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INVERSE PRINCIPALLY CENTROGONAL MATRICES

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Abstract: A real nonsingular $n \times n$ matrix $A = (a_{ij})$ is called *centrogonal* if $A^{-1} = (a_{n+1-i,n+1-j})$, it is called *principally centrogonal* if all leading principal submatrices of A are centrogonal, and it is called *inverse principally centrogonal* if A^{-1} is principally centrogonal. We give a necessary and sufficient condition for a principally centrogonal matrix to be an inverse principally centrogonal matrix.

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1. Introduction

A real nonsingular $n \times n$ matrix $A = (a_{ij})$ is called *centrogonal* if $A^{-1} = (a_{n+1-i,n+1-j})$. A centrogonal matrix is called *principally centrogonal* if all leading principal submatrices of A are centrogonal,and a principally centrog-

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onal matrix, A, is called *inverse principally centrogonal* if A^{-1} is principally centrogonal. Centrogonal matrices were examined in [1], where motivation for the study is given relating them to *persymmetric* ($A^T = A^R$) and *centrosymmetric* ($A^R = A$) matrices. Also, centrogonal matrices are related with binomial coefficients and principal centrogonality of the matrix A, when suitably normalized, shares another significant property with the unit matrix: It has the same characteristic polynomial. If A is a centrogonal matrix then A^{-1} is centrogonal, but if A is a principally centrogonal matrix, A^{-1} is not necessarily principally centrogonal. In this article we investigate the properties of inverse principally centrogonal matrices. We will give necessary and sufficient condition for a principally centrogonal matrix to be an inverse principally centrogonal matrix. In analogy to the fact that the result of transposing a matrix is called the transpose, we call the result of rotating a matrix the *rotate* [1]. Thus, for a square matrix $A = (a_{ij})_{1\le i,j\le n}$ the rotate, denoted A^R, is defined by $A^R = (a_{n+1-i,n+1-j})_{1\le i,j\le n}$. Clearly, for the rotate A^R of A we have that $A^R = JAJ$, where J is the exchange matrix (or a per-identity matrix), i.e.,

$$J = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

For a vector $\boldsymbol{a} = (a_1, a_2, \cdots, a_n)^T$ the lower triangular $n \times n$ matrix $B(\boldsymbol{a})$ is defined by

$$B(\boldsymbol{a}) = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ a_2 & a_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_n & a_{n-1} & \cdots & a_1 \end{pmatrix}.$$

Clearly, for the rotate A^R of A we have that $A^R = JAJ$. The following properties of the rotation operator R was shown in [1]:

$$(A^T)^R = (A^R)^T\,,$$

$$(A^R)^R = A\,,$$

$$(A^{-1})^R = (A^R)^{-1}\,,$$

$$(AB)^R = A^R B^R.$$

2. Inverse Principally Centrogonal Matrices

The properties of centrogonal matrices were examined in [1]. In this paper, we give some conditions for the inverse principally centrogonal matrices. In [1], it was shown that a nonsingular matrix A is *centrogonal* if and only if there exist an $\eta \in \{-1, 1\}$, a nonsingular matrix $B \in \mathcal{R}^{n\times n}$ and a symmetric $n \times n$ permutation matrix P such that $A = \eta B^{-1}PBJ$. In particular, the matrix $\eta A^{-1}A^R$ is a centrogonal matrix if $\eta \in \{-1, 1\}$ and A is nonsingular. A further specialization yields that $\eta B^{-1}B^T$ is centrogonal if $\eta \in \{-1, 1\}$ and B is nonsingular and *persymmetric*, i.e. $B = JB^TJ$. Since Toeplitz matrices are persymmetric, $\eta B^{-1}B^T$ is centrogonal if B is a nonsingluar Toeplitz matrix because of this property.

Principal centrogonality is, of course, a strong condition. Nevertheless, it might be suprising that a matrix is completely determined by its first row.

Theorem 2.1. [1] *Let $A = (a_{ij})$ be a nonsingluar $n \times n$ matrix, $\boldsymbol{a} = (a_{11}, a_{12}, \ldots, a_{1n})^T$ and $B = B(\boldsymbol{a})$. Then A is principally centrogonal if and only if $A = a_{11}B^{-1}B^T$ and $a_{11} \in \{-1, 1\}$.*

Example 2.1. Let

$$A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}.$$

Then A is a principally centrogonal matrix. But

$$A^{-1} = A^R = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix}$$

is not principally centrogonal.

