\varepsilon\to 0} \alpha(E_\varepsilon) = \alpha(\mathbb{S}^{r+\delta})$, there exists $\varepsilon_0 > 0$, such that if $\varepsilon \leq \varepsilon_0$, then $\alpha(E_\varepsilon) < \alpha(\mathbb{S}^r)$. Fix ε, $\varepsilon \leq \varepsilon_0$, there exists a large $N \in \mathbb{N}$ such that

$$\alpha(\widetilde{\Omega}_{\varepsilon,N}) = \alpha(\mathbb{S}^r_N) = \alpha(\mathbb{S}^r).$$

Thus,

$$\alpha(\Omega_\varepsilon) \leq \alpha(E_\varepsilon) < \alpha(\mathbb{S}^r) = \alpha(\widetilde{\Omega}_{\varepsilon,N}).$$

By Theorem 13, there exists a positive ground state solution u of equation (1) in Ω_ε. □

4. Asymptotic Behavior of Solutions

In this section, we present two asymptotic behavior of each solution of equation (1) in the flat flask domains Ω_ε, where $\varepsilon \leq \varepsilon_0$: one is basic and the other is advanced.

Theorem 15. *(Basic) If $u \in H^1_0(\Omega_\varepsilon)$ is a weak solution of equation (1) in Ω_ε, then we have the following results.*

(i) $u \in L^q(\Omega_\varepsilon)$ for all $q \in [2,\infty)$ such that $||u||_{L^q(\Omega_\varepsilon)} \leq c_q p(||u||_{H^1(\Omega_\varepsilon)})$, where c_q is a constant dependent on q and $p(||u||_{H^1(\Omega_\varepsilon)})$ is a polynomial of $||u||_{H^1(\Omega_\varepsilon)}$ in real power,

(ii) $u \in C^{1,\theta}\left(\overline{\Omega_\varepsilon}\right) \cap W^{2,q'}\left(\Omega_\varepsilon\right)$ for some $q' > N$ and $\theta = 2 - \frac{N}{q'} - \left[2 - \frac{N}{q'}\right]$ such that $||u||_{L^\infty(\Omega_\varepsilon)} \leq ||u||_{C^{1,\theta}\left(\overline{\Omega_\varepsilon}\right)} \leq c_{q'}||u||_{W^{2,q'}(\Omega_\varepsilon)} \leq c_q p(||u||_{H^1(\Omega_\varepsilon)})$,

(iii) $\lim_{y\to\infty} u(x,y) = 0$ uniformly in x, where $(x,y) \in \Omega_\varepsilon$,

(iv) $u \in C^{2,\theta}\left(\Omega_\varepsilon\right)$.

Proof. The following estimate is essentially due to Brezis-Kato [2], based on Moser's [9] iteration technique. We can not find this form in any literatures, so we prove, as follows.

(i) Let $\varphi = \varphi_{d,l} = u \min\{|u|^{2d}, l^2\}$, then $\nabla\varphi = \min\{|u|^{2d}, l^2\}\nabla u + \chi_{\{|u|^d \leq l\}} 2d|u|^{2d}\nabla u$.

Since $u \in H_0^1(\Omega_\varepsilon)$, we have

$$\begin{aligned}
\textstyle\int_{\Omega_\varepsilon} |\nabla\varphi|^2 &= \textstyle\int_{\Omega_\varepsilon} \left|\min\{|u|^{2d}, l^2\}\nabla u + \chi_{\{|u|^d \le l\}} 2d|u|^{2d}\nabla u\right|^2 \\
&\le 2\left(\textstyle\int_{\Omega_\varepsilon} \left|\min\{|u|^{2d}, l^2\}\nabla u\right|^2 + \left|\chi_{\{|u|^d \le l\}} 2d|u|^{2d}\nabla u\right|^2\right) \\
&= 2\left(l^4 \textstyle\int_{\{|u|^d \le l\}} |\nabla u|^2 + l^4 \int_{\{|u|^d \ge l\}} |\nabla u|^2\right) \\
&\quad + 8d^2 l^4 \textstyle\int_{\{|u|^d \le l\}} |\nabla u|^2 \\
&\le c \textstyle\int_{\Omega_\varepsilon} |\nabla u|^2 < \infty.
\end{aligned}$$

