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A NOTE ON TRANSITIVE OPERATOR ALGEBRAS

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Abstract: In this paper we are interested in a question introduced by H. Önder (Arıkan) and M. Orhon in [5].

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In Orhon [4], it was proved that B is a complete Boolean algebra of projections on a Banach space Y with finite multiplicity n (i.e., there exist cyclic subspaces $X_1, X_2, ..., X_n$ with pairwise zero intersection such that $Y = \overline{X_1 + ... + X_n)}$, then a weak-operator closed transitive subalgebra A of $L(Y)$ containing B is equal to $L(Y)$. In Önder et al [5] H. Önder(Arıkan) and M. Orhon asked the question whether or not the result of Orhon [4] remains true if we replace B by a unital bounded algebra homomorphism $m : C(K) \to L(Y)$. In this note we will give an affirmative answer to this question.

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For unexplained notion and terminology we refer to Aliprantis et al [1], Radjavi et al [6].

Let X be a Banach space and let K be a compact Hausdorff space, and let $L(X)$ be the algebra of continuous linear operators from X into X. Denote by X' the topological dual of X. We say that X is a Banach $C(K)$-module if the bilinear mapping $C(K) \times X \to X, (a, x) \to a.x$, satisfies the following conditions:

(i) $1.x = x$ for all $x \in X$,$1 \in C(K)$;

(ii) $a.(b.x) = (ab).x$ for all $a, b \in C(K), x \in X$;

(iii) $\|a.x\| \leq \|a\|\|x\|$ for all $a \in C(K), x \in X$.

We establish the following bilinear mappings in two steps:

$$X \times X' \to C(K)' : (x, x') \to \mu_{x,x'}; \mu_{x,x'}(a) = x'(a.x), \tag{1}$$

$$C(K)'' \times X' \to X' : (a, x') \to a.x'; (a.x')(x) = a(\mu_{x,x'}). \tag{2}$$

When $C(K)\times C(K) \to C(K), (a, b) \to a.b$, is taken as the product on $C(K)$, then multiplication on $C(K)''$ defined by $C(K)'' \times C(K)'' \to C(K)'', (a, b) \to (a.b)(c) = b(a.c)$ is known as Arens product on $C(K)''$ which makes $C(K)''$ isomorphic to $C(S)$ with S a hyperstonian, Aliprantis et al [1, Theorem 15.7], Gk [2]. By the bilinear mapping $C(K) \times X \to X$ we define a map $m : C(K) \to L(X), m(a)x = a.x$, which is unital ($m(1) = I$(identity operator)) (norm to strong operator topology) continuous homomorphism. Furthermore, (2) defines a left Banach $C(K)''$-module structure on X' that gives a homomorphism $m^* : C(K)'' \to L(X')$ defined by $m^*(a)(x') = a.x'$. It is called the Arens extension of the module multiplication on X, Gök [3]. On X' we put $\sigma(X', X)$-topology.

We list some properties of m^* from Gök [2], [3].

Lemma 1. *The map m^* has the following properties:*

(1) For each $a \in C(K), m^*(a)$ is the adjoint in $L(X')$ of the operator $m(a)$ in $L(X)$.

(2) m^* is $(weak^*, weak^* - operator)$-continuous.

(3) For each $x' \in X'$ the linear map from $C(K)''$ into X' that sends a to $a.x'$ is $(weak^*, weak^*)$-continuous.

Note that $\overline{m^*(C(K))} = \overline{m^*(C(K)'')}$, where the closure is taken with respect to weak* operator topology.

Let us remember the following well-known definitions. Let M be a Boolean algebra of projections in X. M is called complete in the sense of Bade, if M is complete as a Boolean algebra and $\langle E_\alpha x, x'\rangle \to \langle Ex, x'\rangle$ for all $x \in X$ and all $x' \in X'$, whenever $E_\alpha \uparrow E$ in M. Since S is hyperstonian in our case, the Boolean algebra of projections in $C(S)$ is complete as an abstract Boolean algebra. Since the absolute weak topology $|\sigma|(C(K)'', C(K)')$ on $C(K)''$ is Lebesgue, it follows that for the Boolean algebra B of projections (idempotents) in $C(S)$, $m^*(B)$ is Bade complete in $L(X')$ with respect to the weak* operator topology, Gk [2].