Example 2.2. Let

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

be a 2×2 principally centrogonal matrix. Then by Theorem 2.1, $A = a_{11}B^{-1}B^T$, where $B = \begin{pmatrix} a_{11} & 0 \\ a_{12} & a_{11} \end{pmatrix}$ and $a_{11} \in \{-1, 1\}$. So,

$$A = a_{11}B^{-1}B^T = \begin{pmatrix} 1 & a_{12} \\ -a_{12} & 1 - a_{12}^2 \end{pmatrix},$$

or $\begin{pmatrix} -1 & a_{12} \\ -a_{12} & a_{12}^2-1 \end{pmatrix}$. If A is an inverse principally centrogonal matrix, then $A^{-1}=A^R$ has one of the forms

$$\begin{pmatrix} 1-a_{12}^2 & -a_{12} \\ a_{12} & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} a_{12}^2-1 & -a_{12} \\ a_{12} & -1 \end{pmatrix}.$$

Since A is a centrogonal matrix, we also have

$$A^{-1}=A^R=\begin{pmatrix} a_{22} & a_{21} \\ a_{12} & a_{11} \end{pmatrix}.$$

So we have the following:

$$\begin{pmatrix} a_{22} & a_{21} \\ a_{12} & a_{11} \end{pmatrix} = \begin{pmatrix} 1-a_{12}^2 & -a_{12} \\ a_{12} & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} a_{12}^2-1 & -a_{12} \\ a_{12} & -1 \end{pmatrix}$$

As the same manner, $\hat{B}^{-1}\hat{B}^T = \begin{pmatrix} 1 & a_{21} \\ -a_{21} & 1-a_{21}^2 \end{pmatrix}$ or $\begin{pmatrix} -1 & a_{21} \\ -a_{21} & a_{21}^2-1 \end{pmatrix}$ because $\hat{B}=\begin{pmatrix} a_{22} & 0 \\ a_{21} & a_{22} \end{pmatrix}$. We have the following:

$$\begin{pmatrix} 1-a_{12}^2 & -a_{12} \\ a_{12} & 1 \end{pmatrix} = \begin{pmatrix} 1 & a_{21} \\ -a_{21} & 1-a_{21}^2 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} -1 & a_{21} \\ -a_{21} & a_{21}^2-1 \end{pmatrix},$$

$$\begin{pmatrix} a_{12}^2-1 & -a_{12} \\ a_{12} & -1 \end{pmatrix} = \begin{pmatrix} 1 & a_{21} \\ -a_{21} & 1-a_{21}^2 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} -1 & a_{21} \\ -a_{21} & a_{21}^2-1 \end{pmatrix}.$$

Hence we find that all the 2×2 inverse principally centrogonal matrices are:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & \sqrt{2} \\ -\sqrt{2} & -1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & -\sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix}, \begin{pmatrix} -1 & \sqrt{2} \\ -\sqrt{2} & 1 \end{pmatrix}, \begin{pmatrix} -1 & -\sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix}.$$

We now consider inverse principally centrogonal matrices of order $n>2$. Let $A=(a_{ij})$ be a principally centrogonal matrix. Then by Theorem 1, $A=a_{11}B^{-1}B^T$ and $a_{11}\in\{-1,1\}$. Since A is centrogonal, $A^R=A^{-1}$ and centrogonal. So, if we assume that A^R is principally centrogonal, then A is inverse principally centrogonal. Hence, we have the condition that $A^{-1}=A^R=(a_{11}B^{-1}B^T)^R=a_{11}(B^R)^{-1}(B^R)^T$.

Lemma 2.1. *Let $A = (a_{ij})$ be an $n \times n$ principally centrogonal matrix. If $a_{11} = a_{nn} \in \{-1, 1\}$ and $a_{1k} = 0 = a_{ns}$ for $k = 2, 3, \ldots, n$ and $s = 1, 2, \ldots, n-1$, then A is inverse principally centrogonal.*

Proof. Since A is principally centrogonal, by Theorem 1, $A = a_{11}B^{-1}B^T$, where $B = a_{11}I_n$. So, $A = I$ or $-I$ and $A^{-1} = A^R = A$. □

Theorem 2.2. *Let $A = (a_{ij})$ be an $n \times n$ principally centrogonal matrix. If $a_{11} = a_{12} = \cdots = a_{1n}$, then the matrix A is the companion matrix whose characteristic polynomial is $x^n - x^{n-1} + \cdots + (-1)^n$.*

