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EIGENVALUE ASYMPTOTICS OF SELF-ADJOINT OPERATORS ON A HILBERT SPACE AND AN APPLICATION

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Abstract: We intend to improve some results in [16] on the asymptotics of eigenvalues for a self-adjoint operator on a Hilbert space. Moreover, we give an application of the results on the eigenvalue problems for the Pauli operator.

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1. Introduction

The purpose of the present paper is to improve the results of Mohamed and Parisse [16] and to give an application to the Pauli operator. More precisely, we shall consider the eigenvalue asymptotics for the Dirac operator in the semi-classical sense. The Dirac operator depending on a small parameter $h > 0$ with the magnetic potential $a(x) = (a_1(x), a_2(x), a_3(x))$ and the electric potential $V(x)$ is defined by:

$$P_V^h(a) = \alpha \cdot D^h(a) + \alpha_4 + V(x)I_4 , \tag{1.1}$$

on $L^2(\mathbf{R}^3; \mathbf{C}^4)$, where $D^h(a) = (D_1^h(a), D_2^h(a), D_3^h(a))$, $D_j^h(a) = -ih\partial_j -$

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$a_j(x)$, $\partial_j = \partial/\partial x_j$, $i = \sqrt{-1}$, $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\alpha \cdot D^h(a) = \sum_{j=1}^3 \alpha_j \times D_j^h(a)$. Here α_j $(j = 1, 2, 3, 4)$ are 4×4 Hermitian matrices of the forms

$$\alpha_j = \begin{bmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{bmatrix} \quad (j = 1, 2, 3), \quad \alpha_4 = \begin{bmatrix} I_2 & 0 \\ 0 & -I_2 \end{bmatrix},$$

where I_4 and I_2 are the 4×4 and 2×2 identity matrices, respectively and σ_j $(j = 1, 2, 3)$ are so-called the Pauli matrices:

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

If we assume that the electric potential $V(x)$ is a real valued C^∞ function on $\mathbf{R}^3$ and that there exist $E \in \mathbf{R}, \epsilon > 0, R > 0$ such that $|V(x) - E| \leq 1 - \epsilon$ for $|x| \geq R$, it is well known that the spectrum $\sigma(P_V^h(a))$ of $P_V^h(a)$ in a neighborhood of E is contained in the discrete spectrum $\sigma_{\rm disc}(P_V^h(a))$.

According to Wang [19], we see that the eigenvalue problem for the Dirac operator $P_V^h(a)u^h = E(h)u^h$ is reduced to study that of the Pauli operator $H_P^h v^h = 0$ on $L^2(\mathbf{R}^3; \mathbf{C}^2)$, where

$$\begin{aligned} H_P^h &= H_P^h(a, V, E(h)) \\ &= [D^h(a)^2 + W(h, V, E(h))]I_2 + ih \sum_{j<k} \sigma_k \sigma_j B_{kj}(x). \end{aligned}$$

Here W is a function depending on a, V and $\lambda(h)$ and (B_{kj}) denotes the magnetic field defined by $B_{kj}(x) = \partial_k a_j(x) - \partial_j a_k(x)$. The main purpose of this paper is to study that when H_P^h has the non-degenerate eigenvalue and that what kind of asymptotic expansion of the eigenvalues holds in the semiclassical sense.

In [19], he considered the similar problem for the Dirac operator without magnetic potential and [16] obtained some results for the operator with small magnetic potential, i.e., for $P_V^h(ha)$. In both cases, we can see that the first term in the semiclassical approximation of the operator becomes a harmonic oscillator without the magnetic potential. In our case, however, we need to consider a harmonic oscillator in the uniform magnetic field as the first term. In order to do so, we shall extend the results of [16] on an approximation of eigenvalues of a self-adjoint operator on a abstract Hilbert space and then, as an application, we treat the eigenvalue problem for the Pauli operator.

For the Schrödinger operator, there are many articles in this direction. For example, see Aramaki [2], [3], [4], Helffer and Mohamed [9], Helffer and Sjöstrand [10] and Matsumoto [13].

The plan of this paper is as follows. In Section 2, we consider the eigenvalue problem in the abstract setup and give two theorems, which extend the results of [16]. Section 3 is devoted to the proofs of the theorems. In Section 4, we study an application of the results in Section 2 to the eigenvalues problem for the Pauli operator. In Section 5, we give two examples on the Pauli and the Dirac operators with special electric potentials.

2. Asymptotics of Eigenvalues for a Self-adjoint Operator

In this section, we consider a self-adjoint operator with a small parameter on a separable Hilbert space on $\mathcal{H}$. In order to do so, we summarize the notations. The inner product of $u, v \in \mathcal{H}$ and the norm of an element u in $\mathcal{H}$ are denoted by (u, v) and $\|u\|$, respectively. For a closed subspace E of $\mathcal{H}$, we denote the orthogonal projections onto E and onto the orthogonal complement $E^{\perp}$ by Π_E and $\overline{\Pi}_E$, respectively. Moreover, if E and F are closed subspaces satisfying $F \subset E$, the orthogonal projection onto the orthogonal complement of F in E is denoted by $\Pi_{E\ominus F}$. In general, for any self-adjoint operator H on $\mathcal{H}$, the spectrum and the discrete spectrum of H are denoted by $\sigma(H)$ and $\sigma_{\rm disc}(H)$, respectively and the domain by $D(H)$.

Let H_0 be a self-adjoint operator on $\mathcal{H}$ with domain $D(H_0)$. And let $E_0 \in \sigma_{\rm disc}(H_0)$ and $\mathcal{H}_0$ the associated eigenspace. Assume that there exist a self-adjoint operator H^h with a parameter $h \in (0, h_0]$, sequences $\{H_j\}_{j=1}^{\infty}$, $\{R_j\}_{j=1}^{\infty}$ consisting of symmetric operators with common domain $\mathcal{H}_{\infty}^1$ and a subspace $\mathcal{H}_{\infty}$ of $\mathcal{H}$ such that

$$\mathcal{H}_0 = \operatorname{Ker}(H_0 - E_0) \subset \mathcal{H}_{\infty} \subset \mathcal{H}_{\infty}^1 \subset D(H_0) \cap D(H^h), \tag{2.1}$$

$$\begin{aligned} (H_0 - E_0)^{-1}\overline{\Pi}_{\mathcal{H}_0}(\mathcal{H}_{\infty}) &\subset \mathcal{H}_{\infty}, \\ (H_0 - E_0)^{-1}\overline{\Pi}_{\mathcal{H}_0} H_m(\mathcal{H}_{\infty}) &\subset \mathcal{H}_{\infty}, \end{aligned} \tag{2.2}$$

and moreover, assume that H^h satisfies that

$$H^h \sim \sum_{j=0}^{\infty} h^j H_j. \tag{2.3}$$

Here the symbol $\sim$ means that for any $m \geq 0$, there exists a constant $C_m \geq 0$ such that

$$\|H^h u - \sum_{j=0}^{m} h^j H_j u\| \leq C_m h^{m+1} \|R_{m+1} u\| \quad \text{for } u \in \mathcal{H}_{\infty}. \tag{2.4}$$

Moreover, we assume that there exists $\epsilon_0 > 0$ such that $\sigma(H^h) \cap I_{\epsilon_0} = \sigma_{\rm disc}(H^h) \cap I_{\epsilon_0}$, where $I_{\epsilon_0} = (E_0 - \epsilon_0, E_0 + \epsilon_0)$. Finally, we assume that there exists a constant $C > 0$ such that for any $E(h) \in \sigma_{\rm disc}(H^h) \cap I_{hC}$, the associated normalized eigenfunction u^h, $u^h \in \mathcal{H}^1_\infty$ and for any $m \geq 0$, we have

$$\|H^h u^h - \sum_{j=0}^{m} h^j H_j u^h\| \leq C_m h^{m+1}\,, \tag{2.5}$$

with some constant $C_m > 0$ independent of h.

Under the above situation, we propose problems that when H^h has a non-degenerate eigenvalue with asymptotic expansion in powers of h with E_0 as the first term and what asymptotic expansion of the corresponding eigenfunction holds.

In order to do so, let $p \geq 1$ integer. At the first time, from the hypotheses, note that $S_0 := K_0 = H_0$ has an eigenvalue E_0 with the eigenspace $\mathcal{H}_0$. Next, we define $S_p = H_p$ and assume that $K_p := \Pi_{\mathcal{H}_0} S_p\big|_{\mathcal{H}_0}$ has an eigenvalue E_p with the corresponding eigenspace $\mathcal{H}_p$. Inductively, we can define symmetric operators S_{jp}, real numbers E_{jp} and closed subspaces $\mathcal{H}_{jp}$, as follows. Suppose that for $j \leq r_0$, we can define S_{jp}, E_{jp} and $\mathcal{H}_{jp}$ such that $K_{jp} := \Pi_{\mathcal{H}_{(j-1)p}} S_{jp}\big|_{\mathcal{H}_{(j-1)p}}$ has an eigenvalue E_{jp} with the corresponding eigenspace $\mathcal{H}_{jp}$. Then we have to define an operator $S_{(r_0+1)p}$, a closed subspace $\mathcal{H}_{(r_0+1)p}$ and a real number $E_{(r_0+1)p}$ such that $E_{(r_0+1)p}$ is an eigenvalue of $\Pi_{\mathcal{H}_{r_0 p}} S_{(r_0+1)p}\big|_{\mathcal{H}_{r_0 p}}$. For the purpose, it is convenient to use the following notations. For $i < j$ and $k \geq 0$ integers, we define operators on $\mathcal{H}_\infty$:

$$\begin{aligned} W^{(i,j)} &= -(K_{ip} - E_{ip})^{-1} \Pi_{\mathcal{H}_{(i-1)p} \ominus \mathcal{H}_{ip}} (S_{jp} - E_{jp}), \\ W^{(j)}_{(k)} &= \sum_{\substack{0 \leq i_1 < j_1 \leq \cdots \leq i_s < j_s \leq j \\ \sum_{l=1}^{s} (j_l - i_l) = k}} W^{(i_1,j_1)} W^{(i_2,j_2)} \cdots W^{(i_s,j_s)}. \end{aligned} \tag{2.6}$$

Then it suffices to define

$$S_{(r_0+1)p} = H_{(r_0+1)p} + \sum_{k=1}^{r_0} (H_{kp} - E_{kp}) W^{(r_0)}_{(r_0+1-k)}\,, \tag{2.7}$$

and to let E_{r_0+1} be an eigenvalue of

$$K_{(r_0+1)p} := \Pi_{\mathcal{H}_{r_0 p}} S_{(r_0+1)p}\big|_{\mathcal{H}_{r_0 p}}\,,$$

and $\mathcal{H}_{(r_0+1)p}$ the corresponding eigenspace.

We are in a position to state the main theorem.