Thus, $\varphi \in H_0^1(\Omega_\varepsilon)$. For $d \ge 0$, $l \ge 1$, we have

$$\begin{aligned}
&\int_{\{|u|^d \le l\}} \left|\nabla(|u|^{d+1})\right|^2 \\
\le\ &\int_{\{|u|^d \le l\}} \left|\nabla(|u|^{d+1})\right|^2 + l^2 \int_{\{|u|^d \ge l\}} |\nabla u|^2 \\
=\ &\int_{\Omega_\varepsilon} \left|\nabla(u \min\{|u|^d, l\})\right|^2 \\
=\ &\int_{\Omega_\varepsilon} \left|\min\{|u|^d, l\}\nabla u + \chi_{\{|u|^d \le l\}} d\,|u|^d\,\nabla u\right|^2 \\
\le\ &2\int_{\Omega_\varepsilon} |\nabla u|^2 \min\{|u|^{2d}, l^2\} \\
&+ 2d^2 \int_{\{|u|^d \le l\}} |\nabla u|^2 |u|^{2d}.
\end{aligned} \tag{8}$$

Multiply equation (1) with φ to obtain

$$\int_{\Omega_\varepsilon} \nabla u \cdot \nabla\varphi + \int_{\Omega_\varepsilon} u\varphi = \int_{\Omega_\varepsilon} f(u)\varphi.$$

Thus,

$$\begin{aligned}
&\int_{\Omega_\varepsilon} |\nabla u|^2 \min\{|u|^{2d}, l^2\} + 2d \int_{\{|u|^d \le l\}} |\nabla u|^2 |u|^{2d} \\
&\quad + \int_{\Omega_\varepsilon} u^2 \min\{|u|^{2d}, l^2\} \quad \le c\left(\int_{\Omega_\varepsilon} |u|^{2+2d} + \int_{\Omega_\varepsilon} |u|^{2d+s+1}\right).
\end{aligned} \tag{9}$$

Suppose $u \in L^{2d+s+1}(\Omega_\varepsilon)$. Since $2 \le 2d+2 < 2d+s+1$, write $\frac{1}{2d+2} = \frac{\alpha}{2} + \frac{1-\alpha}{2d+s+1}$, where $0 < \alpha \le 1$. Then

$$||u||_{L^{2d+2}(\Omega_\varepsilon)} \le ||u||^{\alpha}_{L^2(\Omega_\varepsilon)} ||u||^{1-\alpha}_{L^{2d+s+1}(\Omega_\varepsilon)} \le c||u||^{1-\alpha}_{L^{2d+s+1}(\Omega_\varepsilon)}.$$

Then from (9) we have

$$\int_{\Omega_\varepsilon} |\nabla u|^2 \min\{|u|^{2d}, l^2\} + 2d \int_{\{|u|^d \le l\}} |\nabla u|^2 |u|^{2d} \le cp(||u||_{L^{2d+s+1}(\Omega_\varepsilon)}) , \quad (10)$$

where $p(||u||_{L^{2d+s+1}(\Omega_\varepsilon)})$ is a polynomial of $||u||_{L^{2d+s+1}(\Omega_\varepsilon)}$. By (8) and (10) we have

$$\int_{\{|u|^d \le l\}} \left|\nabla(|u|^{d+1})\right|^2 \le c_d p(||u||_{L^{2d+s+1}(\Omega_\varepsilon)}) ,$$

where c_d is a constant dependent on d. Let $l \to \infty$ to obtain that

$$\int_{\Omega_\varepsilon} \left|\nabla(|u|^{d+1})\right|^2 \le c_d p(||u||_{L^{2d+s+1}(\Omega_\varepsilon)}).$$

By the Sobolev inequality, we have

$$||u||^{2(d+1)}_{L^{(d+1)2^*}(\Omega_\varepsilon)} = \left\| |u|^{d+1} \right\|^2_{L^{2^*}(\Omega_\varepsilon)} \le c \left\| \nabla(|u|^{d+1}) \right\|^2_{L^2(\Omega_\varepsilon)} \le c_d p(||u||_{L^{2d+s+1}(\Omega_\varepsilon)}).$$

