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STABILITY OF 1-CODIMENSIONAL ANALYTIC DECOMPOSITIONS

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Abstract: We define an analytic decomposition of a complex manifold X of dimension n to be an equivalence relation $\mathbf{D}$ such that all classes (we call them leaves) are connected analytic subsets of pure codimension one and such that the sheaf of vector fields, which are tangent to all classes, is coherent and has rank $n-1$. Such decompositions occur in a natural way as systems of leaves of certain singular holomorphic foliations. We give sufficient conditions, under which $\mathbf{D}$ is stable in the following sense: for every leaf L and for every compact subset $C \subset X \setminus L$ there exists an open saturated neighborhood U of L satisfying $C \cap U = \emptyset$. In particular, if all leaves are compact, then $\mathbf{D}$ is stable iff $X/\mathbf{D}$ is hausdorff iff $X/\mathbf{D}$ is a Riemann surface iff $\mathbf{D} \subset X \times X$ is analytic.

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1. Introduction

There are different possibilities to define foliations $\mathcal{F}$ on a manifold X. Since we are mainly interested in the leaf space $X/\mathcal{F}$ (this is the space of all leaves), the appropriate approach for us seems to be to regard a foliation as a decomposition of X into disjoint subsets (the leaves). Such an approach was started by Stefan (see [26]); for the theory of leaf spaces of Stefan foliations and further developments in the field of differentiable spaces compare for example Piatkowski and Spallek [18]. If $X/\mathcal{F}$ has a natural complex structure then all leaves must be analytic subsets of X and the function $x \mapsto \dim L_x$ (where L_x is the leaf through x) must be upper semicontinous. Since here we are especially interested in singular holomorphic foliations of codimension 1 it is natural to consider decompositions of a complex manifold into pure 1-codimensional analytic subsets and to assume that these decompositions are coherent in the sense of analytic sheaf theory (not in the sense of Piatkowsky and Spallek [18]). We generalize the notion of stability for such decompositions and prove two theorems concerning the stability of such decompositions: in the first theorem we prove that the theorem "$X/\mathcal{F}$ admits a complex structure if and only if $X/\mathcal{F}$ is Hausdorff", (which is true for regular holomorphic foliations $\mathcal{F}$, see Holmann [11], but not for singular holomorphic foliations in general: let $\mathcal{F}$ be the singular foliation defined by $f : \mathbb{C}^4 \longrightarrow \mathbb{C}^3$, $f(z_1, z_2, w_1, w_2) := (z_1, z_2, z_1 w_2 - z_2 w_1)$, then $X/\mathcal{F} \sim f(\mathbb{C}^4)$ is Hausdorff, but does not admit a complex structure, see Reiffen [23, Example 7.6]), remains true for pure 1-codimensional decompositions $\mathbf{D}$ of X in the following form (see Theorem 1): "If the family of reducible leaves of $\mathbf{D}$ is analytically thin, then the three following conditions are equivalent: $X/\mathbf{D}$ admits a complex structure; $\mathbf{D}$ is stable; $X/\mathbf{D}$ is Hausdorff". In this case, $X/\mathbf{D}$ is a Riemann surface. By Sard's theorem applied to $X \to X/\mathbf{D}$ (see Narasimhan [17]) it is not unnatural to assume that the union of reducible leaves is analytically thin. In the second theorem (see Theorem 2) we prove: "If all leaves of the coherent pure 1-codimensional analytic decomposition $\mathbf{D}$ are compact, then $\mathbf{D}$ is stable and $X/\mathbf{D}$ is a Riemann surface". Note, that real two-codimensional foliations with all leaves compact are not necessarily stable if X is not compact (see Epstein [6] and [7], Edwards, Millet and Sullivan [5], Vogt [27]). Theorem 2 shows that in the holomorphic case of (complex) codimension 1, one has stability if all leaves are compact, even if the decomposition has singularities and X is not compact.

In Section 5 we collect some results that we use in the Sections 2, 3 and 4. We hope that in this way the main ideas of the proofs (mainly in Section 4) become more transparent.

2. Stability

Let X be a locally compact topological space (in particular, X is Hausdorff) and R an equivalence relation on X, let $\pi : X \longrightarrow X/R$ be the projection (quotient map).

Definition 1. The equivalence relation R is *stable*, if all equivalence classes are stable: an equivalence class L is stable, if for every compact subset $C \subset X \setminus L$ there exists an open R-saturated neighborhood U of L satisfying $C \cap U = \emptyset$.

The equivalence relation R is called *simple* respectively *compact*, if all classes $R(x)$, $x \in X$, are connected respectively compact. R is called *open* respectively *proper*, if for any open respectively compact subset $C \subset X$, the saturation $R(C) \subset X$ is also open respectively compact.

A stable class is not necessarily closed; but if all classes are stable, then all classes $R(x)$ are closed: for every $y \notin R(x)$ the class $R(y)$ is stable, hence there exists an open, R-invariant neighborhood U of $R(y)$ such that $U \cap \{x\} = \emptyset$, hence $R(x) \cap U = \emptyset$.

Using this remark, one proves

Lemma 2. (a) *If R is open and stable, then X/R is Hausdorff.*

(b) *If X/R is Hausdorff, then R is stable.*

Lemma 3. *Suppose that R is simple. Then a compact class $L \in X/R$ is stable if and only if for every neighborhood V of L in X there exists an open R-saturated neighborhood U of L contained in V.*

Proposition 4. *Suppose that R is simple and compact. Then the following conditions are equivalent:*

(a) *R is proper.*

(b) *X/R is Hausdorff.*

(c) *R is stable.*

Proof. If (c) is satisfied, then by Lemma 3, the condition ii) of Kaup/Kaup [15, 33 B.4] is satisfied, hence R is proper. □

Example 5. Define $X := \{z \in \mathbb{C} : 1 < |z| < 2\}$, R the equivalence relation on X defined by the projection $x+iy \to x$ and $\mathbf{D}$ the simple equivalence relation on X defined by R (i.e. $\mathbf{D}(z)$ is the connected component of $R(z)$ containing z). Then R is stable, but $\mathbf{D}$ is not; X/R is Hausdorff, $X/\mathbf{D}$ is not.

3. Analytic Decompositions and (Singular) Holomorphic Foliations

Let X be a paracompact connected complex manifold of dimension n.

Definition 6. A 1-*codimensional analytic decomposition* of X is an equivalence relation $\mathbf{D}$ on X such that all equivalence classes of $\mathbf{D}$ are connected, pure 1-codimensional analytic subsets of X. For short we shall often speak of "analytic decompositions", or "decompositions". The equivalence classes $\mathbf{D}(x)$ of $\mathbf{D}$ are also called *leaves* of $\mathbf{D}$ or $\mathbf{D}$-leaves.

In Stein [25], analytic decompositions are defined in an other way: the corresponding quotient has to admit an analytic structure.

An analytic decomposition $\mathbf{D}$ defines a new topology on X, the so called *leaf topology* $\mathcal{T}$ of X: a base of the topology $\mathcal{T}$ are all open subsets $C \Subset A$, where A is a$\mathbf{D}$-leaf (the symbol $\Subset$ means "is an open subset of"). With respect to this topology, X has a second canonical complex structure: every connected component of X with respect to $\mathcal{T}$ is a complex space, biholomorphic to a $\mathbf{D}$-leaf A.

Let $\mathbf{D}$ be an analytic decomposition of X. For an open subset $U \Subset X$ the restriction of $\mathbf{D}$ to U is defined as:

$$(\mathbf{D}|_U)(x) := \text{ the connected component of } \mathbf{D}(x) \cap U \text{ containing } x\ .$$

Note that in general the restriction of $\mathbf{D}$ is not the restriction of $\mathbf{D}$ as equivalence relation $\big($this would be $\mathbf{D} \cap (U \times U) \subset U \times U\big)$. This is the reason, why we distinguish between equivalence relations and decompositions. If we consider $\mathbf{D}$ only as equivalence relation we write $R^{\mathbf{D}}$ instead of $\mathbf{D}$.

A holomorphic mapping $f\colon U \longrightarrow Z$ to a Riemann surface Z, defined on $U \Subset X$, is called a *local integral* for $\mathbf{D}$ if the level sets of f (i.e. the connected components of the fibers of f) are just the leaves of $\mathbf{D}|_U$. The decomposition $\mathbf{D}$ is called *locally integrable* if every point $x \in X$ admits a local integral defined on an open neighborhood of x.