Theorem 2.1. *Assume that there exist integers $p, q \geq 1, r \geq 0$ so that $rp \leq q < (r+1)p$ and assume that*

$$H_k = 0 \quad \text{for } 0 < k < q \quad \text{and } k \neq p, 2p, \ldots, rp. \tag{H}$$

Moreover, assume that when $q = rp$, $\dim \mathcal{H}_{rp} = 1 < \dim \mathcal{H}_{(r-1)p}$ and when $q > rp$, $\dim \mathcal{H}_q = 1 < \dim \mathcal{H}_{rp}$, where $\mathcal{H}_q$ is an eigenspace corresponding to an eigenvalue E_q of $K_q := \Pi_{\mathcal{H}_{rp}} H_q\big|_{\mathcal{H}_{rp}}$. Then H^h has a non-degenerate eigenvalue $E(h)$ of the form:

$$E(h) \sim \sum_{k=0}^{\infty} h^k E_k\,, \tag{2.8}$$

with $E_k = 0$ if $0 < k < q$ and $k \neq p, 2p, \ldots, rp$ and the associated eigenfunction u^h of the form

$$u^h \approx \sum_{k=0}^{\infty} h^k u_k \quad (u_k \in \mathcal{H}_\infty). \tag{2.9}$$

Here the notations $\sim$ and $\approx$ mean that for any large integer m, there exists $C_m > 0$ such that

$$\Big|\lambda(h) - \sum_{k=0}^{m-1} h^k E_k\Big| \;\leq\; C_m h^m, \quad h \in (0, h_0], \tag{2.10}$$

$$\|u^h - \sum_{k=0}^{m-1} h^k u_k\| \;\leq\; C_m h^{m-m_0}\,, \tag{2.11}$$

for some integer $m_0 \geq 0$.

Next, in addition to the above situation, assume that there exists an operator $J : \mathcal{H} \to \mathcal{H}$ satisfying the following:

$$\begin{cases} J(\lambda u + \mu v) = \overline{\lambda} J(u) + \overline{\mu} J(v), \quad \text{for } \lambda, \mu \in \mathbf{C}, u, v \in \mathcal{H}, \\ J^2(u) = -u, \quad (Ju, u) = 0, \quad \text{for } u \in \mathcal{H}, \\ J(D(H_0)) = D(H_0), \quad J(D(H^h)) = D(H^h), \\ JH_0 = H_0 J, \quad JH^h = H^h J, \\ JH_k(u) = H_k J(u), \quad \text{for } u \in \mathcal{H}_\infty, k \geq 1. \end{cases}$$

Then we have:

Theorem 2.2. *Under the situation above, assume that $\dim \mathcal{H}_{rp} = 2 < \dim \mathcal{H}_{(r-1)p}$, when $q = rp$ and $\dim \mathcal{H}_q = 2 < \dim \mathcal{H}_{rp}$ when $q > rp$, where $\mathcal{H}_q$ is an eigenspace corresponding to an eigenvalue E_q of $K_q := \Pi_{\mathcal{H}_{rp}} H_q\big|_{\mathcal{H}_{rp}}$. Then H^h has a double eigenvalue $E(h)$ of the form (2.8) with the corresponding eigenfunctions u^h, Ju^h such that u^h satisfies (2.9).*

Remark 2.3. These theorems are extensions of [13; Theorem 1.1, 1.2 and 1.3] in which the cases, where $r \le 2$ are treated. We shall use the technique in [13], however, we modify them so as to the case, where $r \ge 3$.

3. Proofs of Theorem 2.1 and 2.2.

In this section, we give proofs of Theorems 2.1 and 2.2.

We have to seek the eigenvalue $E(h)$ of H^h and the associated eigenfunction u^h of the forms

$$E(h) \sim \sum_{j=0}^{\infty} h^j E_j, \quad u^h \approx \sum_{j=0}^{\infty} h^j u_j \quad (u_j \in \mathcal{H}_\infty \text{ for } j \ge 0)\,, \tag{3.1}$$

in the sense of (2.8) and (2.9). Substituting (3.1) in (2.5), we must find E_j ($j \ge 1$) and $u_j \in \mathcal{H}_\infty$ ($j \ge 0$) satisfying the following equation that for every integer $j \ge 0$,

$$\sum_{k=0}^{j} (H_k - E_k) u_{j-k} = 0. \qquad (A)_j$$

At the first time, taking the hypothesis (H) into consideration, we consider two equations which are closely related to $(\mathrm{A})_j$.

$$\sum_{k=0}^{j} (H_{kp} - E_{kp}) w_{(j-k)p} = 0, \qquad (A)_{jp}$$

$$\sum_{k=0}^{j} (S_{kp} - E_{kp}) \Pi_{\mathcal{H}_{(k-1)p}} w_{(j-k)p} = 0 \qquad (B)_{jp}\,,$$

where operators S_{kp} are defined by (2.6) and (2.7) and we interpret $\Pi_{\mathcal{H}_{-p}} = I$ (the identity operator on $\mathcal{H}$). Then we have

Proposition 3.1. *Let $1 \le r_0 \le r$ be an integer. Then the following are equivalent:*

$\widetilde{(A)}_{r_0 p}$: *There exist w_{jp} ($0 \le j \le r_0$) such that $(A)_{jp}$ holds for $0 \le j \le r_0$.*

$\widetilde{(B)}_{r_0 p}$: *There exist w_{jp} ($0 \le j \le r_0$) such that $(B)_{jp}$ holds for $0 \le j \le r_0$.*

Proof. It is trivial to see that $(A)_0 = (B)_0$ and so $w_0 \in \mathcal{H}_0$ and therefore it is easily seen that $(A)_p$ is equivalent to $(B)_p$ from the definitions of S_0 and S_p.

We shall prove this proposition by induction on r_0. Thus, assume that there exists r_0 $(1 \leq r_0 < r)$ such that $\widetilde{(A)}_{r_0p}$ is equivalent to $\widetilde{(B)}_{r_0p}$.

Then we need the following lemma.

Lemma 3.2. *The necessary and sufficient condition for* $\widetilde{(B)}_{r_0p}$ *to hold for some* w_{jp} $(j = 0, 1, \ldots, r_0)$ *is that we can find* w_{jp} $(j = 0, 1, \ldots, r_0)$ *satisfying the following*

$$\overline{\Pi}_{\mathcal{H}_{(r_0-j)p}} w_{jp} = \sum_{k=1}^{j} W_{(k)}^{(r_0-j+k)} \Pi_{\mathcal{H}_{(r_0-j+k)p}} w_{(j-k)p} \quad \text{for } 0 \leq j \leq r_0. \qquad \widehat{(B)}_{r_0p}$$

Here for $j = 0$, *we interpret the right hand side as zero, i.e.,* $w_0 \in \mathcal{H}_{r_0p}$.

Proof. We shall also prove this lemma by inducition on r_0. When $r_0 = 1$, assume that $\widetilde{(B)}_p$ holds. Then we have

$$(S_0 - E_0)w_0 = 0 \quad \text{and} \quad (S_0 - E_0)w_p + (S_p - E_p)\Pi_{\mathcal{H}_0} w_0 = 0.$$

From the first equation, we have $w_0 \in \mathcal{H}_0$ and then applying $\Pi_{\mathcal{H}_0}$ from the left to the second equation, by the definition of K_p, we see that $w_0 \in \mathcal{H}_p$. By applying $\overline{\Pi}_{\mathcal{H}_0}$ from the left and using the fact $w_0 \in \mathcal{H}_p$, we have

$$\overline{\Pi}_{\mathcal{H}_0} w_p = -(K_0 - E_0)^{-1} \overline{\Pi}_{\mathcal{H}_0} (S_p - E_p) \Pi_{\mathcal{H}_0} w_0 = W_{(1)}^{(1)} \Pi_{\mathcal{H}_p} w_0.$$

Thus, $\widehat{(B)}_1$ holds. The converse is clear.

We assume that there exist w_{jp} $(j = 0, 1, \ldots, r_0)$ such that $\widetilde{(B)}_{r_0p}$ is equivalent to $\widehat{(B)}_{r_0p}$. Then $(B)_{(r_0+1)p}$ becomes

$$\sum_{j=0}^{r_0+1} (S_{(r_0-j+1)p} - E_{(r_0-j+1)p}) \Pi_{\mathcal{H}_{(r_0-j)p}} w_{jp} = 0.$$

Applying $\Pi_{\mathcal{H}_{r_0p}}$ from the left, we see $w_0 \in \mathcal{H}_{(r_0+1)p}$. Next, applying $\Pi_{\mathcal{H}_{(r_0-1)p} \ominus \mathcal{H}_{r_0p}}$ from the left, it follows from (2.6) and (2.7) that

$$\Pi_{\mathcal{H}_{(r_0-1)p} \ominus \mathcal{H}_{r_0p}} w_p = W^{(r_0, r_0+1)} w_0 = W^{(r_0, r_0+1)} \Pi_{\mathcal{H}_{(r_0+1)p}} w_0. \tag{3.2}$$

By the hypothesis $\widehat{(B)}_p$, we have $\overline{\Pi}_{\mathcal{H}_{(r_0-1)p}} w_p = W_{(1)}^{(r_0)} w_0$. Thus, we have

$$\overline{\Pi}_{\mathcal{H}_{r_0p}} w_p = (W_{(1)}^{(r_0)} + W^{(r_0, r_0+1)}) w_0 = W_{(1)}^{(r_0+1)} w_0.$$

Now, we introduce a convenient notation. For $0 \le i < j$, $k \ge 0$ integers, define

$$W_{(k)}^{(i,j)} = \sum_{\substack{i=i_1<j_1\le\cdots\le i_s<j_s=j \\ \sum_{l=1}^{s}(j_l-i_l)=k}} W^{(i_1,j_1)}W^{(i_2,j_2)}\cdots W^{(i_s,j_s)}. \tag{3.3}$$

Then we note that for $0 \le k < j$,

$$W_{(k)}^{(j)} = \sum_{l=1}^{j-k} W_{(k-l)}^{(j-l-1)} W_{(l)}^{(j-l,j)} + W_{(k)}^{(j-1)}. \tag{3.4}$$

Here we interpret $W_{(l)}^{(j)} = I$, $W_{(l)}^{(i,j)} = I$ for $l \le 0$.

Now we claim that if $\widetilde{(B)}_{(r_0+1)p}$ holds, we can write

$$\Pi_{\mathcal{H}_{(r_0-j)p}\ominus\mathcal{H}_{(r_0-j+1)p}} w_{jp} = \sum_{k=0}^{j-1} W_{(j-k)}^{(r_0-j+1,r_0+1-k)} \Pi_{\mathcal{H}_{(r_0-k+1)p}} w_{kp} \tag{3.5}$$

for $j = 1, 2, \ldots, r_0$.

We prove (3.5) on induction on j. When $j = 1$, it follows from (3.2) that (3.5) holds. Assume that (3.5) holds up to j. Applying $\Pi_{\mathcal{H}_{(r_0-j-1)p}\ominus\mathcal{H}_{(r_0-j)p}}$ to $(B)_{(r_0+1)p}$ and using the induction hypothesis, we see that

$$\begin{aligned}
&\Pi_{\mathcal{H}_{(r_0-j-1)p}\ominus\mathcal{H}_{(r_0-j)p}} w_{(j+1)p} \\
&\quad= \sum_{k=0}^{j} W^{(r_0-j,r_0-k+1)} \Pi_{\mathcal{H}_{(r_0-k)p}} w_{kp} \\
&\quad= \sum_{k=0}^{j} W^{(r_0-j,r_0-k+1)} \Pi_{\mathcal{H}_{(r_0-k+1)p}} w_{kp} \\
&\qquad+ \sum_{k=0}^{j} W^{(r_0-j,r_0-k+1)} \sum_{l=0}^{k-1} W_{(k-l)}^{(r_0-k+1,r_0+1-l)} \Pi_{\mathcal{H}_{(r_0-l+1)p}} w_{lp} \\
&\quad= \sum_{k=0}^{j} W_{(j+1-k)}^{(r_0-j,r_0-k+1)} \Pi_{\mathcal{H}_{(r_0+1-k)p}} w_{kp}.
\end{aligned}$$

Thus, the claim (3.5) holds.