Thus,

$$||u||_{L^{(d+1)2^*}(\Omega_\varepsilon)} \le c_d p(||u||_{L^{2d+s+1}(\Omega_\varepsilon)}).$$

Now iterate, letting $d_0 = 0$ and $2d_i + s + 1 = (d_{i-1} + 1)\frac{2N}{N-2}$, for $i \ge 1$. Since $s + 1 < 2^*$, then we have $d_1 > 0$ and $d_i \ge (\frac{2^*}{2})^{i-1} d_1$. Hence $\lim_{i\to\infty} d_i = \infty$ and

$$||u||_{L^{2d_i+s+1}} = ||u||_{L^{(d_{i-1}+1)2^*}} \le c_{d_i} p(||u||_{L^{2d_{i-1}+s+1}(\Omega_\varepsilon)}).$$

By iterating, we conclude that

$$||u||_{L^{2d_i+s+1}} \le c_{d_i} p(||u||_{L^{s+1}(\Omega_\varepsilon)}) \le c_{d_i} p(||u||_{H^1(\Omega_\varepsilon)}).$$

By the interpolation property, for every $q \in [2, \infty)$, $u \in L^q(\Omega_\varepsilon)$ satisfies

$$||u||_{L^q(\Omega_\varepsilon)} \le c_q p(||u||_{H^1(\Omega_\varepsilon)}).$$

(ii) Since $u \in H^1_0(\Omega_\varepsilon)$ is a weak solution of equation (1), by part *(i)*, $f(u) \in L^q(\Omega_\varepsilon)$ for all $q \in [2, \infty)$. By Gilbarg-Trudinger [6], Theorem 9.15 and Lemma 9.17, the Dirichlet problem

$$\begin{cases} -\triangle v + v = f(u) \text{ in } \Omega_\varepsilon, \\ v \in W^{1,q}_0(\Omega_\varepsilon) \cap H^1_0(\Omega_\varepsilon), \end{cases}$$

has a unique strong solution $v \in W^{2,q}(\Omega_\varepsilon) \cap W_0^{1,q}(\Omega_\varepsilon) \cap H_0^1(\Omega_\varepsilon)$ and

$$\|v\|_{W^{2,q}(\Omega_\varepsilon)} \leq c_q \|f(u)\|_{L^q(\Omega_\varepsilon)}.$$

Thus, u and v satisfy weakly

$$\begin{cases} -\triangle v + v = f(u) \text{ in } \Omega_\varepsilon, \\ -\triangle u + u = f(u) \text{ in } \Omega_\varepsilon. \end{cases}$$

By Gilbarg-Trudinger [6], Corollary 8.2, $u = v \in W^{2,q}(\Omega_\varepsilon) \cap W_0^{1,q}(\Omega_\varepsilon) \cap H_0^1(\Omega_\varepsilon)$ and

$$\|u\|_{W^{2,q}(\Omega_\varepsilon)} \leq c_q \|f(u)\|_{L^q(\Omega_\varepsilon)} \leq c_q p(||u||_{H^1(\Omega_\varepsilon)}),$$

for all $q \in [2,\infty)$. Fix $q' \in [2,\infty)$ such that $q' > N$ and $\theta = 2 - \frac{N}{q'} - \left[2 - \frac{N}{q'}\right]$. By Brezis [1], p. 168, we have $u \in C^{1,\theta}\left(\overline{\Omega_\varepsilon}\right)$ and

$$||u||_{L^\infty(\Omega_\varepsilon)} \leq ||u||_{C^{1,\theta}\left(\overline{\Omega_\varepsilon}\right)} \leq c_{q'}||u||_{W^{2,q'}(\Omega_\varepsilon)} \leq c_{q'} p(||u||_{H^1(\Omega_\varepsilon)}).$$

(iii) Since $u \in H_0^1(\Omega_\varepsilon)$ is a weak solution of equation (1), by part (ii), $u \in C^{1,\theta}\left(\overline{\Omega_\varepsilon}\right) \cap W^{2,q'}(\Omega_\varepsilon)$. For each $t > 0$, apply Brezis [1], p. 168 to obtain

$$||u||_{L^\infty(\Omega_t)} \leq c_{q'}||u||_{W^{2,q'}(\Omega_t)},$$

where $\Omega_t = \{z \in \Omega_\varepsilon \mid |y| > t\}$. Since $\|u\|_{W^{2,q'}(\Omega_t)} = o(1)$ as $t \to \infty$, we have $\lim_{y\to\infty} u(x,y) = 0$ uniformly in x, where $(x,y) \in \Omega_\varepsilon$.