Definition 7. An analytic decomposition $\mathbf{D}$ of X is *regular* at a point $a \in X$ if there exists an open neighborhood U of a and a holomorphic submersion $f\colon U \longrightarrow \mathbb{C}$ that is a local integral for $\mathbf{D}$. By $\operatorname{Reg}\mathbf{D}$ we denote the (open) subset of regular points; then $\operatorname{Sing}\mathbf{D} := X \setminus \operatorname{Reg}\mathbf{D}$ is closed in X. By $\mathbf{D}^{\mathrm{reg}}$ we denote the restriction $\mathbf{D}|_{\operatorname{Reg}\mathbf{D}}$.

In the following, we shall use this notation: $\mathrm{T}X$ is the tangent bundle of X, T_xX the tangent space at $x \in X$; by $\mathcal{O}$ and Θ we denote the sheaf of holomorphic functions, resp. of holomorphic vector fields on X. For $x \in U \Subset X$ and $\theta \in \Theta(U)$, by $\theta(x) \in \mathrm{T}_x X$ we denote the tangent vector defined by θ in x. Since for every $V \Subset U$, $\theta \in \Theta(U)$ and every $f \in \mathcal{O}(V)$ the function $(\theta f)(x) := \theta(x)(f_x)$ is holomorphic on V, θ induces a derivation $\mathcal{O}(V) \to \mathcal{O}(V)$ for all $V \Subset U$, hence a derivation $\theta_x : \mathcal{O}_x \longrightarrow \mathcal{O}_x$ for every $x \in U$. Any holomorphic vector field $\theta \in \Theta(U)$, where $U \Subset \mathbb{C}^n$, is of the form $\theta = \sum_\nu c_\nu \partial_\nu$, where $\partial_\nu := \partial/\partial z_\nu \in \Theta(\mathbb{C}^n)$ and $c_\nu \in \mathcal{O}(U)$.

For an $x \in X$ let A_x be the germ of an analytic subset of X at x and

$$\mathfrak{i}(A_x) := \{f \in \mathcal{O}_x : f|_{A_x} = 0\},$$

the corresponding ideal. A tangent vector $\theta \in \mathrm{T}_x X$ is called *tangent to* A_x or a *tangent vector of* A_x if $\theta(f) = 0$ for all $f \in \mathfrak{i}(A_x)$. The vector space $\mathrm{T}(A_x) \subset \mathrm{T}_x X$ of all vectors tangent to A_x is called the *tangent space* of A_x. If A_x is the germ of a submanifold, then $T(A_x)$ is the tangent space in the usual sense. For an analytic subset $A \subset X$ and $x \in A$ we put $\mathrm{T}_x A := \mathrm{T}(A_x)$. A vector field $\theta \in \Theta(U)$ is tangent to A if $\theta(x) \in \mathrm{T}_x(A)$ for all $x \in A \cap U$; obviously θ is tangent to A iff $\theta\big(\mathfrak{i}(A_x)\big) \subset \mathfrak{i}(A_x)$ for all $x \in A \cap U$.

Let $\mathcal{D} \subset \Theta$ be a coherent subsheaf. For $x \in X$, $\mathcal{D}(x) := \{\theta(x) : \theta \in \mathcal{D}(U), x \in U \Subset X\}$ is a linear subspace of the tangent space $\mathrm{T}_x X$. Define $\mathrm{rank}(\mathcal{D}) := \max\{\dim_{\mathbb{C}} \mathcal{D}(x) : x \in X\}$. Then the *singular locus* $\mathrm{Sing}\,\mathcal{D} := \{x \in X : \dim_{\mathbb{C}} \mathcal{D}(x) < \mathrm{rank}(\mathcal{D})\}$ of $\mathcal{D}$ is a proper analytic subset of X. $\mathcal{D}$ is *involutive*, if $\theta, \eta \in \mathcal{D}$ implies $[\theta, \eta] \in \mathcal{D}$. Then (by Frobenius' theorem, see Narasimhan [17]) $\mathcal{D}$ defines a regular holomorphic foliation on $X \setminus \mathrm{Sing}\,\mathcal{D}$. (In order to be able to distinguish between foliations without, resp. with singularities, we speak of "regular", resp. "singular" foliations.)

A representative of a singular holomorphic foliation on X is a pair $(\mathcal{F}_A, A)$, where $A \subset X$ is a proper analytic subset and $\mathcal{F}_A$ is a regular holomorphic foliation on $X \setminus A$. Two pairs $(\mathcal{F}_A, A)$ and $(\mathcal{F}_B, B)$ are equivalent, if $(\mathcal{F}_A)|_{X \setminus (A \cup B)} = (\mathcal{F}_B)|_{X \setminus (A \cup B)}$. A singular holomorphic foliation $\mathcal{F}$ on X is an equivalence class of such pairs. $\mathcal{F}$ admits a representative $(\mathcal{F}_A, A)$, for which A is minimal; this A is called the singular locus $\mathrm{Sing}\,\mathcal{F}$ of $\mathcal{F}$. By definition, $\mathrm{Reg}\,\mathcal{F} := X \setminus \mathrm{Sing}\,\mathcal{F}$. The singular holomorphic foliation $\mathcal{F}$ is called *coherent*, if there exists an involutive coherent subsheaf $\mathcal{D} \subset \Theta$ such that $(\mathcal{F}_A, A)$ is a representative of $\mathcal{F}$, where $A = \mathrm{Sing}\,\mathcal{D} \subset \mathrm{Sing}\,\mathcal{F}$ and $\mathcal{F}_A$ is the regular foliation defined on $X \setminus A$ by $\mathcal{D}$. A singular foliation $\mathcal{F}$ is coherent if and only if $\mathrm{codim}\,\mathrm{Sing}\,\mathcal{F} \geq 2$ (for a proof see Reiffen [23]). Since we are interested only in coherent singular holomorphic foliations, "singular foliation" and "singular holomorphic foliation" always means "coherent singular holomorphic foliation".

$\mathcal{D} \subset \Theta$ is *complete*, if for any open subset $U \Subset X$ the following holds (put $U' := U \setminus \operatorname{Sing} \mathcal{D}$): if $\theta \in \Theta(U)$ and $\theta|_{U'} \in \mathcal{D}(U')$, then $\theta \in \mathcal{D}(U)$.

An analytic decomposition $\mathbf{D}$ defines an analytic subsheaf $\Theta^{\mathbf{D}}$ of Θ as follows: for $U \Subset X$ define

$$\begin{aligned} \Theta^{\mathbf{D}}(U) &:= \{\theta \in \Theta(U) : \theta \text{ is tangent to all leaves of } \mathbf{D}|_U\} \\ &= \Big\{\theta \in \Theta(U) \ : \ \theta(\mathfrak{a}_x) \subset \mathfrak{a}_x \text{ for all } x \in U\Big\}, \\ &\quad \text{where } \mathfrak{a}_x := \mathfrak{i}(\mathbf{D}(x)_x) \subset \mathcal{O}_x . \end{aligned}$$

Obviously, $\Theta^{\mathbf{D}}$ is involutive.

Definition 8. $\mathbf{D}$ is called *coherent* if $\Theta^{\mathbf{D}} \subset \Theta$ is coherent and $\operatorname{rank}(\Theta^{\mathbf{D}}) = n-1$. In this case, by $\mathcal{F}^{\mathbf{D}}$ we denote the (coherent) 1-codimensional singular holomorphic foliation defined by $\Theta^{\mathbf{D}}$.

An important class of coherent decompositions is given by the following proposition:

Proposition 9. *Let $f : X \longrightarrow \mathbb{C}$ be holomorphic not constant, and $\mathbf{D}^f$ the decomposition of X defined by the level sets of f, i.e. $\mathbf{D}^f(x)$ is the connected component of $f^{-1}\big(f(x)\big)$ containing x.*

(a) *The subsheaf $\Theta^f \subset \Theta$ defined by*

$$\Theta^f(W) := \{\theta \in \Theta(W) : \theta(f|_W) = 0\} \quad \text{for} W \Subset X$$

is coherent and $\operatorname{codim} \operatorname{Sing} \Theta^f \geq 2$.