By the induction hypothesis and the claim above, we have

$$\begin{aligned}
&\overline{\Pi}_{\mathcal{H}_{(r_0-j)p}} w_{jp} \\
&\quad = \sum_{k=1}^{j} W_{(k)}^{(r_0-j+k)} \Pi_{\mathcal{H}_{(r_0-j+k)p}} w_{(j-k)p} \\
&\quad = \sum_{k=1}^{j} W_{(k)}^{(r_0-j+k)} \Pi_{\mathcal{H}_{(r_0+1-j+k)p}} w_{(j-k)p} \\
&\qquad + \sum_{k=1}^{j} W_{(k)}^{(r_0-j+k)} \sum_{l=1}^{j-k} W_{(l)}^{(r_0-j+k+1, r_0-j+k+1+l)} \\
&\times \Pi_{\mathcal{H}_{(r_0-j+k+1+l)p}} w_{(j-k-l)p}.
\end{aligned}$$

Therefore, by (3.5), we obtain

$$\begin{aligned}
&\overline{\Pi}_{\mathcal{H}_{(r_0+1-j)p}} w_{jp} \\
&\quad = \sum_{k=1}^{j} (W^{(r_0-j+1, r_0-j+1+k)} + W_{(k)}^{(r_0-j+k)}) \Pi_{\mathcal{H}_{(r_0+1-j+k)p}} w_{(j-k)p} \\
&\qquad + \sum_{k=2}^{j} \sum_{\substack{k'+l=k \\ k',l\geq 1}} W_{(k')}^{(r_0-j+k')} W_{(l)}^{(r_0-j+k'+1, r_0-j+k+1+l)} \\
&\qquad \times \Pi_{\mathcal{H}_{(r_0-j+k+1+l)p}} w_{(j-k-l)p}
\end{aligned}$$

$$= \sum_{k=1}^{j} W_{(k)}^{(r_0+1-j+k)} \Pi_{\mathcal{H}_{(r_0+1-j+k)p}} w_{(j-k)p}.$$

This completes the proof of Lemma 3.2. □

We continue the proof of Proposition 3.1. We rewrite $\widetilde{(A)}_{(r_0+1)p}$ as the form

$$\begin{aligned}
&(H_0 - E_0) w_{(r_0+1)p} + \sum_{j=0}^{r_0} (H_{(r_0+1-j)p} - E_{(r_0+1-j)p}) \Pi_{\mathcal{H}_{(r_0-j)p}} w_{jp} \\
&\quad + \sum_{j=0}^{r_0} (H_{(r_0+1-j)p} - E_{(r_0+1-j)p}) \overline{\Pi}_{\mathcal{H}_{(r_0-j)p}} w_{jp} = 0.
\end{aligned}$$

By Lemma 3.2, this is equivalent to

$$\begin{aligned}
&(S_0 - E_0)w_{(r_0+1)p} + \sum_{j=0}^{r_0}(H_{(r_0+1-j)p} - E_{(r_0+1-j)p})\Pi_{\mathcal{H}_{(r_0-j)p}} w_{jp} \\
&\quad + \sum_{j=0}^{r_0}\sum_{l=0}^{r_0-j}(H_{(r_0+1-j-l)p} - E_{(r_0+1-j-l)p})W_{(j)}^{(r_0-l)}\Pi_{\mathcal{H}_{(r_0-l)p}} w_{lp} \\
&= (S_0 - E_0)u_{(r_0+1)p} + (S_p - E_p)\Pi_{\mathcal{H}_0} w_{r_0p} + \sum_{j=0}^{r_0-1}\Big[(H_{(r_0+1-j)p} \\
&\quad - E_{(r_0+1-j)p}) + \sum_{l=0}^{r_0-j}(H_{(r_0+1-j-l)p} - E_{(r_0+1-j-l)p})W_{(l)}^{(r_0-j)}\Big] \\
&\quad \times \Pi_{\mathcal{H}_{(r_0-j)p}} w_{jp} = 0.
\end{aligned}$$

Since it follows form (2.7) that

$$\begin{aligned}
&H_{(r_0+1-j)p} - E_{(r_0+1-j)p} + \sum_{l=0}^{r_0-j}(H_{(r_0+1-j-l)p} \\
&\quad - E_{(r_0+1-j-l)p})W_{(l)}^{(r_0-j)} = S_{(r_0+1-j)p} - E_{(r_0+1-j)p},
\end{aligned}$$

this completes the proof of Proposition 3.1. □

Proof of Theorem 2.1. (i) Construction of E_j and u_j.

Let $u_0 \in \mathcal{H}_q$ so that $\|u_0\| = 1$. We search u_j $(j > 0) \in \mathcal{H}_q^{\perp}$.

When $rp < q < (r+1)p$, we can write $(A)_{jp+l}$, as follows. If $j \le r-1$ and $0 \le l < p$,

$$\sum_{k=0}^{j}(H_{kp} - E_{kp})u_{(j-k)p+l} = 0, \tag{3.6}$$

and when $j = r$,

$$\begin{aligned}
&\sum_{k=0}^{r-1}(H_{kp} - E_{kp})u_{(j-k)p+l} + (H_{rp} - E_{rp})u_{rp} \\
&\quad \begin{cases} = 0, & \text{if } l < q - rp, \\ +(H_q - E_q)u_0 = 0, & \text{if } l = q - rp. \end{cases}
\end{aligned} \tag{3.7}$$

If $j \le r-1$, $0 \le l < p$, applying Proposition 3.1 with $w_{jp} = u_{jp+l}$, we see that (3.6) is equivalent to

$$\sum_{k=0}^{j}(S_{kp} - E_{kp})\Pi_{\mathcal{H}_{(k-1)p}} u_{(j-k)p+l} = 0. \tag{3.8}$$

When $j = r$, $(A)_{rp+l}$ is equivalent to

$$\sum_{k=0}^{r}(S_{kp} - E_{kp})\Pi_{\mathcal{H}_{(k-1)p}} u_{(r-k)p+l} \tag{3.9}$$
$$\begin{cases} = 0, & \text{if } l < q - rp \\ +(H_q - E_q)u_0 = 0, & \text{if } l = q - rp. \end{cases}$$

Applying $\Pi_{\mathcal{H}_{(r-2)p}}$ from the left to (3.8), we see that $u_l \in \mathcal{H}_{(r-1)p}$. Moreover, when $l < q - rp$, applying $\Pi_{\mathcal{H}_{(r-1)p}}$ from the left to (3.9), we see that $u_l \in \mathcal{H}_{rp}$ and thereby the restriction to $\mathcal{H}_{(r-1)p}$ of (3.9) holds. When $l = q - rp$, applying $\Pi_{\mathcal{H}_{rp}}$ and $\Pi_{\mathcal{H}_{(r-1)p} \ominus \mathcal{H}_{rp}}$ from the left to (3.9), we get

$$\begin{aligned} &\Pi_{\mathcal{H}_{rp}}(H_q - E_q)u_0 = (K_q - E_q)u_0 = 0, \\ &\Pi_{\mathcal{H}_{(r-1)p} \ominus \mathcal{H}_{rp}} u_l = -(S_{rp} - E_{rp})^{-1}\overline{\Pi}_{\mathcal{H}_{rp}}(H_q - E_q)u_0. \end{aligned} \tag{3.10}$$

Thus, if we define $u_l \in \mathcal{H}_{(r-1)p}$ and

$$\begin{aligned} &\Pi_{\mathcal{H}_{(r-1)p} \ominus \mathcal{H}_{rp}} u_l \\ &= \begin{cases} -(S_{rp} - E_{rp})^{-1}\overline{\Pi}_{\mathcal{H}_{rp}}(H_q - E_q)u_0, & \text{if } l = q - rp\,, \\ 0, & \text{if } l < q - rp\,, \end{cases} \end{aligned}$$

we see that $\overline{\Pi}_{\mathcal{H}_{rp}} u_l$ is determined for $0 \le l \le q - rp$ and that (3.9) restricted to $\mathcal{H}_{(r-1)p}$ holds. Next, we can see that for $0 \le r' \le r$ and $0 \le l' \le l$, $(3.9)_{(r-r')p+l-l'}$ restricted to $\mathcal{H}_{(r-1-r')p}$ holds. In fact, if $r' = l' = 0$, this follows from (3.10). If $l' > 0$, since $l - l' < q - rp$, it follows that $u_{l-l'} \in \mathcal{H}_{rp}$. Therefore, $(3.9)_{(r-r')p+l-l'}$ restricted to $\mathcal{H}_{(r-1-r')p}$ holds. If $l' = 0$ and $r' > 0$, since $u_l \in \mathcal{H}_{(r-1)p} \subset \mathcal{H}_{(r-r')p}$, we see that

$$\begin{aligned} &\Pi_{\mathcal{H}_{(r-r'-1)p}}(S_{(r-r')p} - E_{(r-r')p})\Pi_{\mathcal{H}_{(r-r'-1)p}} u_l \\ &\quad = (K_{(r-r')p} - E_{(r-r')p})u_l = 0. \end{aligned}$$

Now we impose the induction hypotheses: There exists an integer $j \ge q$ such that the following holds:

(i)$_j$ E_{j-k} $(0 \le k \le j)$ and u_{j-q-k} $(0 \le k \le j - q)$ are known.

(ii)$_j$ $\overline{\Pi}_{\mathcal{H}_{rp}} u_{j-rp-k}$ $(0 \le k \le j - rp)$ are known.

(iii)$_j$ $(A)_{j-lp-k}$ restricted to $\mathcal{H}_{(r-l-1)p}$ holds for $0 \le l \le r$ and $0 \le k \le j - lp$ with $\mathcal{H}_{-p} = \mathcal{H}$.

We note that from above arguments, the hypotheses hold for $j = q$ and $(A)_{j-rp-k}$ holds for $0 \le k \le j - rp$.

We now consider $(A)_{j+1}$. At the first time, we treat the case, where $rp < q < (r+1)p$. By Proposition 3.1 and (i) in the induction hypothesis, we see that $(A)_{j+1}$ is equivalent to

$$\sum_{k=0}^{r}(S_{kp} - E_{kp})\Pi_{\mathcal{H}_{(k-1)p}} u_{j+1-kp} + (H_q - E_q)u_{j+1-q} \qquad (3.11)$$
$$+g'_{j+1} - E_{j+1}u_0 = 0\,,$$

where g'_{j+1} is a known element. Here we note that $\overline{\Pi}_{\mathcal{H}_{rp}} u_{j+1-q}$ is known, because of (ii) in the induction hypothesis. Therefore, we can rewrite (3.11) of the form:

$$\sum_{k=0}^{r}(S_{kp} - E_{kp})\Pi_{\mathcal{H}_{(k-1)p}} u_{j+1-kp}$$
$$+ (H_q - E_q)\Pi_{\mathcal{H}_{rp}} u_{j+1-q} + g_{j+1} - E_{j+1}u_0 = 0\,, \quad (3.12)$$

where g_{j+1} is a known element.