(iv) Since $f \in C^1(\mathbb{R}^+)$ and $f(\xi) = 0$ for $\xi < 0$, then f is locally Lipschitz continuous. By part (ii), $u \in C^{1,\theta}\left(\overline{\Omega_\varepsilon}\right)$. Thus, $f \circ u \in C^{0,\theta}(\Omega_\varepsilon)$. By Gilbarg-Trudinger [6, Theorem 9.19], we have $u \in C^{2,\theta}(\Omega_\varepsilon)$. □

In order to assert the advanced asymptotic behavior of solutions, let λ_1 be the first eigenvalue and ϕ_1 the corresponding first positive eigenfunction of the Dirichlet problem $-\Delta\phi_1 = \lambda_1\phi_1$ in $\mathbf{B}^{N-1}(0;r)$, $\phi_1 = 0$ on $\partial\mathbf{B}^{N-1}(0;r)$. Then, we have:

Theorem 16. *(Advanced) Let u be a positive solution of equation (1) in Ω_ε. Then for any $0 < \delta < \min\{1+\lambda_1, 2\}$ there exist $c_1 > 0$, $c_2 > 0$ and $R > s > 0$ such that for $z = (x,y) \in S_R^r$,*

$$c_1\phi_1(x)e^{-\sqrt{1+\lambda_1+\delta}\,y} \leq u(z) \leq c_2\phi_1(x)e^{-\sqrt{1+\lambda_1-\delta}\,y}.$$

Proof. (i) For $0 < \delta < \min\{1+\lambda_1, 2\}$, we choose an $R_1 > s > 0$ such that

$$\delta - \frac{\sqrt{1+\lambda_1+\delta}\,(N-1)}{y} \geq 0, \quad \text{for } y \geq R_1.$$

Define

$$w_\delta(z) = \phi_1(x)e^{-\sqrt{1+\lambda_1+\delta}\,y}, \quad \text{for } z=(x,y)\in \overline{S^r_{R_1}},$$

and $c_1 = \inf_{z\in S^r_{R_1}} \frac{u(x,R_1)}{w_\delta(x,R_1)}$. Thus, $\frac{u(x,R_1)}{w_\delta(x,R_1)} \geq \frac{u(x,R_1)}{\phi_1(x)}$. Since $u(x,y)$ is radially symmetric in x (see Theorem 19 below), then

$$\inf_{z\in S^r_{R_1}} \frac{u(x,R_1)}{w_\delta(x,R_1)} = \inf_{x\in L} \frac{u(x,R_1)}{w_\delta(x,R_1)} \geq \inf_{x\in L} \frac{u(x,R_1)}{\phi_1(x)}, \tag{11}$$

where L is a fixed diameter of $\mathbf{B}^{N-1}(0;r)$. For each $x_0 \in \partial L \subset \partial\mathbf{B}^{N-1}(0;r)$, take a small ball B^0 in $\mathbf{B}^{N-1}(0;r)$ such that $x_0\in\partial B^0$. Note that $\phi_1(x)>0$ for each $x\in B^0$, $\phi_1(x_0)=0$, and $\phi_1(x)\in C^2\left(\overline{\mathbf{B}^{N-1}(0;r)}\right)$. Then by the Hopf Boundary Point Lemma, we have $\frac{\partial\phi_1}{\partial\nu}(x_0)<0$, where ν is the outward unit normal vector of B^0 at x_0. For each $z_1=(x_0,R)\in\partial L\times\mathbb{R}\subset\partial\Omega_\varepsilon$, take a small ball B^1 in Ω_ε such that $z_1\in\partial B^1$. Note that $u(z)>0$ for each $z\in B^1$ and $u(z_1)=0$. By Theorem 15 (ii) and (iv), $u(z)\in C^2(\Omega_\varepsilon)\cap C^{1,\theta}\left(\overline{\Omega_\varepsilon}\right)$. Thus, by the Hopf Boundary Point Lemma, we have $\frac{\partial u}{\partial\overline{\nu}}(z_1)<0$, where $\overline{\nu}=(\nu,0)$ is the outward unit normal vector of B^1 at z_1. Let $u_1(x)=u(x,R_1)$, then