(b) $\Theta^{\mathbf{D}^f} = \Theta^f$*; in particular $\mathbf{D}^f$ is coherent.*

Proof. Θ^f is the kernel of the homomorphism $\Theta \to \mathcal{O}$, $\theta \mapsto \theta(f)$, hence Θ^f is coherent and $\operatorname{Sing} \Theta^f$ is analytic in X. In appropriate local coordinates on $U \Subset X$, we have $df = \sum_\nu f_\nu \, dz_\nu = c \sum_\nu g_\nu \, dz_\nu$, where the germs $g_{1,x}, \dots, g_{n,x}$ are relatively prime in every $x \in U$. Define $\omega := \sum_\nu g_\nu \, dz_\nu$, then the codimension of $\operatorname{Sing}(\omega) := \{x \in U : \omega(x) = 0\}$ is at least two; since $\theta(f) = df(\theta) = 0 \iff \omega(\theta) = 0$, we obtain

$$\Theta^f|_U = \{\theta \in \Theta|_U : \theta(f) = 0\} = \{\theta \in \Theta|_U : \omega(\theta) = 0\},$$

and $\operatorname{codim} \operatorname{Sing} \Theta^f \geq 2$ since $\operatorname{Sing} \Theta^f \subset \operatorname{Sing}(\omega)$. On $U' := U \setminus \operatorname{Sing}(\omega)$, by Frobenius theorem, ω defines a regular holomorphic foliation of codimension one; locally on $V \Subset U'$ we may even suppose that $\omega|_V = dz_1$; then $\Theta^f|_V =$

$\mathcal{O}|_V\partial_2+\cdots+\mathcal{O}|_V\partial_n$, hence $f|_V$ only depends on z_1, i.e. $f(z)=h(z_1)$ for $z\in V$, and $\mathbf{D}^f|_V$ is regular. Hence $\mathbf{D}^f$ is regular on U' and

$$\operatorname{codim}\operatorname{Sing}\mathbf{D}^f\geq 2\ . \tag{+}$$

In order to prove (b), we first suppose $\theta\in\Theta^{\mathbf{D}^f}(U)$. For $a\in U$ define $h:=f-f(a)$. Then $h_a\in\mathfrak{a}_a:=\mathfrak{i}\big(\mathbf{D}^f(a)_a\big)$. Since $\theta(f_a)=\theta(h_a)\in\mathfrak{a}_a$, we obtain $\theta(f_a)(a)=0$; it follows $\theta(f|_U)=0$, hence $\Theta^{\mathbf{D}^f}\subset\Theta^f$. Now suppose that $\theta\in\Theta^f(U)$ is given. We have to show that $\theta(\mathfrak{a}_x)\subset\mathfrak{a}_x$ for all $x\in A$, where $A:=\mathbf{D}^f(a)\cap U$, $a\in U$, and where $\mathfrak{a}\subset\mathcal{O}|_U$ is the sheaf of ideals defining A. For $x\in A\cap\operatorname{Reg}\mathbf{D}^f$, this is evident. Hence, by (+) and Lemma 31 we obtain that $\theta(\mathfrak{a}_x)\subset\mathfrak{a}_x$ for all $x\in A$. □

Corollary 10. *Suppose that* $\mathbf{D}$ *is locally integrable. Then* $\mathbf{D}$ *is coherent.*

We claim:

Proposition 11. *If* $\mathbf{D}$ *is coherent, then*

(a) $\operatorname{Sing}\mathcal{F}^{\mathbf{D}}=\operatorname{Sing}\Theta^{\mathbf{D}}=\operatorname{Sing}\mathbf{D}$ *and* $\operatorname{codim}\operatorname{Sing}\mathbf{D}\geq 2$.

(b) $\Theta^{\mathbf{D}}$ *is complete.*

Proof. The inclusions $\operatorname{Reg}\mathbf{D}\subset\operatorname{Reg}\Theta^{\mathbf{D}}\subset\operatorname{Reg}\mathcal{F}^{\mathbf{D}}$ are obvious. By 12(b) we have $\operatorname{Reg}\Theta^{\mathbf{D}}\subset\operatorname{Reg}\mathbf{D}$, i.e. $\operatorname{Reg}\mathbf{D}=\operatorname{Reg}\Theta^{\mathbf{D}}$, hence we have

$$\operatorname{Sing}\mathcal{F}^{\mathbf{D}}\subset\operatorname{Sing}\Theta^{\mathbf{D}}=\operatorname{Sing}\mathbf{D}.$$

We claim that $\operatorname{Sing}\mathcal{F}^{\mathbf{D}}=\operatorname{Sing}\Theta^{\mathbf{D}}$. For this we have to show that $a\in\operatorname{Reg}\mathcal{F}^{\mathbf{D}}$ implies $a\in\operatorname{Reg}\Theta^{\mathbf{D}}$. Suppose that $a\in\operatorname{Sing}\Theta^{\mathbf{D}}=:S$. By Reiffen [23], S is analytic in X of codimension ≥ 1. We may suppose that $X=P^1\times P^{n-1}$ as in Lemma 12 and that $\mathcal{F}^{\mathbf{D}}$ is given by the projection $p(z_1,\ldots,z_n)=z_1$. Define $L:=\{a_1\}\times P^{n-1}\subset X$. Then (if necessary shrink P^1) we may suppose that there exists $\tilde a\in L$ such that (with $\Lambda:=P^1\times\{(\tilde a_2,\ldots,\tilde a_n)\}$) we have $\Lambda\setminus\{a\}\subset\operatorname{Reg}\Theta^{\mathbf{D}}=\operatorname{Reg}\mathbf{D}$. Then $\mathbf{D}(x)\supset\{x_1\}\times P^{n-1}$ for all $x\in\Lambda\setminus\{a\}$, hence for an arbitrary $x\in X\setminus L$ and $\tilde x:=(x_1,\tilde a_2,\ldots,\tilde a_n)\in\Lambda$ we obtain $x\in\{x_1\}\times P^{n-1}\subset\mathbf{D}(\tilde x)$, hence $\mathbf{D}(x)=\mathbf{D}(\tilde x)$ and

$$\mathbf{D}(x)\supset\{x_1\}\times P^{n-1},\qquad\forall\,x\in X\setminus L.$$

By Lemma 12(a) we obtain that $\mathbf{D}$ is regular on X, a contradiction, and it follows $a\in\operatorname{Reg}\Theta^{\mathbf{D}}$. Now $\operatorname{codim}\operatorname{Sing}\mathbf{D}\geq 2$ since $\operatorname{codim}\operatorname{Sing}\mathcal{F}^{\mathbf{D}}\geq 2$.

In order to prove that $\Theta^{\mathbf{D}}$ is complete, we have to show: if $U\Subset X$, $\theta\in\Theta(U)$ and $\big(\text{with } U':=U\cap\operatorname{Reg}\Theta^{\mathbf{D}}\big)$ $\theta|_{U'}\in\Theta^{\mathbf{D}}(U')$, then $\theta\in\Theta^{\mathbf{D}}(U)$. Let A be a $\mathbf{D}$-leaf. Then by (a), $A\cap U'$ is dense in $A\cap U$ since $\dim A=n-1$; since $\theta(\mathfrak{a}_a)\subset\mathfrak{a}_a$

for all $a \in A \cap U'$, we have $\theta(\mathfrak{a}_a) \subset \mathfrak{a}_a$ for all $a \in A \cap U$ by Lemma 31, hence $\theta \in \Theta^{\mathbf{D}}(U)$. □

Lemma 12. *Let $\mathbf{D}$ be a coherent decomposition of the polycylinder $P = P^1 \times P^{n-1} \Subset \mathbb{C}^n$, $P^k \subset \mathbb{C}^k$, and $\mathbf{B}$ the decomposition $\mathbf{B}(x) := \{x_1\} \times P^{n-1}$ of P. Then $\mathbf{D} = \mathbf{B}$, if one of the following conditions is satisfied:*

(a) $\mathbf{D}(x) \supset \mathbf{B}(x)$ *for all* $x \in P \setminus \mathbf{B}(a) =: \tilde{P}$ *for a certain* $a \in P$.

(b) $\Theta^{\mathbf{D}} = \Theta^{\mathbf{B}} \big(= \mathcal{O}\partial_2 + \cdots + \mathcal{O}\partial_n \big)$.