If we define $E_{j+1} = (g_{j+1}, u_0)$, we see that $(A)_{j+1}$ restricted to $\mathcal{H}_q$ holds. If we define

$$\Pi_{\mathcal{H}_{rp}\ominus\mathcal{H}_q} u_{j+1-q} = -(K_q - E_q)^{-1}\Pi_{\mathcal{H}_{rp}\ominus\mathcal{H}_q} g_{j+1},$$

we see that $(A)_{j+1}$ restricted to $\Pi_{\mathcal{H}_{rp}\ominus\mathcal{H}_q}$ holds. Since we search u_{j+1-q} in $\mathcal{H}_q^{\perp}$, u_{j+1-q} is completely determined and $(\mathrm{i})_{j+1}$ holds. Moreover, we see that $(A)_{j+1}$ restricted to $\mathcal{H}_{rp}$ holds. From these facts, we can rewrite (3.12) of the form:

$$\sum_{k=0}^{r}(S_{kp} - E_{kp})\Pi_{\mathcal{H}_{(k-1)p}} u_{j+1-kp} = f_{j+1}\,, \qquad (3.13)$$

where f_{j+1} is a known element satisfying $\Pi_{\mathcal{H}_{rp}} f_{j+1} = 0$.

Applying $\Pi_{\mathcal{H}_{(r-1)p}\ominus\mathcal{H}_{rp}}$ to (3.13), we have to define

$$\Pi_{\mathcal{H}_{(r-1)p}\ominus\mathcal{H}_{rp}} u_{j+1-q} = (K_{rp} - E_{rp})^{-1}\Pi_{\mathcal{H}_{(r-1)p}} f_{j+1}. \qquad (3.14)$$

If we do so, we see that $(A)_{j+1}$ restricted to $\mathcal{H}_{(r-1)p}$ holds. Thus, we find that $(\mathrm{iii})_{j+1}$ with $l = k = 0$ holds. If $k > 0$, $(\mathrm{iii})_{j+1}$ follows from the induction hypotheses. We consider the case, where $k = 0$ and $0 < l \le r$. In this case, we can write $(A)_{j+1-lp}$ into the form:

$$(S_0 - E_0)u_{j+1-lp} + \cdots + (S_{(r-l)p} - E_{(r-l)p})\Pi_{\mathcal{H}_{(r-l-1)p}} u_{j+1-rp}$$
$$= f_{j+1-lp}\,,$$

where f_{j+1-lp} is known. Here we note that $\Pi_{\mathcal{H}_{(r-l)p}} f_{j+1-lp} = 0$ by the induction hypotheses. Therefore, if we define

$$\Pi_{\mathcal{H}_{(r-l-1)p}\ominus\mathcal{H}_{(r-l)p}} u_{j+1-rp} = (K_{(r-l)p} - E_{(r-l)p})^{-1}\Pi_{\mathcal{H}_{(r-l-1)p}} f_{j+1-lp}, \quad (3.15)$$

for $0 < l \le r$, we see that $(A)_{j+1-lp}$ restricted to $\mathcal{H}_{(r-l-1)p}$ holds. Thus, $(\mathrm{iii})_{j+1}$ holds. By (3.14) and (3.15), $\overline{\Pi}_{\mathcal{H}_{rp}} u_{j+1-rp}$ is completely determined and so $(\mathrm{ii})_{j+1}$ holds.

When $q = rp$, $(A)_{j+1}$ is equivalent to

$$\sum_{k=0}^{r} (S_{kp} - E_{kp})\Pi_{\mathcal{H}_{(k-1)p}} u_{j+1-kp} + g'_{j+1} - E_{j+1}u_0 = 0\,, \quad (3.16)$$

where g'_{j+1} is a known element. If we define $E_{j+1} = (g'_{j+1}, u_0)$, we see that $(A)_{j+1}$ restricted to $\mathcal{H}_{rp}$ holds. Taking (3.16) into consideration, we define

$$\Pi_{\mathcal{H}_{(r-1)p}\ominus\mathcal{H}_{rp}} u_{j+1-rp} = -(K_{rp} - E_{rp})^{-1}\Pi_{\mathcal{H}_{(r-1)p}\ominus\mathcal{H}_{rp}} f_{j+1}\,,$$

where $f_{j+1} = E_{j+1}u_0 - g'_{j+1} \in \mathcal{H}_{rp}^{\perp}$.

Now $(A)_{j+1-p}$ is equivalent to

$$(S_0 - E_0)u_{j+1-p} + \cdots + (S_{(r-1)p} - E_{(r-1)p})\Pi_{\mathcal{H}_{(r-2)p}} u_{j+1-rp} = f_{j+1-p}\,,$$

where f_{j+1-p} is a known element. Since $(A)_{j+1-p}$ restricted to $\mathcal{H}_{(r-1)p}$ holds by the induction hypotheses, we get $\Pi_{\mathcal{H}_{(r-1)p}} f_{j+1-p} = 0$. Therefore, if we define

$$\Pi_{\mathcal{H}_{(r-2)p}\ominus\mathcal{H}_{(r-1)p}} u_{j+1-rp} = -(K_{(r-1)p} - E_{(r-1)p})^{-1}\Pi_{\mathcal{H}_{(r-2)p}} f_{j+1-p},$$

$\Pi_{\mathcal{H}_{(r-2)p}\ominus\mathcal{H}_{rp}} u_{j+1-rp}$ is determined and we see that $(A)_{j+1-p}$ restrictred to $\mathcal{H}_{(r-2)p}$ holds. Thus, $(\mathrm{iii})_{j+1}$ with $l = 1, k = 0$ holds. By the same arguments as the case, where $rp < q$, we reach to the conclusion.

(ii) Non-degeneracy of the eigenvalue $E(h)$.

Since $\|u_0\| = 1$, for all $m \ge 0$,

$$\Big\|\sum_{j=0}^{m} h^j u_j\Big\| - 1 = O(h), \quad \text{as } h \to 0. \quad (3.17)$$

On the other hand, by the construction of E_j and u_j,

$$\Big\|(H^h - \sum_{j=0}^{m} h^j E_j)(\sum_{k=0}^{m} h^k u_k)\Big\| = O(h^{m+1}), \quad \text{as } h \to 0.$$

Thus, for any $m > 0$, there exists $C_m > 0$ such that $\sigma_{\text{disc}}(H^h) \cap I_m(h) \neq \emptyset$, where

$$I_m(h) = \big(\sum_{j=0}^{m} h^j E_j - C_m h^{m+1}, \sum_{j=0}^{m} h^j E_j + C_m h^{m+1}\big).$$

By the hypothesis (2.5), we have $\sigma_{\text{disc}}(H^h) \cap I_m(h) = \sigma(H^h) \cap I_m(h) \neq \emptyset$. We show that for $m \geq q$, the spectral projection of H^h restricted to $I_m(h)$ is of rank one. Let $E(h) \in \sigma_{\text{disc}}(H^h) \cap I_m(h)$ and u^h be an associated normalized eigenfunction, i.e., $H^h u^h = E(h) u^h$, $\|u^h\| = 1$. By (2.4) and (2.5), we see that there exists a constant $C'_m > 0$ such that

$$\|\sum_{j=0}^{m} h^j (H_j - E_j) u^h\| \leq C'_m h^{m+1}. \tag{3.18}$$

By the same arguments as the proof of Proposition 3.1, it follows from (3.18) that

$$\|\sum_{k=0}^{j} h^{kp} (S_{kp} - E_{kp}) \Pi_{\mathcal{H}_{(k-1)p}} u^h\| = O(h^{(j+1)p}), \tag{3.19}$$

as $h \to 0$, for $j = 0, 1, \ldots, rp$ and

$$\|\sum_{k=0}^{r} h^{kp} (S_{kp} - E_{kp}) \Pi_{\mathcal{H}_{(k-1)p}} u^h + h^q (H_q - E_q) \Pi_{\mathcal{H}_{rp}} u^h\| = O(h^{q+1}), \tag{3.20}$$

as $h \to 0$, for $rp < q$. It follows from (3.19) and (3.20) that

$$\begin{aligned} \|\Pi_{\mathcal{H}_{(j-1)p} \ominus \mathcal{H}_{jp}} u^h\| &= O(h^p) \quad (j = 0, 1, \ldots, rp), \\ \|\Pi_{\mathcal{H}_{rp} \ominus \mathcal{H}_q} u^h\| &= O(h). \end{aligned}$$

Therefore, we have

$$\|\overline{\Pi}_{\mathcal{H}_q} u^h\| = O(h) \quad \text{as } h \to 0. \tag{3.21}$$

If we denote the eigenspace of H^h corresponding to the eigenvalue $E(h)$ by $\mathcal{H}_{E(h)}$, we see that (3.21) implies that the mapping $\Pi_{\mathcal{H}_q} : \mathcal{H}_{E(h)} \to \mathcal{H}_q$ is one to one. Thus, $\dim \mathcal{H}_{E(h)} \leq \dim \mathcal{H}_q$. Since $1 = \dim \mathcal{H}_q \leq \dim \mathcal{H}_{E(h)}$, we have $\dim \mathcal{H}_{E(h)} = \dim \mathcal{H}_q$. This completes the proof of Theorem 2.1. □

Proof of Theorem 2.2. We start the proof with the following lemma.

Lemma 3.3. (i) *If T is symmetric operator on $\mathcal{H}_\infty$ commuting with J, then there exists a real number E such that $\Pi_{\mathcal{H}_q} T \Pi_{\mathcal{H}_q} = E\Pi_{\mathcal{H}_q}$.*

(ii) *Let $j > 0$ be an integer. Assume that there exist u_k $(k = 0, 1, \ldots, j)$ with $u_0 \in \mathcal{H}_0, \|u_0\| = 1$ and E_k $(k = 1, 2, \ldots, j)$ such that $(A)_k$ $(k = 0, 1, \ldots, j)$ hold. Then E_k are real numbers.*

Proof. (i) It is well known that there exists $E \in \mathbf{R}$ and $\mathcal{H}_q \ni u \neq 0$ such that $\Pi_{\mathcal{H}_q} T \Pi_{\mathcal{H}_q} u = E\Pi_{\mathcal{H}_q} u$. Since J is commutative with T and $\Pi_{\mathcal{H}_q}$, (i) follows.

(ii) From $(A)_1 : (H_0 - E_0)u_1 + (H_1 - E_1)u_0 = 0$, $E_1 = (H_1 u_0, u_0)$ is a real number, since H_1 is a symmetric operator. Assume that E_l are real numbers for $l \leq k$. Since

$$\left\|(H^h - \sum_{l=0}^{k+1} h^l E_l)(\sum_{j=0}^{k+1} h^j u_j)\right\| = O(h^{k+2}) \quad \text{as } h \to 0,$$

$$\begin{aligned} O(h^{k+2}) &= \left((H^h - \sum_{l=0}^{k+1} h^l E_l)(\sum_{j=0}^{k+1} h^j u_j), \sum_{j=0}^{k+1} h^j u_j\right) \\ &= \left((H^h - \sum_{l=0}^{k} h^l E_l)(\sum_{j=0}^{k+1} h^j u_j), \sum_{j=0}^{k+1} h^j u_j\right) \\ &\quad -h^{k+1} E_{k+1}\left(\sum_{j=0}^{k+1} h^j u_j, \sum_{j=0}^{k+1} h^j u_j\right). \end{aligned}$$

According to the self-adjointness of H^h and induction hypothesis, the first term in the right hand side is real. Therefore Im $E_{k+1} = O(h)$ as $h \to 0$. Thus, we see that E_{k+1} is a real number. This completes the proof of Lemma 3.3. □

We continue the proof of Theorem 2.2. For operators S_{jp} $(0 \leq j \leq r)$ constructed in Section 2, we see that clearly $JS_{jp} \subset S_{jp}J$ and thereby, $JK_{jp} \subset K_{jp}J, J\Pi_{\mathcal{H}_{jp}} = \Pi_{\mathcal{H}_{jp}} J$. We define operators S_j for $j \geq q$ satisfying the following $(C)_j$ for $j \geq q$:

(i) S_l commutes with J for $q \leq l \leq j$.

