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SOLUTIONS OF SOME ANISOTROPIC EQUATIONS

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Abstract: For the problem (Qc) below, in [7] we established the following results:

Theorem A. *If there are two functions ϕ and ψ, which are (Qc)-compatible in Ω with all their second derivatives positive, then:*

i) any classical solution u of (Qc) satisfies $\phi \leq u \leq \psi$ in Ω. Such a solution is unique if the μ_i depend only on x;

ii) if it exists, the solution of (Qc), which has all its second derivatives positive is unique.

Theorem B. *Under the hypotheses (h1) - (h5),*
1) If there are two functions ϕ and ψ, which are (Qc)-compatible in Ω with all their second derivatives positive, then:

i) if the μ_i are independent of u, (Qc) has a unique solution $u \in C^2(\overline{\Omega})$ such that $\phi \leq u \leq \psi$ in Ω;

ii) the same conclusion holds when the μ_i depend also on u but the uniqueness holds only for solutions, whose all second derivatives are all positive.

The object of this note is to complete the work with some extension to the large solutions (as in (Q∞)).

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1. Introduction

In this note, for some $m_0,\ c > 0$ and with $\partial_i = \partial_{x_i} \equiv \partial/\partial x_i$, we consider the following problems:

$$Qu \equiv \sum_{i=1}^{n} \mu_i(x,u)\partial^2_{x_i} u = f(x,u) \quad \text{in} \quad \Omega, \quad u|_{\partial\Omega} = c\,, \tag{Qc}$$

$$Qu \equiv \sum_{i=1}^{n} \mu_i(x,u)\partial^2_{x_i} u = f(x,u) \quad \text{in} \quad \Omega, \quad u|_{\partial\Omega} = +\infty\,, \tag{Q$_\infty$}$$

where:

(h1) $f \in C(\overline{\Omega} \times \mathbb{R}_+)$ and $\mu_i \in C^1(\overline{\Omega} \times \mathbb{R}_+)$ are positive;
(h2) μ_i are decreasing and f increasing in $u > 0$;
(h3) $\mu_i(x,u) > m_0 \quad \forall x \in \overline{\Omega},\ u > 0;\ i = 1, 2, ..., n$;
(h4) $\Omega \in C^{2,\alpha}$ is open and bounded in $\mathbb{R}^n$, $\alpha \in (0,\ 1)$;
(h5) $C_1 t^p \le f(x,t) \le C_2 t^p, \quad \forall x \in \Omega,\ t > 0$ and $p > 1$ for some $C_i > 0$.

The alternative hypothesis on Ω is for it to be open , bounded and to satisfy the uniform external sphere condition [8].

(W) $\exists\{\Omega_m\}_{m\in\mathbb{N}}$ such that $\forall m, \quad \overline{\Omega}_m \subset \Omega_{m+1} \subset \Omega$,

$\bigcup_{m\in\mathbb{N}} \Omega_m = \Omega$ and $\partial\Omega_m$ is a C^∞-submanifold of dimension $n-1$.

Definition 1.1. Let $\phi, \psi \in C^1(\overline{\Omega})$. The function $\phi\ (\psi)$ will be said to be a *subsolution (supersolution)* of the equation in (Qc) if

$$Q\phi - f(x,\phi) \ge 0\ (\ge Q\psi - f(x,\psi)\ ) \quad \text{a.e. in } \Omega.$$

If in addition $\phi \le \psi$ in Ω and $\phi \le c \le \psi$ on $\partial\Omega$, the two functions will be said to be *(Qc)-compatible.*

The large solutions will be obtained as an inductive limits of the solutions $(v_m)_{m\in\mathbb{N}}$, where $\forall m,\ v_m$ solves

$$\sum_{i=1}^{n} \mu_i(x,v)\partial_{ii} v = f(x,u) \quad \textit{in } \Omega_m \qquad v|_{\partial\Omega_m} = \frac{1}{2}(\phi+\psi)|_{\partial\Omega_m}\,, \tag{Em}$$

where $\phi,\ \psi$ are (Q∞)-compatible (i.e. $Q\phi - f(x,\phi) \ge 0 \ge Q\psi - f(x,\psi)$ a.e. in Ω, $\phi \le \psi$; $\lim_{|x|\nearrow\partial\Omega}\phi(x) = \lim_{|x|\nearrow\partial\Omega}\psi(x) = +\infty$) and Ω_m are those in (W) above.