Proof. (a) If $x \in \tilde{P}$, then $\mathbf{B}(x)$ is an irreducible component of $\mathbf{D}(x)$. It is sufficient to prove that $\mathbf{D}(x) = \mathbf{B}(x)$ for all $x \in \tilde{P}$ (since then necessarily also $\{x_1\} \times P^{n-1}$ is a $\mathbf{D}$-leaf). Suppose, that there is $x \in \tilde{P}$ such that $\mathbf{D}(x) \neq \mathbf{B}(x)$; we may suppose, that $\mathbf{D}(x)$ admits an irreducible component $S \neq \mathbf{B}(x)$ such that $x \in S$. Take a 1-dimensional connected local analytic subset $T \subset S$ such that $T \cap \mathbf{B}(x) = \{x\}$. Then $p(T)$ is a neighborhood of x_1 in P^1. Since $T \subset S \subset \mathbf{D}(x)$, we have $\hat{T} := p^{-1}\big(p(T)\big) \subset \mathbf{D}(x)$. This is impossible since $\dim_x \hat{T} = n$ and $\dim_x \mathbf{D}(x) = n - 1$.

(b) Suppose that $A = \mathbf{D}(x)$. Then the ideal $\mathfrak{a}_x = \{f \in \mathcal{O}_x : f|_{A_x} = 0\}$ is of the form $\mathfrak{a}_x = \mathcal{O}_x \cdot f$ for a certain $f \in \mathcal{O}_x$ since $\dim_a A = n - 1$. If $\Theta^{\mathbf{D}} = \Theta^{\mathbf{B}}$, we have $f(x) = 0$ and $\partial_\nu f \in \mathfrak{a}_x$ for $2 \leq \nu \leq n$. By Lemma 13 we obtain that $f(x_1, z_2, \ldots, z_n) = 0$ if $|z_\nu - x_\nu| < \epsilon$ for $2 \leq \nu \leq n$. Hence $A \supset \mathbf{B}(x)$ and (b) is a consequence of (a) since $x \in P$ was chosen arbitrarily. □

Lemma 13. *Suppose $a \in \mathbb{C}^n$, $f, \alpha \in \mathcal{O}_a$ such that $f(a) = 0$ and $\partial_\nu f = \alpha f \in \mathcal{O}_a$ for one ν. Then $f(a_1, \ldots, a_{\nu-1}, a_\nu + t, a_{\nu+1}, \ldots, a_n) = 0$ for small $t \in \mathbb{C}$.*

Proof. We may suppose that $a = 0$ and $\nu = 1$; put $t := z_1$ and $\zeta := (z_2, \ldots, z_n)$. Then f admits the decomposition $f(t, \zeta) = f_0(t) + \tilde{f}(t, \zeta)$, where f_0 contains all terms, which do not depend on ζ and $\tilde{f}$ all terms, which depend on ζ. In the same way we write $\alpha(t, \zeta) = \alpha_0(t) + \tilde{\alpha}(t, \zeta)$ and we obtain

$$\begin{aligned} \partial_t f(t,\zeta) &= f_0'(t) + \partial_t \tilde{f}(t,\zeta) \;=\; \alpha(t,\zeta) f(t,\zeta) \\ &= \alpha_0(t) f_0(t) + \alpha_0(t) \tilde{f}(t,\zeta) + \tilde{\alpha}(t,\zeta) f(t,\zeta) \end{aligned}$$

Hence $f_0'(t) = \alpha_0(t) f_0(t)$; since $f_0(0) = 0$, we obtain $f_0 \equiv 0$, hence $f(t, 0) = \tilde{f}(t, 0) \equiv 0$. □

If R is an analytic equivalence relation on X, then all connected components of all classes $R(x)$ are analytic in X. If in addition $\dim_x R(x) = n - 1$ for all $x \in X$, then this leads to a coherent decomposition:

Proposition 14. *Let R be an analytic equivalence relation on X such that all classes $R(x)$ have pure dimension $n-1$. Then the corresponding decomposition* $\mathbf{D}$ *(the leaves of* $\mathbf{D}$ *are the connected components of the classes of R) is coherent.*

Proof. By Proposition 28, $\mathbf{D}$ is locally integrable. Hence $\mathbf{D}$ is coherent by Corollary 10. □

If $\mathbf{D}$ is a coherent 1-codimensional analytic decomposition, then in general $\mathbf{D} \subset X \times X$ is not analytic (e.g. the decomposition $\mathbf{D}$ on $X := \mathbb{C}^2 \setminus \{0\}$ defined by by $f(z) := z_1 z_2$). However, since $\mathbf{D}$ is open (see Corollary 21) then by Proposition 16, $\mathbf{D}$ is analytic in $X \times X$ if and only if the quotient $X/\mathbf{D}$ is a complex space; if $\mathbf{D}$ is regular, then by Proposition 15 these conditions are satisfied if and only if the quotient $X/\mathbf{D}$ is Hausdorff.

We used the following results (see Holmann [11] for Proposition 15 and Bohnhorst Reiffen [4] for Proposition 16):

Proposition 15. *Let $\mathcal{F}$ be a regular holomorphic foliation on the complex manifold X. Then the leaf space $X/\mathcal{F}$ is a complex space if and only if $X/\mathcal{F}$ is Hausdorff.*

Proposition 16. *Let R be an open analytic equivalence relation on the manifold X. Then the quotient X/R is a complex space.*

If $\mathcal{D} \subset \Theta$ is a coherent involutive subsheaf of rank d and $\mathcal{F}$ the corresponding coherent singular foliation, then the notion of an $\mathcal{F}$-leaf is well defined (see Reiffen [20] and [23]): first for $U \Subset X$ define the subalgebra

$$\mathcal{O}^{\mathcal{D}}(U) := \left\{ f \in \mathcal{O}(U) : \theta(f|_V) = 0 \ \forall \theta \in \mathcal{D}(V), V \Subset U \right\} \subset \mathcal{O}(U).$$

A local analytic subset A of X is a *local $\mathcal{F}$-leaf*, if the following conditions are satisfied:

(i) $f|_{U \cap A}$ is locally constant for all $f \in \mathcal{O}^{\mathcal{D}}(U)$, $U \Subset X$.

(ii) A is connected and $\dim_a A \geq d$ for all $a \in A$

(iii) If $B \subset X$ is locally analytic and has (with A replaced by B) properties (i) and (ii), and if $x \in A \cap B$, then $B_x \subset A_x$.

Define $X^\rho \supset \operatorname{Reg} \mathcal{F}$ to be the union of all local $\mathcal{F}$-leaves. Then there is a unique topology on X^ρ such that the set of all local $\mathcal{F}$-leaves is a basis of this topology. Now, an $\mathcal{F}$-leaf is just a connected component of X^ρ with respect to this topology. X^ρ is a complex space in a natural way. We say that $\mathcal{F}$ has leaves everywhere if $X^\rho = X$.

Now let $\mathcal{F}^{\mathbf{D}}$ be the holomorphic foliation defined by the coherent decomposition $\mathbf{D}$. Then the notion of "leaf" is the same for the decomposition $\mathbf{D}$ and the corresponding foliation $\mathcal{F}^{\mathbf{D}}$:

Proposition 17. *Let $\mathbf{D}$ be a coherent decomposition of X. Then all $\mathbf{D}$-leaves are $\mathcal{F}^{\mathbf{D}}$-leaves. In particular, $\mathcal{F}^{\mathbf{D}}$ has leaves everywhere.*

The proof is an easy consequence of:

Lemma 18. *Let $\mathbf{D}$ be a coherent decomposition of X, $U \Subset X$ and $f \in \mathcal{O}(U)$. Then the following conditions are equivalent:*

(a) $\big(f - f(x)\big)_x \in \mathfrak{a}_x$ *for all* $x \in U$, *where* $\mathfrak{a}_x := \mathfrak{i}\big(\mathbf{D}(x)_x\big)$

(b) $f|_{A\cap U}$ *is locally constant for all* $\mathbf{D}$*-leaves* A.

(c) $\theta(f|_V) = 0$ *for all* $\theta \in \Theta^{\mathbf{D}}(V), V \Subset U$.