(ii) There exists a real number E_l for $q \leq l \leq j$ such that $\Pi_{\mathcal{H}_q} S_l \Pi_{\mathcal{H}_q} = E_l \Pi_{\mathcal{H}_q}$.

(iii) $(A)_j$ is equivalent to

$$\sum_{k=0}^{r}(S_{kp}-E_{kp})\Pi_{\mathcal{H}_{(k-1)p}}u_{j-kp}\Big(+(S_q-E_q)\Pi_{\mathcal{H}_{rp}}u_{j-q}\Big)^{\#} \\ +(S_j-E_j)u_0=0.$$

Here and hereafter, we interpret that the parenthesis $(\cdot)^{\#}$ vanishes when $q=rp$.

In fact, at the first time, we define $S_q=H_q$ if $rp<q$. $(A)_{q+1}$ is equivalent to

$$\sum_{k=0}^{r}(S_{kp}-E_{kp})\Pi_{\mathcal{H}_{(k-1)p}}u_{j-kp}\Big(+(S_q-E_q)\Pi_{\mathcal{H}_{rp}}u_{j-q}\Big)^{\#} \\ +(H_{q+1}-E_{q+1})u_0=0.$$

Therefore, if we define $S_{q+1}=H_{q+1}$, it follows from Lemma 3.3 and symmetricity of S_{q+1} that there exists a real number E_{q+1} such that

$$\Pi_{\mathcal{H}_q}S_{q+1}\Pi_{\mathcal{H}_q}=E_{q+1}\Pi_{\mathcal{H}_q}.$$

Thus, $(C)_{q+1}$ holds.

Lemma 3.4. *Assume that for $j_0>q$, there exist operators S_j and $E_j\in\mathbf{R}$ $(q<j\le j_0)$ satisfying $(C)_j$ hold for $j\le j_0$. Then we can construct linear operators T_k on $\mathcal{H}_\infty$ for $0\le k\le j_0-q$, which commute with J such that $u_k=T_ku_0$ for $0\le k\le j_0-q$.*

Proof. Put $T_0=I$ (the identity) and assume that there exist T_k for $0\le k\le j_0-q-1$ such that $u_k=T_ku_0$. Then $(C)_{j_0}$ is equivalent to

$$\sum_{k=0}^{r}(S_{kp}-E_{kp})\Pi_{\mathcal{H}_{(k-1)p}}u_{j_0-kp}\Big(+(S_q-E_q)\Pi_{\mathcal{H}_{rp}}u_{j_0-q}\Big)^{\#} \\ +(S_{j_0}-E_{j_0})u_0=0.$$

Applying $\Pi_{\mathcal{H}_{rp}\ominus\mathcal{H}_{\mathcal{H}_q}}$ from the left to this equation, we see

$$\Pi_{\mathcal{H}_{rp}\ominus\mathcal{H}_{\mathcal{H}_q}}u_{j_0-q}=T^{(q)}_{j_0-q}u_0\,,$$

where

$$T^{(q)}_{j_0-q}=-(K_q-E_q)^{-1}\Pi_{\mathcal{H}_{rp}\ominus\mathcal{H}_{\mathcal{H}_q}}(S_{j_0}-E_{j_0}).$$

Next, we consider $(C)_{j_0-(q-rp)}$:

$$\sum_{k=0}^{r-1}(S_{kp}-E_{kp})\Pi_{\mathcal{H}_{(k-1)p}}u_{j_0-q+rp-kp}+(S_{rp}-E_{rp})\Pi_{\mathcal{H}_{(r-1)p}}u_{j_0-q}$$
$$\Big(+(S_q-E_q)\Pi_{\mathcal{H}_{rp}}u_{j_0-q+rp-q}\Big)^{\#}$$
$$+(S_{j_0-(q-rp)}-E_{j_0-(q-rp)})u_0=0.$$

When $rp<q$, it follows from the induction hypothesis that we can write $u_{j_0-q+rp-q}=T_{j_0-q+rp-q}u_0$. Applying $\Pi_{\mathcal{H}_{(r-1)p}\ominus\mathcal{H}_{\mathcal{H}_{rp}}}$ from the left to this equation, we see

$$\Pi_{\mathcal{H}_{(r-1)p}\ominus\mathcal{H}_{\mathcal{H}_{rp}}}u_{j_0-q}=T^{(rp)}_{j_0-q}u_0\,,$$

where

$$\begin{aligned}T^{(rp)}_{j_0-q} &= -(K_{rp}-E_{rp})^{-1}\big[(S_q-E_q)\Pi_{\mathcal{H}_{rp}}T_{j-q_0+rp+q}\\ &\quad +(S_{j_0-(q-rp)}-E_{j_0-(q-rp)})\big]u_0\,,\end{aligned}$$

which is commutative with J. Repeating this procedure, we can write $\Pi_{\mathcal{H}_{(k-1)p}\ominus\mathcal{H}_{\mathcal{H}_{kp}}}u_{j_0-q}=T^{(kp)}_{j_0-q}u_0$ for $k=0,1,\dots,r$. Thus, we see that

$$\overline{\Pi}_{\mathcal{H}_q}u_{j_0-q}=\Big(\sum_{k=0}^{r}T^{(kp)}_{j_0-q}+T^{(q)}_{j_0-q}\Big)u_0.$$

Since we search $u_k\in\mathcal{H}_q^{\perp}$ for $k>0$, it suffices to define

$$T_{j_0-q}=\sum_{k=0}^{r}T^{(kp)}_{j_0-q}+T^{q}_{j_0-q}.$$

This completes the proof. □

Now, we return to the proof of Theorem 2.2. By Lemma 3.4, we can write $(A)_{j_0+1}$ as follows.

$$\sum_{k=0}^{r}(S_{kp}-E_{kp})\Pi_{\mathcal{H}_{(k-1)p}}u_{j_0+1-kp}\Big(+(S_q-E_q)\Pi_{\mathcal{H}_{rp}}u_{j_0+1-q}\Big)^{\#}$$
$$+\Big(\sum_{k=q+1}^{j_0}(H_k-E_k)T_{j_0+1-k}+H_{j_0+1}\Big)u_0-E_{j_0+1}u_0=0.$$

Therefore, we define

$$S_{j_0+1} = H_{j_0+1} + \sum_{k=q+1}^{j_0} (H_k - E_k)T_{j_0+1-k}$$

which clearly commutes with J. If we define $E_{j_0+1} = (S_{j_0+1}u_0, u_0)$, then $(A)_{j_0+1}$ is equivalent to $(C)_{j_0+1}$. Therefore, by the same arguments of the proof of Theorem 2.1, if we choose $u_0 \in \mathcal{H}_q$ with $\|u_0\| = 1$, we can construct E_j $(0 \leq j \leq j_0 + 1)$ and $u_j \in \mathcal{H}_q^{\perp}$ $(0 < j < j_0 + 1)$ such that $(A)_{j_0+1}$ holds. By Lemma 3.3, we see that E_{j_0+1} is a real number. This completes the proof of Theorem 2.2. □

4. An Application to the Pauli Operator

In this section, we apply the results in the preceding section to the Pauli operator. The Pauli operator with the magnetic vector potential $a(x)$ and the electric scalar potential $V(x)$ is defined by

$$H_P^h = H_P^h(a, V) = H_{0,0}^h I_2 - h\sigma \cdot B(x) \quad \text{on } L^2(\mathbf{R}^3; \mathbf{C}^2)\,, \tag{4.1}$$

where $H_{0,0}^h = (-ih\nabla + a(x))^2 + V(x)$ on $L^2(\mathbf{R}^3)$, h is a small parameter, I_2 is the 2×2 identity matrix, $B(x) = \nabla \times a(x)$ is the magnetic field, $i = \sqrt{-1}$ and $\sigma = (\sigma_1, \sigma_2, \sigma_3)$, σ_j are the Pauli matrices as in Introduction.

We assume:

(A.1) The magnetic field $B(x)$ is smooth and has the constant direction.

Under this assumption, if we choose the direction as x_3-axis, we may assume that $B(x) = (0, 0, b(x'))$, where $x = (x_1, x_2, x_3) = (x', x_3)$ and moreover, that the magnetic potential a is of the form $a(x) = (a_1(x'), a_2(x'), 0)$, where $b(x') = \partial_1 a_2(x') - \partial_2 a_1(x')$. For these facts, see Iwatsuka and Tamura [11] and [12]. Then (4.1) becomes

$$H_P^h = H_P^h(a, V) = \begin{bmatrix} H_+^h - h^2\partial_3^2 + V & 0 \\ 0 & H_-^h - h^2\partial_3^2 + V \end{bmatrix}, \tag{4.2}$$

where $H_{\pm}^h = (D_1^h)^2 + (D_2^h)^2 \mp hb \quad (j = 1, 2)$. Since we can write $H_{\pm}^h = (D_1^h \pm iD_2^h)^*(D_1^h \pm iD_2^h)$, we see $H_P^h(a, 0) \geq 0$. If $b(x') \geq c > 0$, we see that $H_-^h \geq hc > 0$. On the other hand, according to Shigekawa [17], H_+^h has zero as an eigenvalue with infinite multiplicity. Moreover, by Cycon, Froese, Kirsh and Simon, the non-zero spectra of H_+^h and H_-^h are coincide.