$$\frac{\partial u_1}{\partial\nu}(x_0) = \nabla u_1(x_0)\cdot\nu = \nabla u(z_1)\cdot\overline{\nu} = \frac{\partial u}{\partial\overline{\nu}}(z_1) < 0.$$

By the L'Hospital rule, we have

$$\lim_{\substack{x\in\mathbf{B}^{N-1}(0;r)\\ x\to x_0\\ \text{normally}}} \frac{u(x,R_1)}{\phi_1(x)} = \lim_{h\to 0^-} \frac{u_1(x_0+h\nu)}{\phi_1(x_0+h\nu)} = \frac{\frac{\partial u_1}{\partial\nu}(x_0)}{\frac{\partial\phi_1}{\partial\nu}(x_0)} > 0.$$

Define

$$\frac{u(x_0,R_1)}{\phi_1(x_0)} = \lim_{\substack{x\in\mathbf{B}^{N-1}(0;r)\\ x\to x_0\\ \text{normally}}} \frac{u(x,R_1)}{\phi_1(x)},$$

then $\frac{u(x,R_1)}{\phi_1(x)}:\overline{L}\to\mathbb{R}$ is continuous and $\frac{u(x,R_1)}{\phi_1(x)}>0$ for $x\in\overline{L}$. Thus, by (11), $c_1>0$. Let $v_1(z)=c_1w_\delta(z)$ for $z\in\overline{S^r_{R_1}}$, then $v_1(z)\leq u(z)$ for $z\in\overline{S^r_{R_1}}$ and $y=R_1$. For $z\in S^r_{R_1}$, we have

$$\begin{aligned}\triangle(v_1-u)(z)-(v_1-u)(z) &= (\triangle v_1(z)-v_1(z))+(-\triangle u(z)+u(z))\\ &= v_1(z)\left(\delta-\frac{\sqrt{1+\lambda_1+\delta}\,(N-1)}{y}\right)+f(u)\geq 0.\end{aligned}$$

By the maximum principle, we have $v_1(z) \le u(z)$ for $z \in S^r_{R_1}$.

(ii) For $0 < \delta < \min\{1+\lambda_1, 2\}$, by $(f5)$ and Theorem 15 (iii), we choose an $R > R_1 > 0$ such that $f(u) \le \frac{\delta}{2}u$, for $y \ge R$. Define

$$\begin{aligned} w_{-\delta}(z) &= \phi_1(x)e^{-\sqrt{1+\lambda_1-\delta}\,y}, \quad \text{for } z=(x,y) \in \overline{S^r_R}, \\ \frac{1}{c_2} &= \sup_{z \in S^r_{R_1}} \frac{w_{-\delta}(z)}{u(z)} > 0. \end{aligned}$$

It is similar to part (i), $\frac{1}{c_2} < \infty$. Let $v_2(z) = c_2 w_{-\delta}(z)$ for $z \in \overline{S^r_R}$, then $v_1(z) \le u(z)$ for $z \in \overline{S^r_R}$ and $y = R$. For $z \in S^r_R$, we have

$$\begin{aligned} &-\Delta(u-v_2)(z) + (u-v_2)(z) = (-\Delta u(z) + u(z)) + (\Delta v_2(z) - v_2(z)) \\ &\qquad = f(u(z)) + (-\delta - \frac{\sqrt{1+\lambda_1-\delta}(N-1)}{y})v_2(z) \le \frac{\delta}{2}(u-v_2)(z). \end{aligned}$$

Therefore

$$-\Delta(u-v_2)(z) + (1-\frac{\delta}{2})(u-v_2)(z) \le 0.$$

As in part (i), we obtain $u(z) \le v_2(z)$ for $z \in S^r_R$. $\square$

5. Symmetry of Solutions

We apply the "moving plane" method in Li-Ni [7] to prove the symmetry of each solution of equation (1) in the flat flask domain Ω_ε, where $\varepsilon \le \varepsilon_0$, as follows.