Proof. Since all $\mathbf{D}$-leaves have pure dimension $n-1$ and codim Sing $\mathbf{D} \geq 2$, these conditions are satisfied for f iff they are satisfied for $f|_{U'}$, where $U' := U \cap \operatorname{Reg}\mathbf{D}$. For U' the equivalence is evident: we may suppose that $\mathbf{D}$ is given by the projection $(z_1, \ldots, z_n) \mapsto z_1$. □

In Corollary 10 we stated that locally integrabel decompositions are coherent. Now (using a theorem of Mattei-Moussu) we can prove the converse, i.e. that coherent decompositions are locally integrable:

Proposition 19. *Every coherent analytic decomposition is locally integrable.*

Proof. We may suppose that $a \in \operatorname{Sing}\mathbf{D} = \operatorname{Sing}\mathcal{F}$, where $\mathcal{F} = \mathcal{F}^{\mathbf{D}}$ is the holomorphic foliation corresponding to $\mathbf{D}$. Let A be the $\mathbf{D}$-leaf through a and U an open neighborhood of a such that $A \cap U$ has only a finite number of irreducible components $C_1, \ldots, C_N$; we may suppose that $a \in C_\nu$ for all ν. Put $U' := U \cap \operatorname{Reg}\mathcal{F}$. Since codim Sing $\mathcal{F} \geq 2$, $C'_\nu := C_\nu \cap U'$ is connected, hence a leaf of $\mathcal{F}|_{U'}$. The C'_ν are the only $\mathcal{F}|_{U'}$-leaves such that $a \in \overline{C'_\nu}$. Hence the theorem of Mattei-Moussu (see Mattei and Moussu [16] and Reiffen [21]) guarantees the existence of a local $\mathcal{F}$-integral defined on an open neighborhood $\tilde{U}$ of a; we may suppose that $\tilde{U} = U$. By Reiffen [23, 6.13], the level sets of f are the leaves of $\mathcal{F}|_U$, by proposition 17 also the leaves of $\mathbf{D}|_U$, hence f is a local integral of $\mathbf{D}$. □

4. The Main Results

We shall say that a 1-codimensional coherent analytic decomposition $\mathbf{D}$ of X is *generically irreducible*, if

$$\operatorname{Rdh}(\mathbf{D}) := \bigcup_{\substack{A \text{ is a reducible} \\ \mathbf{D}-\text{leaf}}} A$$

is analytically thin in X (i.e. locally contained in a proper analytic subset).

With this notation we can formulate our main results:

Theorem 1. *Let $\mathbf{D}$ be a 1-codimensional coherent and generically irreducible analytic decomposition of X. Then* $\operatorname{Rdh}(\mathbf{D})$ *is the (locally finite) union of all reducible leaves of $\mathbf{D}$ and hence analytic in X; moreover, the following conditions are equivalent:*

(a) $\mathbf{D}$ *is stable.*

(b) $X/\mathbf{D}$ *is Hausdorff.*

(c) $X/\mathbf{D}$ *is a Riemann surface.*

(d) $\mathbf{D}$ *is an analytic equivalence relation.*

Theorem 2. *Let $\mathbf{D}$ be a 1-codimensional compact coherent analytic decomposition of X. Then the statements* (a) - (d) *of Theorem 1 hold.*

In Reiffen [21] and [22], the foliation-version of Theorem 1 was proven under the stronger assumption of generic regularity. The proof used methods of hyperbolic geometry, which work only in codimension one. The foliation-version of Theorem 2 in its substance is also published in Reiffen [21] and [22]; in this paper methods are developed that can be applied also in codimension bigger then one.

Proof of Theorem 1. Let us remember our previous remark, that several details of the following proof are given in Section 5. By proposition 20, $\mathbf{D}$ is open. Hence, by Lemma 2, conditions (a) and (b) are equivalent. Since $\mathbf{D}$ is generically irreducible, we obtain (see Proposition 25) that $\operatorname{Rdh}(\mathbf{D})$ is an analytic subset of X, the connected components of $\operatorname{Rdh}(\mathbf{D})$ being exactly the reducible leaves of $\mathbf{D}$.

Hence $X' := X \setminus \operatorname{Rdh}(\mathbf{D})$ is open in X and $\mathbf{D}$-saturated. We define $X'' := X' \cap \operatorname{Reg}(\mathbf{D})$, $\mathbf{D}' := \mathbf{D}|_{X'}$ and $\mathbf{D}'' := \mathbf{D}|_{X''}$.

Since all leaves have dimension $n-1$ and since $\operatorname{codim} \operatorname{Sing} \mathbf{D} \geq 2$, all leaves intersect $\operatorname{Reg}(\mathbf{D})$. Hence there is a canonical surjection $\alpha' : X''/\mathbf{D}'' \longrightarrow X'/\mathbf{D}'$ such that the following diagram commutes:

$$\begin{array}{ccc} X'' & \Subset & X' \\ \big\downarrow{\scriptstyle \pi''} & & \big\downarrow{\scriptstyle \pi'} \\ X''/\mathbf{D}'' & \xrightarrow{\ \alpha'\ } & X'/\mathbf{D}' \end{array}$$

The mapping α' is continuous and open (since $\mathbf{D}$ is open), α' is injective since all $\mathbf{D}'$-leaves are irreducible, hence α' is a homeomorphism; α' is an isomorphism of ringed spaces since all leaves of $\mathbf{D}'$ are irreducible (see Lemma 26). By hypothesis $X/\mathbf{D}$ is Hausdorff, hence $X''/\mathbf{D}'' \sim X'/\mathbf{D}' \Subset X/\mathbf{D}$ is Hausdorff, by Proposition 15 the quotient $X''/\mathbf{D}''$ and hence $X'/\mathbf{D}'$ are normal complex spaces, in our case Riemann surfaces. In order to prove that $X/\mathbf{D}$ is a Riemann surface, by Proposition 16 it is sufficient to prove that the equivalence relation $\mathbf{D} \subset X \times X$ is an analytic subset of $X \times X$, since $\mathbf{D}$ is open by Proposition 20. Since X' is $\mathbf{D}$-saturated, we have $\mathbf{D}' = \mathbf{D} \cap (X' \times X')$. Now $\mathbf{D} \subset \overline{\mathbf{D}'}$ since $\mathbf{D}$ is open, and $\mathbf{D}$ is closed in $X \times X$ since $X/\mathbf{D}$ is Hausdorff; hence $\mathbf{D} = \overline{\mathbf{D}'}$. The dimension of $\mathbf{D}'$ is $n + (n-1) = 2n-1$, since all $\mathbf{D}'$-classes have pure dimension $n-1$. Furthermore $\mathbf{D}'$ is analytic outside of the analytic subset $\operatorname{Rdh}(\mathbf{D}) \times \operatorname{Rdh}(\mathbf{D}) \subset X \times X$ of dimension $(n-1)+(n-1) < 2n-1$; by the singularity theorem of Remmert-Stein-Thullen, the closure $\mathbf{D} = \overline{\mathbf{D}'}$ is analytic. □

Proof of Theorem 2. We show that all leaves L are stable. First we show

1. *If $L \cap \operatorname{Sing} \mathbf{D} = \emptyset$, then L is stable.*

Since L is a compact leaf of the regular decomposition $\mathbf{D}^{\text{reg}} := \mathbf{D}|_{X^{\text{reg}}}$, L is stable by Proposition 27.

Hence we may suppose that $L \cap \operatorname{Sing} \mathbf{D} \neq \emptyset$ and that U is a neighborhood of L. We have to find an open saturated neighborhood V of L contained in U. For this we may assume that $\overline{U}$ is compact; by Corollary 21 we may assume that

$$\overline{U} \cap \operatorname{Sing} \mathbf{D} \subset L. \tag{$*$}$$

In a second step we prove:

2. *Suppose that $W \cap \mathbf{D}(\operatorname{Sing} \mathbf{D}) = L$ for a certain open neighborhood W of L. Then L is stable.*

First observe that $\mathbf{D}(W) \cap \mathbf{D}(\operatorname{Sing} \mathbf{D}) = L$. The set $\widehat{W} := \mathbf{D}(W) \setminus L$ is open in X (since $\mathbf{D}$ is open, see Proposition 20) and saturated, and the restriction

$\mathbf{D}|_{\hat{W}}$ is regular and compact; by 1 $\mathbf{D}|_{\widehat{W}}$ is stable, so $\widehat{W}/(\mathbf{D}|_{\widehat{W}})$ is Hausdorff. We may assume that $\overline{U} \subset W$ for our neighborhood U according to $(*)$. Then $\mathbf{D}(\partial U)$ is compact by Proposition 4, hence $V := \mathbf{D}(U) \setminus \mathbf{D}(\partial U)$ is an open $\mathbf{D}$-saturated neighborhood of L contained in U.