Proof. Let $A = [a_{ij}]$ be a principally centrogonal matrix of order n. By Theorem 1, $A = a_{11}B^{-1}B^T$, where $B = B(\boldsymbol{a})$ and $a_{11} \in \{-1, 1\}$. Since the entries of A have the property that $a_{11} = a_{12} = \cdots = a_{1n}$, so $B = (b_{ij})$, where $b_{ij} = a_{11}, (i \geq j)$ and $b_{ij} = 0$, otherwise. Hence $A = a_{11}B^{-1}B^T = C \quad \text{or} - C$, where the matrix have the form;

$$C = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ -1 & 0 & \cdots & 0 \\ 0 & -1 & & \\ \vdots & & & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \qquad \square$$

Theorem 2.3. *Let $A = (a_{ij})$ be an $n \times n$ principally centrogonal matrix. If the matrix A is symmetric, then $A = \operatorname{diag}(d_1, d_2, \ldots, d_n)$, where $d_i = +1$ for all i or $d_i = -1$ for all i.*

Proof. Let $A = [a_{ij}]$ be a principally centrogonal matrix of order n. By Theorem 1, $A = a_{11}B^{-1}B^T$, where $B = B(\boldsymbol{a})$ and $a_{11} \in \{-1, 1\}$. Since A is symmetric $A = A^T$ and thus $a_{11}B^{-1}B^T = a_{11}(B^T)^T(B^T)^{-1} = a_{11}B(B^T)^{-1}$. So $B^2 = (B^T)^2$. Since $B = B(\boldsymbol{a})$ is lower triangular and B^2 is also lower triangular, we have that B^2 is a diagonal matrix. Thus, $B = \operatorname{diag}(a_{11}, a_{11}, \ldots, a_{11})$. □

Example 2.3. There are nontivial inverse pincipally centrogonal matrices, for example, let

$$F = \begin{pmatrix} 1 & 2 & 2 \\ -2 & -3 & -2 \\ 2 & 2 & 1 \end{pmatrix}.$$

It can be easily checked that F is an inverse principally centrogonal matrix.

Theorem 2.4. *Let $A = (a_{ij})$ be an $n \times n$ principally centrogonal matrix with decomposition $A = a_{11}B^{-1}B^T$ and $a_{11} = a_{nn} \in \{-1, 1\}$, where $B = B(\boldsymbol{a})$. Suppose that $\hat{B} = B(\hat{\boldsymbol{a}})$, where $\hat{\boldsymbol{a}} = (a_{nn}, a_{nn-1}, \ldots,$*

a_{n1}). *Then A is inverse principally centrogonal matrix if and only if the following matrix equation holds:*

$$B^{-1}B^TJ = \pm J\hat{B}^{-1}\hat{B}^T$$

Proof. Let A be an $n \times n$ inverse principally centrogonal matrix with decomposition $A = a_{11}B^{-1}B^T$ and $a_{11} = a_{nn} \in \{-1, 1\}$, where $B = B(\boldsymbol{a})$. Since A is principally centrogonal, by Theorem 1 and properties of the rotate R,

$$A^{-1} = A^R = a_{11}(B^{-1}B^T)^R = a_{11}J(B^{-1}B^T)J,$$

where $a_{11} \in \{-1, 1\}$. Also, since A is inverse principally centrogonal, A^{-1} is principally centrogonal matrix. Hence $A^{-1} = a_{nn}\hat{B}^{-1}\hat{B}^T$, where $a_{nn} \in \{-1, 1\}$. Since $a_{11}, a_{nn} \in \{-1, 1\}$, $a_{11} = a_{nn}$ or $a_{11} = -a_{nn}$. It follows that

$$(B^{-1}B^T)^R = \pm\hat{B}^{-1}\hat{B}^T\,,$$

or

$$B^{-1}B^TJ = \pm J\hat{B}^{-1}\hat{B}^T.$$

Conversely, if we assume that hold the matrix equation $B^{-1}B^TJ = \pm J\hat{B}^{-1}1\hat{B}^T$. Since A is principally centrogonal matrix $A^{-1} = A^R = a_{11}(B^{-1}B^T)^R = a_{11}J(B^{-1}B^T)J$. But since $B^{-1}B^TJ = \pm J\hat{B}^{-1}\hat{B}^T$, $A^{-1} = \pm a_{11}\hat{B}^{-1}\hat{B}^T$. This means that A^{-1} is a principally centrogonal matrix. The proof is complete. □

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