Let $u(x,y)$ be a solution of equation (1) in Ω_ε. By Theorem 15 and the Schauder regularity, u is a C^2 solution. Define

$$\begin{aligned} &T_\lambda = \{(x,y) = (x_1, x_2, \cdots, x_{N-1}, y) \in \Omega_\varepsilon \mid x_1 = \lambda\}, \\ &\Sigma_\lambda = \Omega_\varepsilon \cap \{(x,y) \mid x_1 < \lambda\}. \end{aligned}$$

For any $(x,y) = (x_1, x_2, \cdots, x_{N-1}, y) \in \Omega_\varepsilon$, set

$$(x^\lambda, y) = (2\lambda - x_1, x_2, \cdots, x_{N-1}, y),$$

that is to say, (x^λ, y) is the reflection of (x,y) with respect to T_λ. Let Λ be the collection of all $\lambda \in (-s, 0)$ such that the following statements hold:

$$\begin{cases} u(x,y) < u(x^\lambda, y) & \text{for all } (x,y) \in \Sigma_\lambda, \\ u_{x_1}(x,y) > 0 & \text{on } T_\lambda. \end{cases}$$

Lemma 17. *For some* $0 < \delta < s$, $(-s, -s+\delta) \subset \Lambda$.

Proof. Given $\lambda \in (-s, 0)$, set $v^{\lambda}(x,y) = u(x,y) - u(x^{\lambda}, y)$ for $(x,y) \in \Sigma_{\lambda}$, then $v^{\lambda}(x,y) = 0$ for $(x,y) \in T_{\lambda}$, and $v^{\lambda}(x,y)$ satisfies

$$\Delta v^{\lambda}(x,y) + c_{\lambda}(x,y) v^{\lambda}(x,y) = 0, \tag{12}$$

where $c_{\lambda}(x,y) = \frac{f(u(x,y)) - f(u(x^{\lambda},y))}{u(x,y) - u(x^{\lambda},y)} - 1 = f'(\zeta_{\lambda}) - 1$, where ζ_{λ} is in between $u(x,y)$ and $u(x^{\lambda}, y)$. By Lemma 3 $(f5)$, $\lim_{y \to 0^+} f'(y) = 0$. Take $y_0 > 0$ such that if $0 < y \le y_0$, then $f'(y) < 1$. By Theorem 15 (iii), $\lim_{y \to \infty} u(x,y) = 0$ uniformly in x, we can choose δ, $s > \delta > 0$ such that $s - \delta < y < s$ implies $u(x,y) \le y_0$ uniformly in y. Claim that if $-s < \lambda < -s + \delta$, then $v^{\lambda}(x,y) \le 0$ in Σ_{λ}. Otherwise, suppose there exists λ such that $-s < \lambda < -s + \delta$, $v^{\lambda}(x,y) > 0$ for some $(x,y) \in \Sigma_{\lambda}$. Since $\lim_{y \to \infty} v^{\lambda}(x,y) = 0$ uniformly in x, $v^{\lambda}(x,y)$ achieves its maximum at $(x_{\lambda}, y_{\lambda}) \in \Sigma_{\lambda}$, then

$$\nabla v^{\lambda}(x_{\lambda}, y_{\lambda}) = 0, \quad \{v^{\lambda}_{ij}(x_{\lambda}, y_{\lambda})\} \le 0.$$

But

$$\Delta v^{\lambda}(x_{\lambda}, y_{\lambda}) \le 0, \quad (f'(\zeta_{\lambda}) - 1) v^{\lambda}(x_{\lambda}, y_{\lambda}) = c_{\lambda}(x,y) v^{\lambda}(x_{\lambda}, y_{\lambda}) < 0,$$

which contradicts to equation (12). Thus, for $-s < \lambda < -s + \delta$, $v^{\lambda}(x,y) \le 0$ in Σ_{λ}. Applying the maximum principle and Hopf Boundary Point Lemma, for $-s < \lambda < -s + \delta$, $v^{\lambda}(x,y) < 0$ in Σ_{λ} and $v^{\lambda}_{x_1}(x,y) > 0$ for $(x,y) \in T_{\lambda}$. Hence $u_{x_1}(x,y) > 0$ for $(x,y) \in T_{\lambda}$. Then, $(-s, -s + \delta) \subset \Lambda$. □

Lemma 18. *If* $(-s, \lambda] \subset \Lambda$, *then there exists* $\tau > 0$ *such that* $[\lambda, \lambda + \tau) \subset \Lambda$.