Note that the hypothesis of 2 is satisfied if e.g. $\operatorname{Sing}\mathbf{D}$ has only a finite number of connected components (since any such component is contained in a leaf by 21); this is the case, if X is compact. Hence, in those cases, the proof of Theorem 2 is complete.

Now we come to the general case, in which there is no W as in 2. Let U be chosen according to $(*)$.

Suppose that L is not stable. Then $R(V) \not\subset U$ for every open neighborhood V of L since R is an open equivalence relation. Hence, there exist points $b_\nu \in U \setminus L$ converging to a point $b \in L$ with the following property: let $L_\nu \subset U$ be the leaf of $\mathbf{D}|_U$ through b_ν, then (the closure is taken with respect to $\overline{U}$) $\overline{L_\nu} \cap \partial U \neq \emptyset$ for all ν. We may suppose that the compact subsets $\overline{L_\nu}$ of $\overline{U}$ converge (with respect to the Hausdorff metric on the complete metric space $\mathcal{K}(\overline{U})$ of non void compact subsets of $\overline{U}$; note that we may assume that the topology of X is induced by a metric on X) to a certain $\Lambda \in \mathcal{K}(\overline{U})$ (see Barnsley [2]; this Λ is independent of the metric defining the topology of X). Certainly Λ is connected (since all $\overline{L_\nu}$ are), $L \subset \Lambda$ (since the equivalence relation defined by $\mathbf{D}|_U$ on U is open) and $\Lambda \cap \partial U \neq \emptyset$. Furthermore, $\Lambda \cap U$ is $\mathbf{D}|_U$-saturated since $\mathbf{D}|_U$ is open.

3. $\Lambda \subset \mathbf{D}(\operatorname{Sing}\mathbf{D})$.

To prove this suppose that there exists $x \in \Lambda$ such that the $\mathbf{D}$-leaf M through x does not meet $\operatorname{Sing}\mathbf{D}$. Then, by 1, M is stable, hence there are disjoint open neighborhoods A of M and B of L, we may suppose that A is $\mathbf{D}$-saturated. Hence, $L_\nu \subset A$ for $\nu \geq \nu_0$ — a contradiction.

Since the analytic set $\operatorname{Sing}\mathbf{D}$ has at most countably many connected components and each such component is contained in a leaf, by 3 the set Λ is of the form

$$\Lambda = L \cup \bigcup_{\nu\in\mathbb{N}} \overline{M_\nu}\,,$$

where M_ν is a leaf of $\mathbf{D}|_U$, contained in a $\mathbf{D}$-leaf that meets $\operatorname{Sing}\mathbf{D}$, such that $M_\nu \neq L$ for all ν and $M_\nu \neq M_\mu$ if $\nu \neq \mu$.

We prove:

4. *It is possible to choose U in such a way that (with our previous notations) $\overline{M_\nu} \cap \overline{M_\mu} = \emptyset$ for all $\nu \neq \mu$.*

For a proof fix $b \in L$. There exists a proper $\mathcal{C}^\infty$ -mapping $\rho: U \longrightarrow \mathbb{R}_{\geq 0}$ such that $\rho(b) = 0$ (embed U as a closed differentiable submanifold of $\mathbb{R}^N$ and restrict $\big(\mathrm{dist}(x,0)\big)^2$ to U). By Sard's theorem (see Narasimhan [17]), there exists $r > 0$ such that $L \subset \{x \in U : \rho(x) < r\}$ and such that r is a regular value of $\rho|_{M_\nu}$ for all ν.

Define $\tilde{U} := \{x \in U : \rho(x) < r\} \subset U$. We may assume that there are points $b_\nu \in M_\nu \cap \tilde{U}$ such that $b_\nu \to b$. Let N_ν be the connected component of $M_\nu \cap \tilde{U}$ containing b_ν. We may suppose that the sequence $\big(\overline{N_\nu}\big)_{\nu \in \mathbb{N}}$ converges to $\tilde{\Lambda} \in \mathcal{K}(\overline{U})$. Then $\overline{N_\nu} \cap \overline{N_\mu} = \emptyset$ if $\nu \neq \mu$. In order to prove this put $N := N_\nu$ for an arbitrary ν and take an $a \in \overline{N} \setminus N$. Since $\mathbf{D}$ is regular in a by $(*)$, we may suppose that $P = \{z \in \mathbb{C}^n : |z_k| < \epsilon_k \ \forall k\}$ is an open neighborhood of $a = 0$, that $\mathbf{D}|_P = \{P_t : t \in \mathbb{C}, |t| < \epsilon_1\}$, where $P_t := \{z \in P : z_1 = t\}$, and that $N \cap P = P_0$. Since $\rho(0) = r$ and r is a regular value of ρ, we can suppose that ρ is one component of a real $\mathcal{C}^\infty$-coordinate system on an open neighborhood of 0. Considering $\mathbf{D}$ near 0 as a $\mathcal{C}^\infty$-foliation, we get $\overline{N} \cap \overline{N_\mu} = \emptyset$ for all $\mu \neq \nu$. This proves 4: just take $\tilde{U}$ and $\tilde{\Lambda}$ instead of U and Λ.

Now take any point $a \in (\Lambda \setminus L) \cap U$ and an open neighborhood D of a in U, which is biholomorphic to $D' \times D'' \Subset \mathbb{C} \times \mathbb{C}^{n-1}$ such that the projection $z = (z', z'') \mapsto z'$ is a local chart for $\mathbf{D}$ (note that, by $(*)$, $\mathbf{D}$ is regular in all points of $U \setminus L$). Then $S := \Lambda \cap D' \subset D'$ is countable and has no isolated points: if b in S is isolated, say $b \in M_\nu$, then using 4 and using that $\mathbf{D}$ is open we conclude that there is an open neighborhood A of $\overline{M_\nu}$ in $\overline{U}$ such that $A \cap \Lambda = \overline{M_\nu}$. But then $\overline{M_\nu}$ is open and closed in Λ, which is impossible since Λ is connected. Now any nonempty locally closed subset $S \subset \mathbb{C}$ without isolated points is not countable (see Alexandroff and Hopf [1, page 121]), hence our assumption, that L is not stable, is false. □

In Reiffen [21], by more sophisticated methods, a more general result is proven for certain foliations.

5. Some Propositions

In this section we always suppose that $\mathbf{D}$ is a 1-codimensional coherent analytic decomposition of the paracompact, connected n-dimensional manifold X.

Proposition 20. $\mathbf{D}$ *is locally integrable and open, more precisely: For every $a \in X$ there exist an open connected neighborhood U of a in X and a holomorphic function $f: U \longrightarrow \mathbb{C}$ with the following properties:*

(a) $f(a) = 0$, *the fiber $f^{-1}(0)$ has only finitely many irreducible components* *and they all contain a.*

(b) $U \cap \operatorname{Sing} \mathbf{D} \subset f^{-1}(0)$.

(c) *f is simple, i.e. all fibers of f are connected.*

(d) *The fibers of f are the leaves of $\mathbf{D}|_U$.*

Proof. $\mathbf{D}$ is locally integrable by Proposition 19. Let $f : U \longrightarrow \mathbb{C}$ be such a local integral. Then (b) follows from the fact that f is locally constant on the singular set of f; (c) can be satisfied by Reiffen [19, Addendum]; then the level sets of f are just the fibers of f and (d) follows. By Holmann, Kaup and Reiffen [10, proposition 2.20] we get that $\mathbf{D}$ is open. □

As a consequence of Proposition 20 we obtain:

Corollary 21. *For $a \in \operatorname{Sing}(\mathbf{D})$ let Z be the connected component of $\operatorname{Sing} \mathbf{D}$ containing a. Then $Z \subset \mathbf{D}(a)$.*

In the following the operator $\widehat{\ }$ plays an important role: for any subset $\mathcal{M} \subset \operatorname{Reg}(\mathbf{D})/\mathbf{D}^{\mathrm{reg}}$ we define

$$\widehat{\mathcal{M}} := \bigcup_{L \in \mathcal{M}} \overline{L} \subset X$$

(here $\overline{L}$ is the closure of $L \subset \operatorname{Reg}(\mathbf{D})$ with respect to X).