The main results are the following theorem.

Theorem C. *Let the hypotheses (h1)-(h5) be satisfied. Then:*

i) if they exist, the two distinct solutions of ($Q\infty$) do not intersect one each other;

ii) given two C^2 domains Ω_1 and Ω_2 such that $\overline{\Omega}_1 \subset\subset \Omega_2$, the respective large solutions (if they exist) satisfy $U_1 > U_2$ in Ω_1;

iii) large solutions having the same estimates at the boundary (i.e. $\exists \gamma \in C(\mathbb{R}_+)$ and $\nu > 0$ with $\gamma(x) \simeq dist(x, \partial\Omega)$ such that $\lim_{\partial\Omega} \gamma(x)^\nu u(x) = C_u < +\infty$) coincide.

Theorem D. *Under the same hypotheses, ($Q\infty$) has a solution $u \in C^2(\Omega)$, which is bounded in any compact subset of Ω.*

Moreover, if $B(0, R_1) \subset \Omega \subset B(O, R)$, where $B()$ are balls centered at the origine, then for $\min_\Omega u(x) = a$,

$$a^{p-1} R^2 > \frac{2(p+1)}{(p-1)^2} \quad \text{and} \quad a^{p-1} R_1^4 \leq \frac{8(p+1)}{(p-1)^2}. \tag{Ed}$$

2. Proof of Theorems

For ease writing set $f(w) := f(x, w)$, $\mu_i(w) := \mu_i(x, w)$ and assume that Ω contains the origine.

Proof of Theorem C. i) For such solutions u and v,

$$\sum_{i=1}^{n} \{\mu_i(u)\partial_{ii}(u - v) + [\mu_i(u) - \mu_i(v)]\partial_{ii} v\} + f(x, v) - f(x, u) = 0 \quad \text{in} \quad \Omega. \tag{2.1}$$

If there is $B \subset \Omega$ with non void interior such that $u > v$ in B and $(u - v) = 0$ on ∂B then at the point in B, where $u - v$ reaches its maximum, the left hand side of (2.1) is strictly negative which is a contradiction.

ii) Assume that there is $B \subset \Omega_1$ as in i) above, where we suppose that $u = U_2 \geq v = U_1$ in B. Similarly, we reach a contradiction whence $U_1 > U_2$ in Ω_1.

iii) Let U and V be two solutions having the same estimate at the boundary. With $u := U/C_U$; $v := V/C_V$, either $u > v$ in Ω and $(u - v)|_{\partial\Omega} = 0$ or this holds on a proper subset of Ω. Here (2.1) reads

$$\sum_i \{\mu_i(C_U u)\partial_i^2[C_U u - C_V v] + [\mu_i(C_U u) - \mu_i(C_V v)]C_V \partial_i^2 v\} + f(C_v v) - f(C_U u) = 0\,,$$

and as before we have a contradiction at the point, where $u - v$ reaches its maximum, which here is an interior point of Ω, the two solutions coincide. □

Before proving Theorem D, we show the following lemma.

Lemma 2.1. *Let (h5) be satisfied. For any* $a > 0$ *and* $p > 1$ *define for* $F(t) = \int_0^t f(s)ds$,

$$R_a = \frac{1}{\sqrt{2}} \int_a^\infty \frac{dt}{\sqrt{F(t) - F(a)}}. \tag{Ra}$$

Then the problem

$$u'' = f(u) := u^p,\ r \in (0,\ R_a);\quad u'(0) = 0;\quad u(0) = a \tag{2.2}$$

has a unique large solution U_a *(i.e.* $\lim_{r\nearrow R_a} U_a(r) = +\infty$*) such that*

$$\begin{aligned} &(R_a - r)^{2/(p-1)}\, U_a(r) = O(1) \qquad \text{in} \quad I_a := (0,\ R_a) \\ &\text{and} \quad \frac{2(p+1)}{(p-1)^2} < a^{p-1}\, R_a^2\,. \end{aligned} \tag{2.3}$$