Proof. Suppose not, there exists a decreasing sequence $\lambda_k \to \lambda$ and a sequence $\{(x_k, y_k)\}$ of points in Σ_{λ_k} such that $v^{\lambda_k}(x_k, y_k) = u(x_k, y_k) - u(x_k^{\lambda_k}, y_k) > 0$. By the compactness of $\overline{\mathbf{B}^{N-1}(0; s)}$, there is a subsequence $\{(x_k, y_k)\}$ such that $x_k \to \overline{x} \in \overline{\mathbf{B}^{N-1}(0; s)}$. Then we have two possibilities as shown in Case 1 and 2.

Case 1. $y_k \to \infty$. From $\lim_{y_k \to \infty} u(x_k, y_k) = 0$, we may assume

$$\begin{aligned} &v^{\lambda_k}(x_{\lambda_k}, y_{\lambda_k}) = \max_{(x,y) \in \overline{\Sigma_{\lambda_k}}} v^{\lambda_k}(x,y), \\ &\nabla v^{\lambda_k}(x_{\lambda_k}, y_{\lambda_k}) = 0, \\ &\{v^{\lambda_k}_{ij}(x_{\lambda_k}, y_{\lambda_k})\} \le 0. \end{aligned}$$

As in Lemma 17, we obtain a contradiction.

Case 2. $y_k \to \overline{y}$. We have $(x_k, y_k) \to (\overline{x}, \overline{y}) \in \overline{\Sigma_\lambda}$. Thus, $v^\lambda(\overline{x}, \overline{y}) \geq 0$. Clearly $(\overline{x}, \overline{y}) \notin \Sigma_\lambda$ since $v^\lambda(x, y) < 0$ in Σ_λ. If $(\overline{x}, \overline{y}) \in T_\lambda$ then $u_{x_1}(\overline{x}, \overline{y}) < 0$, which contradicts to $\lambda \in \Lambda$. Moreover, $(\overline{x}, \overline{y}) \notin \partial\Omega_\varepsilon \cap \overline{\Sigma_\lambda}$ since if $(\overline{x}, \overline{y}) \in \partial\Omega_\varepsilon \cap \overline{\Sigma_\lambda}$ then $0 = u(\overline{x}, \overline{y}) \geq u(\overline{x}^\lambda, \overline{y}) > 0$, a contraction. We conclude that Case 2 is impossible. □

Theorem 19. *Let $u(x, y)$ be a solution of equation (1) in Ω_ε. Then u is radially symmetric in x: that is to say, there is a function $h : [0, t] \times [-t, \infty) \to \mathbb{R}$ such that $u(x, y) = h(|x|, y)$.*

Proof. Let $\mu = \sup\{\lambda \in (-s, 0) \mid (-s, \lambda) \subset \Lambda\}$. Then $\mu \notin \Lambda$. If not, by Lemma 18 we would have $[\mu, \mu + \epsilon) \subset \Lambda$, which contradicts to the definition of μ. We claim that $\mu = 0$. Suppose not, $\mu \in (-s, 0)$. By continuity we have $u(x, y) \leq u(x^\mu, y)$ for all $(x, y) \in \Sigma_\mu$, then by the maximum principle we have $u(x, y) \equiv u(x^\mu, y)$ for all $(x, y) \in \Sigma_\mu$, which is impossible. Thus, $\mu = 0$. By reversing the x_1 axis, we conclude that $u(x, y)$ is symmetric with respect to the hyperplane T_0 and $u_{x_1}(x, y) < 0$ for $x_1 > 0$. Since the x_1 direction can be chosen arbitrarily, we conclude that $u(x, y)$ is radially symmetric in x. □

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