Lemma 22. *Given $a \in U \Subset X$ and $f : U \longrightarrow \mathbb{C}$ with the properties of Proposition 20, then the following holds:*

(a) *$L \cap f^{-1}(0) \neq \emptyset$ for every $\mathbf{D}^{\mathrm{reg}}$-leaf L with $a \in \overline{L}$.*

(b) *The set $\mathcal{L}_a := \{L \in \operatorname{Reg}(\mathbf{D})/\mathbf{D}^{\mathrm{reg}};\ \overline{L} \ni a\}$ is finite and $f^{-1}(0) \subset \widehat{\mathcal{L}_a}$; especially $\widehat{\mathcal{L}_a}$ is an open $\mathcal{T}$-neighborhood of a.*

The mapping $\operatorname{Reg}(\mathbf{D}) \ni x \mapsto \mathbf{D}(x) \in X/\mathbf{D}$ is constant on all $\mathbf{D}^{\mathrm{reg}}$-leaves. This induces a map α, such that the following diagram commutes:

$$\begin{array}{ccc} \operatorname{Reg}(\mathbf{D}) & \Subset & X \\ \big\downarrow \pi & & \big\downarrow \pi \\ \operatorname{Reg}(\mathbf{D})/\mathbf{D}^{\mathrm{reg}} & \xrightarrow{\ \alpha\ } & X/\mathbf{D} \end{array}$$

Proposition 23.

(a) *Let L be a $\mathbf{D}^{\text{reg}}$-leaf. Then L is a purely $(n-1)$-dimensional closed submanifold of* $\text{Reg}(\mathbf{D})$ *and $\overline{L}$ is a purely $(n-1)$-dimensional irreducible analytic subset of X.*

(b) *Let L be a $\mathbf{D}^{\text{reg}}$-leaf. Then $\overline{L} \subset \alpha(L)$.*

(c) *For $L, M \in \text{Reg}(\mathbf{D})/\mathbf{D}^{\text{reg}}$ we define*

$$L \sim M \iff \begin{cases} \exists L_1, \dots, L_N \in \text{Reg}(\mathbf{D})/\mathbf{D}^{\text{reg}}, \\ \text{with } L_1 = L, L_N = M, \overline{L_k} \cap \overline{L_{k+1}} \neq \emptyset \text{ for } 1 \leq k < N. \end{cases}$$

This defines an equivalence relation $\sim$ on $\text{Reg}(\mathbf{D})/\mathbf{D}^{\text{reg}}$.

(d) $L \sim M \longrightarrow \alpha(L) = \alpha(M)$

(e) *If $\mathcal{A} \subset \text{Reg}(\mathbf{D})/\mathbf{D}^{\text{reg}}$ is an equivalence class with respect to $\sim$, then $\widehat{\mathcal{A}}$ is an $\mathbf{D}$-leaf.*

(f) *For two $\mathbf{D}^{\text{reg}}$-leaves L, M the following holds: $\alpha(L) = \alpha(M) \iff L \sim M$.*

(g) *For every $\mathbf{D}$-leaf L we have the following decomposition*

$$L = \bigcup_{M \in \alpha^{-1}(L)} \overline{M} = \widehat{\alpha^{-1}(L)}$$

into irreducible components, from which follows

$$L \cap \text{Reg}(\mathbf{D}) = \bigcup_{M \in \alpha^{-1}(L)} M.$$

Proof. (a) follows from the singularity theorem of Remmert-Thullen-Stein (compare Siu [24]), since $\dim \text{Sing}\,\mathcal{F} < \dim L$.

(d) It suffices to prove that $\alpha(L) = \alpha(M)$, if $\overline{L} \cap \overline{M} \neq \emptyset$. This we get, as follows:

$$\overline{L} \cap \overline{M} \underset{\substack{\uparrow \\ \text{(b)}}}{\subset} \alpha(L) \cap \alpha(M) \neq \emptyset, \text{ consequently } \alpha(L) = \alpha(M).$$

(e) From (d) follows $\alpha(L) = \alpha(M)$ for $L, M \in \mathcal{A}$; consequently $\widehat{\mathcal{A}}$ is completely contained in a $\mathbf{D}$-leaf because of (b). It suffices to show that $\widehat{\mathcal{A}}$ is open with respect to the topology $\mathcal{T}$, because this implies immediately that $\widehat{\mathcal{A}}$ is also $\mathcal{T}$-closed: for every $b \in X \setminus \widehat{\mathcal{A}}$ there exists an equivalence class $\mathcal{B}$ with $b \in \widehat{\mathcal{B}}$ and $\widehat{\mathcal{B}} \cap \widehat{\mathcal{A}} = \emptyset$; then $\widehat{\mathcal{B}}$ is a $\mathcal{T}$-open neighborhood of b in the complement of $\widehat{\mathcal{A}}$. In order to prove that $\widehat{\mathcal{A}}$ is $\mathcal{T}$-open we choose for $a \in \widehat{\mathcal{A}}$ an integral $f: U \longrightarrow \mathbb{C}$ with $a \in U$ as in Lemma 22. We have $\mathcal{L}_a \subset \mathcal{A}$ because of Lemma 22(b), consequently

$$f^{-1}(0) \subset \widehat{\mathcal{L}_a} \subset \widehat{\mathcal{A}}.$$

Since $f^{-1}(0)$ is a $\mathcal{T}$-neighborhood of a, the same holds for $\widehat{\mathcal{A}}$.

(f) $\Longleftarrow$ follows from (d) and $\Longrightarrow$ follows from (e): namely, if $L \not\sim M$, then L and M lie in disjoint equivalence classes $\mathcal{A}$ respectively $\mathcal{B}$ and $\alpha(L) = \widehat{\mathcal{A}} \neq \widehat{\mathcal{B}} = \alpha(M)$.

(g) $\mathcal{A} := \alpha^{-1}(L)$ is an $\sim$-equivalence class in $\mathrm{Reg}(\mathbf{D})/\mathbf{D}^{\mathrm{reg}}$ and $\bigcup_{M \in A} \overline{M} = \widehat{\mathcal{A}} = L$ is the decomposition of L (with respect to the natural complex structure of $(X, \mathcal{T})$) into irreducible components. □

Proposition 24. *There are at most countably many* $\mathbf{D}^{\mathrm{reg}}$*-leaves* L *with* $\overline{L} \cap \operatorname{Sing} \mathcal{F} \neq \emptyset$.

Proof. The analytic set $\operatorname{Sing} \mathcal{F}$ has at most countably many connected components. Because of Corollary 21 each of them is completely contained in a $\mathbf{D}$-leaf; for each of these $\mathbf{D}$-leaves M there exist at most countably many $\mathbf{D}^{\mathrm{reg}}$-leaves L with $\overline{L} \cap M \neq \emptyset$ because of 23(g). □

As a consequence we obtain

Proposition 25. *If* $\mathbf{D}$ *is generically irreducible, then* $\mathrm{Rdh}(\mathbf{D})$ *is a proper analytic subset of* X*. The connected components of* $\mathrm{Rdh}(\mathbf{D})$ *are exactly the reducible* $\mathbf{D}$*-leaves.*

Proof. We define $\mathcal{A} = \{L \in \mathrm{Reg}(\mathbf{D})/\mathbf{D}^{\mathrm{reg}};\ \alpha(L) \text{ is reducible }\}$. Then $\mathrm{Rdh}(\mathbf{D}) = \widehat{\mathcal{A}} = \bigcup_{L \in \mathcal{A}} \overline{L}$ because of 23(g). Since $\overline{L}$ is analytic in X for $L \in \mathcal{A}$, in order to prove the analyticity of $\mathrm{Rdh}(\mathbf{D})$ it suffices to show that the family $\overline{\mathcal{A}} := \{\overline{L}; L \in \mathcal{A}\}$ is locally finite. For every $x_0 \in \overline{\mathrm{Rdh}(\mathbf{D})}$ according to our assumption there exists a neighborhood $U \Subset X$ of x_0 and a purely $(n-1)$-dimensional analytic set $B \subset U$ with $\mathrm{Rdh}(\mathbf{D}) \cap U \subset B$. We may suppose that the set $\mathcal{B} = \{B_1, \ldots, B_N\}$ of irreducible components of B is finite. For every L from $\mathcal{A}$ there exists a (probably empty) subset $\mathcal{B}_L$ of $\mathcal{B}$ with $\overline{L} \cap U = \bigcup_{B_\nu \in \mathcal{B}_L} B_\nu$. Since $\mathcal{B}_L \cap \mathcal{B}_M = \emptyset$ for different $L \neq M$ from $\mathcal{A}$, there are at most finitely many $L \in \mathcal{A}$ with $\overline{L} \cap U \neq \emptyset$. Thus, the local finiteness of $\overline{\mathcal{A}}$ is shown and $\mathrm{Rdh}(\mathbf{D})$ is analytic. Because of 23(e) every connected component of $\mathrm{Rdh}(\mathbf{D})$ is a $\mathbf{D}$-leaf.