We shall consider the case, where $H^h_{0,0}$ is a perturbation of the harmonic oscillator. In order to do so, we assume that the electric potential V is smooth and of the form:

$$V(x) = |x|^2 + W(x)\,, \qquad (V.1)$$

where $0 \le W(x) \in C^\infty(\mathbf{R}^3)$ is an even function and has only one zero at the origin and satisfy

$$W(x) = O(|x|^4) \quad \text{as } |x| \to 0.$$

For the magnetic field b, assume that there exists an even integer $q = 2r \ge 0$ such that for constants $C > 0$ and $C_1 > 0$ such that

$$C^{-1}|x'|^q \le b(x') \le C|x'|^q \quad \text{for } |x'| \le C_1. \qquad (A.2)$$

According to Helffer and Mohamed [8], by a gauge transformation, we may assume that there exists a constant $C_1 > 0$ such that

$$|a_j(x')| \le C_1 |x'|^{q+1}.$$

Under the above assumptions, we can write the Taylor expansions of a_j, b and V, as follows. For a large integer N,

$$\begin{aligned}
a_j(x') &= \sum_{k=q+1}^{2N-1} \widetilde{a}_j^{(k)}(x') + \widehat{a}_j^{(2N)}(x') \quad (j = 1, 2),\\
b(x') &= \sum_{k=q}^{2N-1} \widetilde{b}^{(k)}(x') + \widehat{b}^{(2N)}(x'),\\
V(x) &= |x|^2 + \sum_{k=2}^{2N-1} \widetilde{W}^{(2k)}(x) + \widehat{W}^{(2N)}(x)\,,
\end{aligned}$$

where

$$\begin{aligned}
\widetilde{a}_j^{(k)}(x') &= \sum_{|\alpha'|=k} \frac{(x')^{\alpha'}}{\alpha'!}(\partial^{\alpha'} a_j)(0), \quad \widehat{a}_j^{(2N)}(x') = O(|x'|^{2N}),\\
\widetilde{b}^{(k)}(x') &= \sum_{|\alpha'|=k} \frac{(x')^{\alpha'}}{\alpha'!}(\partial^{\alpha'} b)(0), \quad \widehat{b}^{(2N)}(x') = O(|x'|^{2N}),\\
\widetilde{W}^{(2k)}(x) &= \sum_{|\alpha|=2k} \frac{x^{\alpha}}{\alpha!}(\partial^{\alpha} W)(0), \quad \widehat{W}^{(2N)}(x) = O(|x|^{2N})\,,
\end{aligned}$$

as $|x'|$ or $|x| \to 0$. Here $\alpha' = (\alpha_1, \alpha_2) \in (\mathbf{Z}_+)^2$ and $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in (\mathbf{Z}_+)^3$. As in Aramaki [3] and [4], using a unitary transformation $(U_h f)(x) = h^{-3/4} f(h^{-1/2}x)$ for $f \in L^2(\mathbf{R}^3; \mathbf{C}^2)$ and expanding $(U_h)^* H_P^h U_h$ with respect to powers of h, we can write

$$(U_h)^* H_P^h U_h \sim h \sum_{j=0}^{\infty} h^{j/2} H_j \,, \tag{4.3}$$

where

$$H_j = H_{j,0} I_2 - \sigma_3 b^{(j)} \quad \text{for } j \geq 0 \,,$$

and

$$\begin{aligned} H_{0,0} &= (-i\partial_1 + a_1^{(1)})^2 + (-i\partial_2 + a_2^{(1)})^2 - \partial_3^2 + |x|^2, \\ H_{j,0} &= -i[\partial_1 a_1^{(j+1)} + a_1^{(j+1)}\partial_1 + \partial_2 a_2^{(j+1)} + a_2^{(j+1)}\partial_2] + \widetilde{W}^{(2+j)} \,, \end{aligned}$$

for $j \geq 1$. Here the symbol $\sim$ in (4.3) means that for every integer $N > 1$, there exist an integer $M = M(N)$ and a constant $C_N > 0$ such that

$$\left\| \left((U_h)^* H_P^h U_h - h \sum_{j=0}^{N-1} h^{j/2} H_j \right) u \right\| \leq C_N h^{(N+2)/2} \| (-\Delta + |x|^2)^M u \| \,,$$

for all $u \in \mathcal{S}(\mathbf{R}^3; \mathbf{C}^2)$, which denotes the Schwartz space with values in $\mathbf{C}^2$.

At the first time, we consider the case, where $q = 0$, i.e., $b = b(0) > 0$. According to the hypothesis (A.2) and a gauge transformation, we may take

$$H_0 = H_{0,0} I_2 - \begin{bmatrix} b & 0 \\ 0 & -b \end{bmatrix} ,$$

where $H_{0,0} = (-i\partial_1 - bx_2/2)^2 + (-i\partial_2 + bx_1/2)^2 - \partial_3^2 + |x|^2$. Thanks to Matsumoto and Ueki [15] and [4], the spectrum of $H_{0,0}$ are discrete and $\sigma(H_{0,0}) = \{\sum_{j=1}^3 (2\alpha_j + 1)s_j; \alpha = (\alpha_1, \alpha_2, \alpha_3) \in (\mathbf{Z}_+)^3\}$, including multiplicities, where $s_1 = \frac{1}{2}(\sqrt{4 + b^2} - b), s_2 = 1, s_3 = \frac{1}{2}(\sqrt{4 + b^2} + b)$. Thus, since $H_{0,0}$ has the non-degenerate ground state energy $e_1 = 1 + \sqrt{4 + b^2}$, H_0 has the non-degenerate ground state energy $E_1 = e_1 - b$. Therefore, it follows from Theorem 2.1 that H_P^h has the non-degenerate ground state energy $E_1(h)$ satisfying $E_1(h) = h(E_1 + O(h^{1/2}))$ as $h \to 0$. If $0 < b < 1/\sqrt{2}$, the second eigenvalue $E_2(h)$ of H_P^h is also non-degenerate and satisfies $E_2(h) = h(E_2 + O(h^{1/2}))$ as $h \to 0$, where $E_2 = e_1 + b = b + 1 + \sqrt{b^2 + 4}$ and we have

$$0 < E_2(h) - E_1(h) \leq 2bh + O(h^{3/2}) \quad \text{as } h \to 0.$$

If $b > 1/\sqrt{2}$, the second eigenvalue $E_2(h)$ is also non-degenerate and satisfies $E_2(h) = h(E_2 + O(h^{1/2}))$ as $h \to 0$, where $E_2 = e_1 - b + 2s_1 = 1 - 2b + 2\sqrt{b^2+4}$ and we have

$$0 < E_2(h) - E_1(h) \le 2\sqrt{b^2+4}h + O(h^{3/2}) \quad \text{as } h \to 0.$$

Next, we consider the case, where $q = 2r > 0$. In this case, it is well known that the harmonic oscillator $H_{0,0} = -\Delta + |x|^2$ has only the discrete spectrum

$$\sigma(H_{0,0}) = \sigma_{\mathrm{disc}}(H_{0,0}) = \{\mu_\alpha = 2|\alpha| + 3; \alpha = (\alpha_1, \alpha_2, \alpha_3) \in (\mathbf{Z}_+)^3\},$$

including multiplicities and that $L^2(\mathbf{R}^3)$ has the complete orthonormal basis consisting of the eigenfunctions $\phi_\alpha(x)$ corresponding to μ_α. Moreover, we can write $\phi_\alpha(x) = \phi_{\alpha_1}(x_1)\phi_{\alpha_2}(x_2)\phi_{\alpha_3}(x_3)$, where

$$\phi_j(t) = (\sqrt{k!}2^{k/2})^{-1} H_j(t)\phi_0(t),$$

$\phi_0(t) = \pi^{-1/4}\exp(-t^2/2)$ and $H_j(t)$ denotes the j-th Hermite polynomial:

$$H_j(t) = \sum_{r=0}^{[j/2]} 2^{j-r}(-1)^r(2r-1)!! \begin{pmatrix} j \\ 2r \end{pmatrix} t^{j-2r} \quad (j \ge 0), \tag{4.4}$$

where $[\cdot]$ denotes the Gaussian symbol. Since $\phi_j(t)$ satisfies the equation

$$\phi_{j+1}(t) - \frac{\sqrt{2}}{\sqrt{j+1}} t\phi_j(t) + \frac{\sqrt{j}}{\sqrt{j+1}}\phi_{j-1}(t) = 0 \quad (j \ge 0), \tag{4.5}$$

with $\phi_{-1} = 0$, we can write $t^j\phi_k = \sum_{l=0}^{j} C^j_{k,k-j+2l}\phi_{k-j+2l}$ for integers $j \ge 0, k \ge 0$, where $C^j_{k,l} = (t^j\phi_k, \phi_l)$. Here we can compute $C^j_{k,l}$, as follows. For $l = 0, 1, \ldots$, by using a well known equality

$$\int_{-\infty}^{\infty} t^l e^{-t^2} dt = \Gamma\left(\frac{l+1}{2}\right) = \begin{cases} (2l'-1)!!/2^{l'} \text{ if } l \text{ is even: } l = 2l' \\ l'! \text{ if } l \text{ is odd: } l = 2l'+1 \end{cases}$$

we see that

$$\begin{aligned} C^j_{k,l} \;&=\; 2^{-j/2}\frac{1}{\sqrt{k!}}\frac{1}{\sqrt{l!}}\sum_{r=0}^{[k/2]}\sum_{s=0}^{[l/2]}(-1)^{r+s}(2r-1)!!\,(2s-1)!! \\ &\quad\times \begin{pmatrix} k \\ 2r \end{pmatrix}\begin{pmatrix} l \\ 2s \end{pmatrix}(j+k+l-2r-2s-1)!!. \end{aligned} \tag{4.6}$$

Clearly we have $C^j_{k,l} = C^j_{l,k}$ if $l \geq 0$. For multi-indices $\alpha, \beta, \gamma \in (\mathbf{Z}_+)^3$, we define $C^\alpha_{\beta,\gamma} = C^{\alpha_1}_{\beta_1,\gamma_1} C^{\alpha_2}_{\beta_2,\gamma_2} C^{\alpha_3}_{\beta_3,\gamma_3}$. Then we see that

$$x^\alpha \phi_\beta = \sum_{\gamma \leq \alpha} C^\alpha_{\beta,\beta-\alpha+2\gamma} \phi_{\beta-\alpha+2\gamma}.$$

From now on, for every multi-indices γ, δ, we define a differential operator:

$$\partial^{(k)}_{\gamma,\delta} = \sum_{*(1)} \frac{C^\beta_{\gamma,\delta}}{\beta!} \partial^\beta \,, \tag{4.7}$$

where the sum $*(1)$ runs over all multi-indices β satisfying that $|\beta| = k, \gamma - \delta \leq \beta, \delta - \gamma \leq \beta$ and all components of $\delta - \gamma + \beta$ are even.

Clearly, we see that the ground state energy $E_0 = 3$ of $H_0 = H_{0,0} I_2$ is double and the corresponding eigenspace $\mathcal{H}_0 = \mathcal{H}_{00} \oplus \mathcal{H}_{00}$, where $\mathcal{H}_{00} = [\phi_0]$. Since W is an even function, we have

$$\begin{aligned} H_{2j+1} &= 0, \\ H_{2j} &= \widetilde{W}^{(j+2)} I_2, \quad \text{for } 1 \leq j < r, \\ H_{2r} &= \Big[-i\{\partial_1 a_1^{(2r+1)} + a_1^{(2r+1)} \partial_1 + \partial_2 a_2^{(2r+1)} + a_2^{(2r+1)} \partial_2\} \\ &\quad + \widetilde{W}^{(2r+2)} \Big] I_2 - \sigma_3 b^{(2r)}. \end{aligned}$$

We shall apply Theorem 2.1 with $p = 2$. Since we can write $S_{2j} = S_{2j,0} I_2$, we see that $\mathcal{H}_0 = \mathcal{H}_2 = \cdots = \mathcal{H}_{2(r-1)}$. Therefore, we have $W^{(i,j)} = 0$ for $i = 1, 2, \ldots, r-1$ and so $W^{(j)}_{(k)} = W^{(0,k)}$. Thus, we can write

$$S_{2j} = H_{2j} + \sum_{k=1}^{j-1} (H_{2k} - E_{2k})(-(K_0 - E_0))^{-1} \overline{\Pi}_{\mathcal{H}_0} (S_{2(j-k)} - E_{2(j-k)}) \,,$$

for $1 \leq j \leq r$. Then we have:

