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# A NECESSARY AND SUFFICIENT CONDITION FOR REAL RANDOM FUNCTION TO BE STATIONARY

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**Abstract:** In this paper we proposed a necessary and sufficient condition for real random function to be stationary. The real random variables are (ZERO) mean, of (UNITE) variance and independent, and the real functions are contiuous.

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**Key Words:** random function, real random variables

## 1. Introduction

Let us recall that the $X = \{X(t), t \in R\}$ random functions is stationary (in the strict sense) when, for any $\{X(t_1+\nu), X(t_2+\nu), \ldots, X(t_n+\nu)\}$, $n-$dimensional law is independent of $\nu$.

In the following theorem we study the stationarity of the random function $X$ defined by

$$X\left(t\right) = \sum_{n \in Z} X_n \nu_n \left(t\right), \qquad t \in R,$$

where:

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(A1) $X_n, \quad n \in Z$ are mutually independent real variables, with the same probability law such that $E(X_n) = 0, E(X_n^2) = 1$.

(A2) $\nu_n(t)$ are real function, contiuous on $R$ and obey the conditions:

$$\begin{gathered} \nu_n(n) = 1, \\ \nu_n(k) = 0, \qquad k \in Z, \qquad k \neq n, \\ \sum_{n \in Z} \nu_n^2(t) < \infty, \qquad t \in R. \end{gathered} \tag{1}$$

This last condition is necessary and sufficient [1], [2] for the $X(t)$ quadratic mean and almost everywhere existence.

## 2. Theorem

A necessary and sufficient condition for $X(t) = \sum_{n \in Z} X_n \nu_n(t)$ to be defined as a stationary random function is that the following conditions are satisfied:

$(I)$ $\nu_n(t) = \nu_0(t-n), \quad \forall\, t \in R, n \in Z,$

$(II)$ $\exists s(h)$ taking the values 0 and 1 and such that:

$$\nu_0(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iht} s(h)\, dh,$$

$$\sum_{k \in Z} s(h + 2k\pi) = 1, \quad \forall h \in R,$$

$(III)$ $X_0$ is normal.

## 3. Proof

The $(I)$ condition is necessary. Following (1), it can actually by written:

$$E[X(t)X(t+\tau)] = \sum_{n \in Z} \nu_n(t)\nu_n(t+\tau), \tag{2}$$

given that the $X_n$ are (ZERO) mean, of (UNITE) variance and independent (A1). In order for $X$ to be stationary, it is necessary for (2) to be independent of $t$. In particular, for $t$ integer and $t = 0$, according to (A2):

$$E[X(t)X(t+\tau)] = \nu(n+t) = \nu_0(t) \tag{3}$$

hence $(I)$ .

The $(II)$ condition is necessary. First of all, the $\nu_0(\tau)$ continuity (A2) makes it possible to apply the Bochner-Kinchine Theorem [3], [4]:

$$\nu_0(\tau) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} e^{ih\tau} ds(h), \tag{4}$$

where $s(h)$ is real, nondecreasing and such that $s(-\infty) = 0, s(\infty) = 2\pi$.

Relation (4) can be written in the following way:

$$\nu_0(t) = \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} e^{ih\tau} d\alpha_\tau(h),$$

$$d\alpha_\tau(h) = \sum_{k\in Z} e^{2i\pi k\tau} ds(h + 2k\pi), \qquad \tau \in R. \tag{5}$$

In particular,

$$\nu_0(n) = \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} e^{ihn} d\alpha_0(h),$$

$$d\alpha_0(h) = d\alpha_n(h) = \sum_{k\in Z} ds(h + 2k\pi), \qquad n \in Z\,. \tag{6}$$

On the other had, according to (A2):

$$\nu_0(n) = \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} e^{ihn} dh, \qquad n \in Z. \tag{7}$$

The unicity of bounded measure transforms [5] then implies:

$$d\alpha_0(h) = \sum_{n\in Z} ds(h + 2n\pi) = dh, \tag{8}$$

which shows that $s(h)$ is absolutely contiuous and that its derivative w.r.t $(h)$ satisfies:

$$\alpha_0'(h) = \sum_{n\in Z} s(h + 2n\pi) = 1\,. \tag{9}$$

It remains for us to demonstrate that $s(h)$ can only take the values 0 and 1. We first note that according to (5), (6) and (9), $\alpha_\tau(h)$ is absolutely continuous for any $\tau \in R$. According to (5),

$$\left|\alpha_\tau'(h)\right| \le \alpha_0(h) = 1,$$

and $e^{ih\tau}\alpha_\tau^{'}(h)$ is periodic with regard to $h$ of period $2\pi$.