Lemma 26. *Suppose that $\mathbf{D}$ is generically irreducible. Then, for every $U \Subset X$, the morphism of ringed spaces $U/\big(R^{\mathbf{D}}|_U\big) \to \pi(U) \Subset X/\mathbf{D}$ (where $\pi : X \longrightarrow X/\mathbf{D}$ is the canonical projection) is an isomorphism of ringed spaces.*

Note that as mentioned at the beginning of Section 3, in general $R^{\mathbf{D}|_U} \neq R^{\mathbf{D}}|_U$.

Proof of lemma 26. Obviously the natural mapping $U/(R^{\mathbf{D}}|_U) \to \pi(U)$ is a homeomorphism and a morphism of ringed spaces. We have to show: if $V \Subset U$ and if $f \in \mathcal{O}(V)$ is $\mathbf{D}$-invariant, then the canonical continuation F of f to $W := \mathbf{D}(V) \Subset X$ is holomorphic on W. Certainly F is continuous, since $\mathbf{D}$ is open. Put $\mathrm{Reg}(\mathbf{D}) := \mathrm{Reg}(\mathbf{D})$, $X^{\mathrm{irr}} := X \setminus \mathrm{Rdh}(\mathbf{D})$ (note that $\mathrm{Rdh}(\mathbf{D})$ is analytic by Proposition 25) and $V_0 := V \cap \mathrm{Reg}(\mathbf{D}) \cap X^{\mathrm{irr}} \Subset X$. Since Lemma 26 is true for regular decompositions, certainly F is holomorphic on $R^{\mathrm{reg}}(V_0) \Subset X$, where $R^{\mathrm{reg}} \subset \mathrm{Reg}(\mathbf{D}) \times \mathrm{Reg}(\mathbf{D})$ is the equivalence relation defined by $\mathbf{D}^{\mathrm{reg}} = \mathbf{D}|_{\mathrm{Reg}(\mathbf{D})}$. Now $\mathbf{D}(x) \cap \mathrm{Reg}(\mathbf{D}) = R^{\mathrm{reg}}(x)$ for all $x \in \mathrm{Reg}(\mathbf{D})$, for which $\mathbf{D}(x)$ is irreducible, since in this case $\mathbf{D}(x) \cap \mathrm{Reg}(\mathbf{D})$ is connected. Hence $R^{\mathrm{reg}}(V_0) = W \cap \mathrm{Reg}(\mathbf{D}) \cap X^{\mathrm{irr}}$, and F is holomorphic on W by Riemann's continuation theorem. □

Proposition 27. *Suppose that L is a compact leaf of the regular, pure 1-codimensional analytic decomposition $\mathbf{D}$ of the complex manifold X. If there exists a local 1-dimensional submanifold S of X, which cuts L transversally in a point $x \in L$ such that $S \cap L'$ is finite for any $\mathbf{D}$-leaf L', then the holonomy group $\mathcal{H}(L)$ of L is finite and there exists an open $\mathbf{D}$-saturated neighborhood U of L such that all leaves in U are compact and the restriction $\mathbf{D}|_U$ is compact and stable.*

For a proof see Kaup [13].

We use the following notations: $d_f(a) := \dim_a f^{-1}\big(f(a)\big)$ for a holomorphic mapping $f : Z \longrightarrow Y$ between complex spaces and $a \in Z$. It is well known that d_f is upper semi-continuous. If Z is pure dimensional, then ρ_f, defined by $\rho_f(a) := \dim_a Z - \dim_a f^{-1}\big(f(a)\big)$, is lower semi-continuous. We put $\rho(f) := \max_a \rho_f(a)$. For a subset $T \subset X \times X$, by $p_T, q_T : T \longrightarrow X$ we denote the canonical projections.

Proposition 28. *Suppose that R is an analytic equivalence relation on X with pure d-dimensional equivalence classes. Then there exists an open holomorphic mapping $\pi : X \longrightarrow \tilde{X}$ onto a normal complex space $\tilde{X}$ such that the connected components of the fibers of π are the connected components of the equivalence classes of R.*

Proof. Let T be an irreducible component of R, $t := \dim T$. Then for all $(x, y) \in T$ we have $d_{p_T}(x, y) \leq d$. Since d_{p_T} is upper semi-continuous and since $\dim R(x) = d$ for all $x \in X$, we get $d_{p_T}(x, y) = d$ for all $(x, y) \in T$. Note that $t \leq n + d$. By Lemma 29, p_T is open if and only if $t = n + d$. If $t < n + d$ then $p_T(T)$ has Lebesgue measure zero by Lemma 30. Therefore, $\tilde{R} := \{(x, y) \in R : \dim_{(x,y)} R = n + d\}$, the union of all irreducible components of R of dimension $n + d$, is precisely the set of all points of R, at which p_R is open.

Obviously $\tilde{R}$ is symmetric. There exists at least one irreducible component T of R containing Δ. For such a component we get that p_T is surjective, $\rho(p_T) = n$, $\dim T = n + d$, $T \subset \tilde{R}$. Therefore $\Delta \subset \tilde{R}$. Now suppose $(a, b), (b, c) \in \tilde{R}$. Since p_R is open in the points (a, b) and (b, c), we conclude that p_R is also open in (a, c). Therefore we get $(a, c) \in \tilde{R}$. Hence $\tilde{R}$ is an open analytic equivalence relation. By Proposition 16, the quotient $\tilde{X} := X/\tilde{R}$ is a normal complex space. The projection $\pi : X \longrightarrow \tilde{X}$ is an open holomorphic mapping. Each irreducible component of a fiber of π is a d-dimensional analytic subset of an equivalence class $R(x)$, $x \in X$ and therefore an irreducible component of an equivalence class. □

Lemma 29. *Suppose that $f : Z \longrightarrow Y$ is holomorphic, Z an irreducible and Y a connected locally irreducible complex space. Then f is open if and only if $\dim Z = \dim Y + d_f(x)$ for all $z \in Z$.*

Lemma 30. *Let Z be an irreducible complex space and $f : Z \longrightarrow X$ holomorphic. If f is not open in any point $z \in Z$, then $f(Z)$ is a Lebesgue zero set in X.*

Proof. The lemma is certainly correct by Sard's theorem if Z is a manifold. In the general case prove the lemma by induction: consider the regular part of Z and the singular locus of Z. □

Lemma 31. *Let A be an analytic subset of X and $\mathfrak{a} \subset \mathcal{O}$ the sheaf of ideals defining A. Suppose that for a $\theta \in \Theta(X)$ we have that $\theta(\mathfrak{a}_x) \subset \mathfrak{a}_x$ for all x in the open dense subset B of A. Then $\theta(\mathfrak{a}_x) \subset \mathfrak{a}_x$ for all $x \in A$.*

Proof. The result is an easy consequence of the following two facts:

$$\theta(\mathfrak{a}_x) \subset \mathfrak{a}_x \ \forall x \in B$$
$$\iff \Big(\forall V \Subset X, \forall f \in \mathcal{O}(V) : f|_{V \cap B} = 0 \longrightarrow (\theta f)|_{V \cap B} = 0\Big)$$

and

$$f|_{V \cap B} = 0 \iff f|_{V \cap A} = 0 .$$

□

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