The link between $a > 0$ *and* R_a *is one-to-one.*

Proof. The initial value problem in $\mathbb{R}_+$

$$u'' = u^p;\quad u'(0) = 0;\quad u(0) = a > 0\,,$$

has a unique solution in a maximal support $D \subseteq \mathbb{R}_+$. The solution satisfies $u'(r)^2 = 2\{F(u) - F(a)\}$. Because $u' \geq 0$, we have for $r \in D$

$$\int_a^{u(r)} = \frac{du}{\sqrt{F(u) - F(a)}} = \sqrt{2}\quad r. \tag{2.4}$$

The solution cannot be bounded (in which case $D = \mathbb{R}_+$) as $u'(r)^2 = 2\{F(u) - F(a)\}$. Thus, the solution in the maximal support is large. As $p > 1$, the maximal support is

$$D := [0, R_a] \text{ say, where } R_a = \frac{1}{\sqrt{2}} \int_a^\infty \frac{dt}{\sqrt{F(t) - F(a)}} < +\infty.$$

Let V be the smallest large solution in I_a (see [1]) of

$$L(v) := v'' + \frac{(n-1)v'}{r} = v^p \quad r \in I_a; \quad v'(0) = 0. \tag{2.5}$$

The function $V(r)$ is obtained as the inductive limit as $k \nearrow \infty$ of the sequence $(v_k)_{k\in\mathbb{N}}$, where each v_k is the solution of

$$L(v) = v^p \text{ in } I_a; \quad v(R_a) = k\,, \tag{2.5a}$$

and has positive first and second derivatives.

For any k, the set $A_k := \{r \in I_a | U_a(r) \leq k\}$ is a proper subset of I_a whence $\forall k \in \mathbb{N}$, $U_a|_{A_k} > v_k$ and $U_a \geq V$ in I_a.

In fact, if that was not true, there should be $J \subset I_a$ such that $U_a < V$ in J and $(U_a - V)|_{\partial J} = 0$.

As $(U_a - V)'' = (1-n)V'/r + U_a^p - V^p < U_a^p - V^p < 0$ in J, at the point where $U_a - V$ reaches its minimum, we would have a contradiction.

From the estimate of V (see [1]), $\quad U_a(r) \geq \frac{C}{\{R_a - r\}^{2/(p-1)}}$.

For $v(r) = A/\{R^2 - r^2\}^b$, $\quad b = 2/(p-1)$, $R := R_a$, $\quad v', v'' \geq 0$ and

$$\triangle v = v'' + \frac{(n-1)v'}{r} \leq \frac{8(p+1)}{(p-1)^2 A^{p-1}} v^p; \quad r \geq 0. \tag{2.5b}$$

Obviously $\triangle U_a \geq U_a^p$ whence for $A \geq \{8(p+1)/(p-1)^2\}^{1/(p-1)}$, $\quad U_a$ and v are (Q∞)-compatible in $B(0, R_a)$ for the Laplacian operator and

$$U_a(r) \leq v(r) = A/(R^2 - r^2)^{2/(p-1)}. \qquad \square$$

Proof of Theorem D. The hypotheses on the μ_i imply that there is $m_1 > 0$ such that

$$m_0 \triangle g \leq Qg := \sum_1^n \mu_i(x, g)\partial_{ii} g \leq m_1 \triangle g \tag{2.6}$$

for any $g \in C^2(\mathbb{R})$ with positive first and second derivatives. Thus with appropriate choices of $f(t) = Kt^p$ in different equations above, taking R in (2.5) such that $B(0, R) \subset \Omega$ and R_a such that $\Omega \subset B(0, R_a)$, U_a and v will provide a couple of (Q∞)-compatible functions in Ω. (Ed) is obtained from easy calculations. $\square$

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