Finally, by arranging (3) and (5):

$$\nu_n(t) = \nu_0(\tau - n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ihn}\left[e^{ih\tau}\alpha_\tau^{'}(h)\right] dh\,. \tag{10}$$

Relation (10) express the fact that $\nu_0(\tau - n)$ is the $n$-th Fourier coefficient of $e^{ih\tau}\alpha_\tau^{'}(h)$. To this function, the *Parseval equality* can be applied [5]:

$$\sum_{n\in Z}\nu_0^2(\tau - n) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\alpha_\tau^{'}(h)\right|^2 dh, \tag{11}$$

As the $X(t)$ random function is assumed stationary, we have:

$$E\left[X^2(t)\right] = E\left(X_0^2\right) = 1, \quad \forall t \in R,$$

hence, according to (10), (11):

$$1 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\alpha_\tau^{'}(h)\right|^2 dh. \tag{12}$$

We have seen above that $\left|\alpha_\tau^{'}(h)\right|^2 \leq 1$ almost. Then (12) implies that, for any $\tau : \left|\alpha_\tau^2(h)\right| = 1$ almost every where with regard to $h$.

Lastly, for almost every $h, \alpha_\tau^{'}(h)$ is a characteristic function in the probabilistic sense, according to (5). More precisely, it is the characteristic function of a random variable taking the values $2k\pi$, $k \in Z$ with the probability $s(h + 2k\pi)$. The fact that $\left|\alpha_\tau^{'}(h)\right| = 1$ almost everywhere implies that it is degenerate [2]. So, for each $h$, (5) becomes:

$$\alpha_\tau^{'}(h) = e^{2i\pi k(h)\tau} s(h + 2k\pi(h)) \tag{13}$$

with

$$s(h + 2j\pi) = 0, \quad \text{for } j \neq k(h)\ ,$$

$$s(h + 2\pi k(h)) = 1\,.$$

This shows that $(II)$ is necessary. An alternative possibility for the proof can be deduced from [6].

The $(III)$ condition is necessary. It is immediately deduced from Laha and Lukacs [7], [8].

$\nu_0(t)$ is actually continuous and then takes all $[0,1]$ interval values (A2). As a result, for a certain value of $t$:

$$X(t_0) = \sum_{n \in Z} a_n X_n,$$

where at least two of the $a_n$ are not zero and where $\sum_{n \in Z} a_n^2 = 1$.

It is enough to ensure that the law common to the $X_n$ is normal law.

Now, $(I), (II)$ and $(III)$ are sufficient. $(I)$ and $(II)$ ensure stationarity of $X$ in the wide sense. In fact,

$$\nu_0(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{[iht + 2i\pi k(h)t]} dh, \qquad k(h) \in Z\,.$$

Now, $\nu_0(t-n)$ is the Fourier coefficient associated with $e^{[iht+2i\pi k(h)t]}$.

By applying the Parseval identity (see [5]):

$$\sum_{n \in Z} \nu_0(t-n)\,\nu_0(t+\tau-n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{[-ih\tau - 2i\pi k(h)\tau]} dh.$$

Thus, $E[X(t)\,X(t+\tau)]$ does not depend on $t$. The the random function $X$ is stationary in the wide sense.

The $(III)$ condition allows us to affirm that $X$ is a Gaussian random function. Indeed, the Levy Continuity Theorem [2] implies that for any $n$ the random vector $[X(t), X(t+\tau_1), \ldots]$ is Gaussian in this case, stationarity in the strict sense is equivalent to stationarity in the wide sense.

## 4. Conclusion

Insofar as $E(X_n)$ and $\mathrm{Var}(X_n)$ exist, the stationarity (in the strict sense) of the random function defined by $X(t) = \sum_{n \in Z} X_n \nu_n(t)$ is a restricting property. At the first order and the upper orders, it confers a particular shape on the interpolation function $\nu_0(\tau) = \nu_n(\tau+n)$ which is also the autocorrelation function $E[X(t)\,X(t+\tau)]$.

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