Definition 2. Let $M \subseteq L(X)$ be a Boolean algebra of projections and $x \in X$. The cyclic space, $M[x]$, generated by x is the closed subspace $M[x] = \overline{\text{span}}\{Ex : E \in M\}$ of X. If $M[x] = X$, then x is called a cyclic vector for M and M itself is called cyclic. Similar definitions can be done for Banach $C(K)$-modules. Let X be a Banach $C(K)$-module and let B be idempotents in $C(K)''$ and let $x' \in X'$ and let $M = m^*(B)$. The cyclic space, $M[x']$, generated by x' is the weak* closed subspace $M[x'] = \overline{\text{span}}\{Ex' = e.x' : E \in m^*(B), e \in B\}$ of X'. If $M[x'] = X'$, then x' is called a cyclic vector for B and B itself is called cyclic. We say that X' has finite multiplicity n if there exist finite cyclic subspaces $M[x'_1], M[x'_2], ..., M[x'_n]$ in X' with pairwise zero intersection such that $X' = \overline{M[x'_1] + ... + M[x'_n]}$, where the closure is taken with respect to weak* topology. For Boolean algebras of projections on Banach spaces we refer to the papers Rall [7], Rosenthal et al [8], Rosenthal et al [9] and so on.

Let A be an algebra in $L(X)$. We say that A is a transitive algebra if only A-invariant subspaces are $\{0\}$ and X.

Lemma 3. *Let A be a weakly closed transitive algebra in $L(X)$ and let $A^* = \{T' \in L(X') : T \in A\}$, where T' is $(\sigma(X', X), \sigma(X', X))$ continuous adjoint of T. Then, $A^* = \overline{A^*} = \overline{A}^*$ is a weak*-operator closed transitive algebra in $L(X'[\sigma(X', X)])$.*

Proof. Equality $\overline{A} = A$ implies that $\overline{A}^* = A^*$. By the definition of the closure operation we have that $\overline{A}^* \subseteq \overline{A^*}$. For the reverse inclusion, let us take $T \in \overline{A^*}$. Then there exists a net (T^*_α) in A^*, (T_α) in A such that $T^*_\alpha \to T$ in weak* operator topology. This means that $x(T^*_\alpha x') \to x(Tx')$ for each $x \in X, x' \in X'$. By $x' \circ T = T'x'$, we get $(x' \circ T)x = x'(Tx)$ and $(T'x')x = x'(Tx)$ and so $x'(T_\alpha x) \to x'(T'x)$. This implies $T' \in \overline{A} = A$. Therefore, $T \in \overline{A}^* = A^*$,i.e., $\overline{A^*} \subseteq \overline{A}^*$. Hence, $\overline{A}^* = \overline{A^*}$. Transitivity of A implies that $\overline{A}^*$ is a transitive algebra. For this, let $Y \subset X'$ be a $\sigma(X', X)$ closed subspace different from $\{0\}$ and X', and $T'Y \subseteq Y$ for each $T' \in A^*, T \in A$. Then taking polar of both sides, we get $TY^\circ \subseteq Y^\circ$, Y° is $\sigma(X, X')$-closed subspace in X. So, this contradicts the fact that A is a transitive algebra. Therefore, A^* is a transitive operator algebra. □

Now we can state and prove the main result of this paper that gives a solution to a question raised in Önder et al [5].

Theorem 4. *Let $m : C(K) \to L(X)$ be a unital bounded homomorphism and let A be a transitive weakly closed algebra containing $m(C(K))$. Assume that X' has finite multiplicity n. Then $A = L(X)$.*

Proof. Let $A^* = \{T' \in L(X') : T \in A\}$, where T' is $(\sigma(X', X), \sigma(X', X))$ continuous adjoint of $T \in A$. Since $m^*(C(K)) \subseteq A^*$, we have that $m^*(B) \subseteq \overline{m^*(C(S))} = \overline{m^*(C(K))} \subseteq \overline{A^*}$. Since $m^*(B)$ is a Bade complete Boolean algebra, it follows from Orhon[4] that $\overline{A^*} = L(X')$. Equality $\overline{A} = A$ implies that $\overline{A^*} = \overline{A}^* = A^*$ by Lemma 3. Let $T \in L(X)$. Then $T' \in L(X') = \overline{A^*}$. There exist a net $\{T'_\alpha\}$ in A^*, $\{T_\alpha\}$ in A such that $T'_\alpha \to T'$ in the weak* operator topology. This means that $x(T'_\alpha x') \to x(T'x')$ for each $x' \in X', x \in X$. Therefore, $x'(T_\alpha x) \to x'(Tx)$ for each $x' \in X', x \in X$. $\{T_\alpha\}$ in $\overline{A} = A$ implies that $T \in A$. The reverse inclusion $A \subseteq L(X)$ is obvious. Hence, $A = L(X)$. □

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