Lemma 4.1. *For $0 \leq j \leq r-1$, we can write*

$$\begin{aligned} &S_{2j,0} - E_{2j} \\ &\quad = \sum_{\substack{k_1 + \cdots + k_l = j \\ k_i \geq 1, l \geq 1}} (-1)^{l-1} (H_{2k_1,0} - E_{2k_1})(K_{0,0} - E_0)^{-1} \overline{\Pi}_{\mathcal{H}_{00}} \\ &\qquad \times (H_{2k_2,0} - E_{2k_2})(K_{0,0} - E_0)^{-1} \overline{\Pi}_{\mathcal{H}_{00}} (H_{2k_3,0} - E_{2k_3}) \end{aligned}$$

$$\begin{aligned}&\times\cdots\\&\times(H_{2k_{l-1},0}-E_{2k_{l-1}})(K_{0,0}-E_0)^{-1}\overline{\Pi}_{\mathcal{H}_{00}}(H_{2k_l,0}-E_{2k_l}).\end{aligned}$$

Proof. For $j=1$, it is trivial. For $j=2$, since we can write

$$S_{4,0}-E_4=-(H_{2,0}-E_2)(K_{0,0}-E_0)^{-1}\overline{\Pi}_{\mathcal{H}_{00}}(H_{2,0}-E_2)\\ +(H_{4,0}-E_4),$$

the lemma holds for $j=2$. Assume that the lemma holds up to $j-1$. For j, it follows from (2.7) that we can write

$$\begin{aligned}S_{2j,0}-E_{2j} &= H_{2j,0}-E_{2j}-\sum_{k=1}^{j-1}(H_{2k,0}-E_{2k})\\ &\quad\times(K_{0,0}-E_0)^{-1}\overline{\Pi}_{\mathcal{H}_{00}}(S_{2(j-1-k),0}-E_{2(j-1-k)}).\end{aligned}$$

Thus, by the induction hypothesis,

$$\begin{aligned}&S_{2j,0}-E_{2j}\\&=H_{2j,0}-E_{2j}-\sum_{k=1}^{j-1}(H_{2k,0}-E_{2k})(K_{0,0}-E_0)^{-1}\overline{\Pi}_{\mathcal{H}_{00}}\\&\quad\times\sum_{\substack{k_1+\cdots+k_l=j-k\\k_i\geq1,l\geq1}}(-1)^{l-1}(H_{2k_1,0}-E_{2k_1})(K_{0,0}-E_0)^{-1}\overline{\Pi}_{\mathcal{H}_{00}}\\&\quad\times\cdots\\&\quad\times(H_{2k_{l-1},0}-E_{2k_{l-1}})(K_{0,0}-E_0)^{-1}\overline{\Pi}_{\mathcal{H}_{00}}(H_{2k_l,0}-E_{2k_l})\\&=\sum_{\substack{k_1+\cdots+k_{l+1}=j\\k_i\geq1,l\geq1}}(-1)^{l}(H_{2k_1,0}-E_{2k_1})(K_{0,0}-E_0)^{-1}\overline{\Pi}_{\mathcal{H}_{00}}\\&\quad\times\cdots\\&\quad\times(H_{2k_l,0}-E_{2k_1})(K_{0,0}-E_0)^{-1}\overline{\Pi}_{\mathcal{H}_{00}}(H_{2k_{l+1},0}-E_{2k_{l+1}}).\end{aligned}$$

This completes the proof. □

Since $H_{2j,0}=\widetilde{W}^{(2j+2)}$ for $1\leq j\leq r-1$, we see that

$$\begin{aligned}K_{2j,0} &= \Pi_{\mathcal{H}_{00}}S_{2j,0}\,|_{\mathcal{H}_{00}}\\ &= \sum_{\substack{k_1+\cdots+k_l=j\\k_i\geq1,l\geq2}}(-1)^{l-1}\Pi_{\mathcal{H}_{00}}\widetilde{W}^{(2k_1+2)}(K_{0,0}-E_0)^{-1}\overline{\Pi}_{\mathcal{H}_{00}}\\ &\quad\times\cdots\\ &\quad\times(\widetilde{W}^{(2k_{l-1}+2)}-E_{2k_{l-1}})(K_{0,0}-E_0)^{-1}\overline{\Pi}_{\mathcal{H}_{00}}\widetilde{W}^{(2k_l+2)}\,|_{\mathcal{H}_{00}}\end{aligned}$$

$$+\Pi_{\mathcal{H}_{00}}\widetilde{W}^{(2j+2)}|_{\mathcal{H}_{00}}\ .$$

From this fact, we can derive:

Lemma 4.2. *For $0 \le j \le r$, if we put*

$$\begin{aligned} E'_{2j} &= \sum_{\substack{k_1+\cdots+k_l=j\\ k_i\ge 1, l\ge 1}} (-1)^{l-1} \sum_{0<|\gamma_i|\le 2k_i+2} (\partial_{0,\gamma_1}^{(2k_1+2)}W)(0) \\ &\quad \times \prod_{i=1}^{l-2} (2|\gamma_i|)^{-1}((\partial_{\gamma_i,\gamma_{i+1}}^{(2k_{i+1}+2)}W)(0) - \delta_{\gamma_i,\gamma_{i+1}}E_{2k_{i+1}}) \\ &\quad \times (2|\gamma_{l-1}|)^{-1}(\partial_{\gamma_{l-1},0}^{(2k_l+2)}W)(0), \end{aligned}$$

we have

$$\begin{aligned} E_{2j} &= E'_{2j} \quad \text{for } 0 \le j \le r-1, \\ E_{2r} &= E'_{2r} \pm \delta_{jr}2^{-2r}\frac{1}{r!}(\Delta'^{r}b)(0)\,, \end{aligned}$$

where $\Delta' = \partial_1^2 + \partial_2^2$.

Summing up and applying Theorem 2.1, we get the following.

Proposition 4.3. *Assume that (A.1), (A.2) with $q = 2r > 0$ and (V.1) hold. Then H_P^h has the first two non-degenerate eigenvalues $E^{\pm}(h)$ and we can find $E_{2j}^{\pm}$ for $j \ge r+1$ such that $E^{\pm}(h)$ has the following asymptotic expansion:*

$$E^{\pm}(h) \sim h\left(\sum_{j=0}^{r-1} h^j E_{2j} + \sum_{j=r}^{\infty} h^j E_{2j}^{\pm}\right),$$

where E_{2j} for $j \le r-1$ and $E_{2r}^{\pm}$ are given in the above lemma.

5. Examples

In this section, we consider two examples.

(1) At the first time, we consider the Pauli operator (4.1). For the brevity of computation, we assume that $W(x)$ in (V.1) is a function only of $|x|^2$ and (A.1) holds for $q = 2r \le 6$. Thus, we write

$$0 \le W(x) = \widetilde{W}^{(4)} + \widetilde{W}^{(6)} + \widetilde{W}^{(8)} + O(|x|^{10}) \quad \text{as } |x| \to 0\,, \tag{5.1}$$

where $\widetilde{W}^{(2j)} = a_{2j}|x|^{2j}$ for $2 \le j \le 4$.

When $r = 0$, by the arguments in the previous section, we see that H_P^h has the non-degenerate ground state energy $E(h)$ satisfying

$$E(h) = h(1 + \sqrt{4 + b(0)^2} - b(0) + O(h)) \quad \text{as } h \to 0.$$

When $r > 0$, we see that $H_{2j} = H_{2j,0}I_2$ for $0 \le j \le r-1$ and $H_{2r} = H_{2r,0}I_2 - \sigma_3 b^{(2r)}(x)$. Therefore, we can write

$$\begin{aligned} H_{0,0} &= -\Delta + |x|^2, \\ H_{2j,0} &= \widetilde{W}^{(2j+2)}(x) \quad \text{for } 1 \le j \le r-1, \\ H_{2r,0} &= -i\{\partial_1 a_1^{(2r+1)} + a_1^{(2r+1)}\partial_1 + \partial_2 a_2^{(2r+1)} + a_2^{(2r+1)}\partial_2\} \\ &\quad + \widetilde{W}^{(2r+2)}(x). \end{aligned}$$

In general, we note that for any integer $k \ge 0$,

$$(\partial_{0,0}^{(2k)} W)(0) = 2^{-2k}\frac{1}{k!}(\Delta^k W)(0). \tag{5.2}$$

When $r = 1$, by Proposition 4.2 and 4.3, it follows from (5.2), (4.6) and (4.7) that

$$E_0 = 3, \quad E_2^{\pm} = (\partial_{0,0}^{(4)} W)(0) \pm \frac{1}{4}(\Delta' b)(0) = 2^{-2}\cdot 3\cdot 5a_4 \pm \frac{1}{4}(\Delta' b)(0).$$

Thus, we see that H_P^h has two first non-degenerate eigenvalues $E^{\pm}(h)$ satisfying

$$E^{\pm}(h) = h(3 + E_2^{\pm}h + O(h^2)) \quad \text{as } h \to 0.$$

When $r = 2$, by the similar arguments, we see that $E_0 = 3, E_2 = 2^{-2}\cdot 3\cdot 5a_4$ and

$$\begin{aligned} E_4^{\pm} &= -\sum_{\substack{0<|\gamma|\le 4 \\ \gamma:\text{even}}} (\partial_{0,\gamma}^{(4)} W)(0)(2|\gamma|)^{-1}(\partial_{\gamma,0}^{(4)} W)(0) \\ &\quad + (\partial_{0,0}^{(6)} W)(0) \pm 2^{-5}({\Delta'}^2 b)(0) \\ &= -2^{-4}\cdot 3\cdot 5\cdot 11a_4^2 + 2^{-3}\cdot 3\cdot 5\cdot 7a_6 \pm 2^{-5}({\Delta'}^2 b)(0). \end{aligned}$$

Thus, we see that H_P^h has two first non-degenerate eigenvalues $E^{\pm}(h)$ satisfying

$$E^{\pm}(h) = h(3 + E_2 h + E_4^{\pm}h^2 + O(h^3)) \quad \text{as } h \to 0.$$

Next, we consider the case, where $r = 3$. Then we have

$$\begin{aligned} E_2 &= = 2^{-2} \cdot 3 \cdot 5 a_4, \\ E_4 &= -2^{-4} \cdot 3 \cdot 5 \cdot 11 a_4^2 + 2^{-3} \cdot 3 \cdot 5 \cdot 7 a_6. \end{aligned}$$

In order to calculate $E_6^{\pm}$, we put $E_6^{\pm} = I_{21} - I_{22} - 2I_1 + I_3 \pm I_4$, where

$$\begin{aligned} I_1 &= \sum_{0<|\gamma|\le 4} (2|\gamma|)^{-1} (\partial_{0,\gamma}^{(6)} W)(0) (\partial_{\gamma,0}^{(4)} W)(0)\,, \\ I_{21} &= \sum_{\substack{0<|\gamma|\le 4 \\ 0<|\delta|\le 4}} (\partial_{0,\gamma}^{(4)} W)(0) (2|\gamma|)^{-1} (\partial_{\gamma,\delta}^{(4)} W)(0) (2|\delta|)^{-1} (\partial_{\delta,0}^{(4)} W)(0), \\ I_{22} &= E_2 \sum_{0<|\gamma|\le 4} (2|\gamma|)^{-2} \{(\partial_{0,\gamma}^{(4)} W)(0)\}^2, \\ I_3 &= 2^{-11} \cdot 3^{-1} (\Delta^4 W)(0), \\ I_4 &= 2^{-7} \cdot 3^{-1} ({\Delta'}^3 b)(0). \end{aligned}$$

By using (5.2), (4.6), we can get the values, as follows.

$$\begin{aligned} I_1 &= 2^{-4} \cdot 3^3 \cdot 5 \cdot 7 a_4 a_6, \\ I_2 &= I_{21} - I_{22} = 2^{-7} \cdot 29 \cdot 241 a_4^3, \\ I_3 &= 2^{-9} \cdot 3^3 \cdot 5 \cdot 7 a_8. \end{aligned}$$

Thus, we see that H_P^h has the first two non-degenerate eigenvalues $E^{\pm}(h)$ satisfying

$$E^{\pm}(h) = h(E_0 + hE_2 + h^2 E_4 + h^3 E_6^{\pm} + O(h^4)) \quad \text{as } h \to 0.$$

(2) Next, we consider the Dirac operator (1.1) with a special electric potential $V(x)$ which is of the form $V(x) = -\exp(-|x|^2/2)$. Such operator of this type were considered in [4]. We also assume that the magnetic field has constant direction, i.e., $\nabla \times a(x) = (0, 0, b(x')) = B(x')$ and that (A.2) with $q = 2r \le 6$ holds. According to [19] and [4], we see that the eigenvalue problem for the Dirac operator : $P_V^h(a)u^h = E(h)u^h$, where $E(h)$ is of the form $E(h) \sim \sum_{k=0}^{\infty} h^{k/2} E_k$ is reduced to the one for the Pauli operator : $H_P^h u_+^h = 0$, where

$$\begin{aligned} H_P^h = & \big[D^h(a)^2 + h^2 \big\{ \frac{3}{4} (\nabla V(x) \cdot \nabla V(x)) \big(1 - (V(x) - E(h))\big)^{-2} \\ & + \frac{1}{2} (\Delta V(x)) \big(1 - (V(x) - E(h))\big)^{-1} \big\} \\ & + 1 - (V(x) - E(h))^2 \big] I_2 + ih \sum_{j<k} \sigma_k \sigma_j B_{kj}(x). \end{aligned}$$

Here we denote $u^h = (u^h_+, u^h_-) \in L^2(\mathbf{R}^3; \mathbf{C}^4) = L^2(\mathbf{R}^3; \mathbf{C}^2) \oplus L^2(\mathbf{R}^3; \mathbf{C}^2)$. If we use the unitary transformation U_h for H^h_P, for every large integer m, we have

$$\widetilde{H}^h_P = (U_h)^* H^h_P U_h \sim h \sum_{j=0}^{\infty} h^{j/2}(H_j - 2E_j)\,,$$

in the sense of (4.3), where

$$\begin{aligned} H_0 &= \left[D(B^{(0)})^2 + 2V^{(2)}\right] I_2 - \sigma_3 b^{(0)}, \\ H_l &= \left[D(B^{(0)} \cdot a^{(l+1)} + a^{(l+1)} \cdot D(B^{(0)}) + W_{l+2}\right] I_2 - \sigma_3 b^{(l)}\,, \end{aligned}$$

for $l \geq 1$ and $D_j(B^{(0)}) = i\partial_j - (B(0)x)_j/2$ for $j = 1, 2, 3$ and W_{l+2} is a polynomial of x of order $l + 2$. Here, as in the preceding section, we denote $f^{(l)}(x) = \sum_{|\alpha|=l} \frac{x^\alpha}{\alpha!}(\partial^\alpha f)(0)$ for any smooth function f. More precisely, we can derive

$$\begin{aligned} W_{l+2} &= \sum_{j=1}^{3} \sum_{\substack{l_1+l_2=l+2 \\ l_i \geq 2}} a_j^{(l_1)} a_j^{(l_2)} \\ &\quad + \sum_{\substack{l_1+\cdots+l_{i+2}=l \\ l_i \geq 2}} (i+1) 2^{-i-4} \cdot 4 \nabla V^{(l_1)} \cdot \nabla V^{(l_2)} \\ &\quad \times (V^{(l_3)} - E_{l_3-2}) \cdots (V^{(l_{i+2})} - E_{l_{i+2}-2}) \\ &\quad + \sum_{\substack{l_1+\cdots+l_{i+1}=l \\ l_i \geq 2}} (i+1) 2^{-i-2} \Delta V^{(l_1)} \\ &\quad \times (V^{(l_2)} - E_{l_2-2}) \cdots (V^{(l_{i+1})} - E_{l_{i+1}-2}) \\ &\quad + 2V^{(l+2)} + \sum_{\substack{l_1+l_2=l+2 \\ l_i \geq 2}} (V^{(l_1)} - E_{l_1-2})(V^{(l_2)} - E_{l_2-2}). \end{aligned}$$

By the above formula, we see that

$$\begin{aligned} W_4(x) &= -2^{-1} \cdot 3|x|^2 + 3, \\ W_6(x) &= -2^{-2} \cdot 3^{-1}|x|^6 + 2^{-3} \cdot 3|x|^4 - 2^{-3} \cdot 5|x|^2 + 2^{-4} \cdot 3^2, \\ W_8(x) &= 2^{-5}|x|^8 - 2^{-4}|x|^6 + 2^{-4}|x|^4 - 2^{-5} \cdot 29|x|^2 \\ &\quad + 2^{-7} \cdot 3 \cdot 131. \end{aligned}$$

When $r = 0$, we see that $P^h_V(a)$ has the non-degenerate eigenvalue $E(h)$ satisfying $E(h) = \frac{1}{2}h(1 + \sqrt{4 + b(0)^2} - b(0) + O(h))$ as $h \to 0$.

When $r=1$,

$$2E_2^{\pm} = 2^{-3}\cdot 3 \pm \frac{1}{4}(\Delta' b)(0).$$

Thus, we see that $P_V^h(a)$ has two non-degenerate eigenvalues $E^{\pm}(h)$ satisfying

$$E^{\pm}(h) = \frac{1}{2}h(3 + E_2^{\pm}h + O(h^2)) \quad \text{as } h \to 0.$$

When $r=2$, by the similar arguments, we see that $2E_2 = 2^{-3}\cdot 3$ and

$$2E_4^{\pm} = 2^{-5}\cdot 3^3 \pm 2^{-5}(\Delta'^2 b)(0).$$

Thus, we see that $P_V^h(a)$ has two non-degenerate eigenvalues $E^{\pm}(h)$ satisfying

$$E^{\pm}(h) = \frac{1}{2}h(3 + 2^{-3}\cdot 3h + E_4^{\pm}h^2 + O(h^3)) \quad \text{as } h \to 0.$$

We consider the case, where $r=3$. By Lemma 4.2, it is easily seen (cf. [4]) that $2E_0 = 3, 2E_2 = 2^{-3}\cdot 3, 2E_4 = 2^{-5}\cdot 3^3$. For the value of $E_6^{\pm}$, we have to calculate $E_6^{\pm} = -2I_1 + I_{21} - I_{22} + I_3 \pm I_4$, where

$$\begin{aligned}
I_1 &= (W_4(K_{00}-E_0)^{-1}\overline{\Pi}_{\mathcal{H}_{00}}W_6\phi_0, \phi_0),\\
I_{21} &= (W_4(K_{00}-E_0)^{-1}\overline{\Pi}_{\mathcal{H}_{00}}W_4(K_{00}-E_0)^{-1}\overline{\Pi}_{\mathcal{H}_{00}}W_4\phi_0, \phi_0),\\
I_{22} &= E_2(W_4(K_{00}-E_0)^{-2}\overline{\Pi}_{\mathcal{H}_{00}}W_4\phi_0, \phi_0),\\
I_3 &= (W_8\phi_0, \phi_0),\\
I_4 &= (b^{(6)}\phi_0, \phi_0).
\end{aligned}$$

Using the formula $\Delta^j|x|^{2j} = (2j+1)!$, we easily have $I_3 = 2^{-9}\cdot 3^2\cdot 13^2$ and $I_4 = 2^{-7}\cdot 3^{-1}(\Delta'^3 b)(0)$. Next, we calculate I_1 and $I_2 = I_{21} - I_{22}$. In order to do so, it is convenient to use the following notation. Let $\epsilon_1 = (1,0,0,), \epsilon_2 = (0,1,0), \epsilon_3 = (0,0,1)$. By (5.2), (4.6), we can write

$$\begin{aligned}
|x|^2\phi_0 &= 2^{-5/2}\sum_{j=1}^{3}\phi_{2\epsilon_j},\\
\overline{\Pi}_{\mathcal{H}_{00}}|x|^4\phi_0 &= 2^{-1/2}\cdot 3^{1/2}\sum_{j=1}^{3}\phi_{4\epsilon_j} + \sum_{j<k}\phi_{2\epsilon_j+2\epsilon_k}\\
&\quad +2^{-1/2}\cdot 5\sum_{j=1}^{3}\phi_{2\epsilon_j},
\end{aligned}$$

$$
\begin{aligned}
\overline{\Pi}_{\mathcal{H}_{00}}|x|^6\phi_0 &= 2^{-1}\cdot 3\cdot 5^{1/2}\sum_{j=1}^{3}\phi_{6\epsilon_j} + 2^{-1}\cdot 3^{3/2}\sum_{j\neq k}\phi_{4\epsilon_j+2\epsilon_k} \\
&\quad + 2^{-1/2}\cdot 3\phi_{(2,2,2)} + 2^{-3/2}\cdot 3^{3/2}\cdot 7\sum_{j=1}^{3}\phi_{4\epsilon_j} \\
&\quad + 2^{-1}\cdot 3\cdot 7\sum_{j<k}\phi_{2\epsilon_j+\epsilon_k} + 2^{-5/2}\cdot 3\cdot 5\cdot 7\sum_{j=1}^{3}\phi_{2\epsilon_j}.
\end{aligned}
$$

Since $(K_{0,0}-E_0)^{-1}\phi_\alpha = (2|\alpha|)^{-1}\phi_\alpha$ for $\alpha \neq 0$ and $E_2 = 2^{-3}\cdot 3$, we have

$$
\begin{aligned}
I_1 &= 2^{-9}\cdot 3^2\cdot 5\cdot 23, \\
I_{21} &= -2^{-11}\cdot 3^3, \\
I_{22} &= 2^{-9}\cdot 3, \\
I_3 &= 2^{-9}\cdot 3^2\cdot 13^2, \\
I_4 &= 2^{-7}\cdot 3^{-1}(\Delta'^3 b)(0).
\end{aligned}
$$

Thus, we see that $P_V^h(a)$ has two non-degenerate eigenvalues $E_6^{\pm}(h)$ satisfying

$$
E_6^{\pm}(h) = h\big(E_0 + E_2 h + E_4 h^2 + E_6^{\pm} h^3 + O(h^4)\big) \quad \text{as } h \to 0\,,
$$

where

$$
\begin{aligned}
E_0 &= 2^{-1}\cdot 3, \\
E_2 &= 2^{-3}\cdot 3, \\
E_4 &= 2^{-5}\cdot 3^3, \\
E_6^{\pm} &= 2^{-12}\cdot 3^3\cdot 5\cdot 59 \pm 2^{-7}\cdot 3^{-1}(\Delta'^3 b)(0).
\end{aligned}
$$

Remark 5.1. In the previous paper [4], we obtained the similar results for $r \leq 2$ in the case, where the magnetic potential is of the form $a(x') = (a_1(x'), a_2(x'), 0)$, where $a_1(x') = a_{10}x_2 + a_{11}x_1^2x_2 + a_{12}x_1^4x_2, a_2(x') = a_{20}x_1 + a_{21}x_1x_2^2 + a_{22}x_1x_2^4$. The result for $r = 3$ seems to be new.

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