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REPRESENTATIONS OF HOLOMORPHIC GENERATORS AND DISTORTION THEOREMS FOR SPIRALLIKE FUNCTIONS WITH RESPECT TO A BOUNDARY POINT

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Abstract: In this paper we consider some classes of holomorphic generators of one-parameter semigroups of holomorphic self-mappings of the open unit disk, which have boundary fixed points. In particular, we establish an infinitesimal version of the Julia-Caratheodory Theorem. Further, we apply these results to the study of classes of spirallike functions with respect to a boundary point.

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1. Representations of Holomorphic Generators

Let Δ be the open unit disk in the complex plane $\mathbb{C}$ and let $\mathrm{Hol}\,(\Delta, \mathbb{C})$ denote the set of all holomorphic functions on Δ that take values in $\mathbb{C}$.

By $\mathrm{Hol}\,(\Delta)$ we denote the semigroup (with respect to composition operation) of all holomorphic self-mappings of Δ.

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A family $S=\{F_t\}_{t\geq 0}\subset \mathrm{Hol}\,(\Delta)$ is said to be *one-parameter semigroup (or flow)* on Δ if

(i) $F_{t+s}=F_t\circ F_s,\ t,s\geq 0.$

(ii) $F_0=I$ - the restriction of the identity mapping on Δ.

It is well known (see, for example, [4], [21] and [25]) that if S satisfies the continuity condition at $t=0^+$:

$$\lim_{t\to 0^+} F_t(z)=z,\quad z\in\Delta\,,$$

then it is also differentiable at $t\geq 0$ and the limit

$$\lim_{t\to 0^+}\frac{z-F_t(z)}{t}=f(z),\qquad z\in\Delta$$

defines a holomorphic mapping on Δ. This mapping f is called the *(infinitesimal) generator* of S.

Also, it follows by the semigroup properties (i) and (ii) that the function $u(t,z):=F_t(z)$ is the solution of the Cauchy problem:

$$\begin{aligned}\frac{\partial u\,(t,z)}{\partial t}+f\,(u\,(t,z)) &= 0,\quad t\geq 0,\\ u\,(o,z) &= z,\quad z\in\Delta.\end{aligned}\tag{1.1}$$

The family of all holomorphic generators on Δ will be denoted by $G(=G\,(\Delta))$.

It can be shown that if an element F_t of S generated by $f\in G$ is an automorphism of Δ then all elements of S are automorphisms, too, i.e., S can be extended to a one-parameter group. In this case we write that f belongs to the class $G_{aut}\,(=G_{aut}\,(\Delta))$ .The class G_{aut} is known to be a real Banach space while G is a real cone.

A well known representation of the class G is due to E.Berkson and H.Porta [4](see also [1], [2] and [25]).

Namely, $f\in G$ *if and only if there exist* $\tau\in\overline{\Delta}$ *and* $p\in\mathrm{Hol}\,(\Delta,\mathbb{C})$ *with* $\mathrm{Re}\,p(z)\geq 0,\ z\in\Delta,$ *such that*

$$f(z)=(z-\tau)(1-z\overline{\tau})\;p(z)\,.\tag{1.2}$$

It is clear that if $\tau\in\Delta$, and $f\neq 0$ identically, then τ is a unique null point of f in Δ. In addition, if $S=\{F_t\}_{t\geq 0}\subset\mathrm{Hol}\,(\Delta)$ is the flow generated by f , then it follows by the uniqueness of the solution of the Cauchy problem (1.1) that τ is a common fixed point of S, i.e.,

$$F_t(\tau)=\tau,\qquad \text{for all}\quad t\geq 0\,.\tag{1.3}$$

Moreover, if S does not contain an elliptic automorphism of Δ or the identity mapping on Δ, the point $\tau \in \overline{\Delta}$ in (1.2) is the so-called Denjoy-Wolff point of $S = \{F_t\}_{t\geq 0}$, i.e.,

$$\lim_{t\to\infty} F_t(z) = \tau \qquad \text{for all} \quad z \in \Delta \tag{1.4}$$

(see, for example [4], [1], [19], [2] and [25]).

If $\tau \in \partial\Delta$ then it is sometimes also called *the boundary Wolff point of S or sink point* (see, for example, [10], [24], [22] and [20]) because of the invariance property:

$$\frac{|F_t(z) - \tau|^2}{1 - |F_t(z)|^2} \leq \frac{|z-\tau|^2}{1-|z|^2}, \qquad t \geq 0\,, \tag{1.5}$$

which means that for each $t \geq 0$ and any horocycle $D(\tau, \kappa)$:

$$D(\tau,\kappa) = \left\{ z \in \Delta : \frac{|z-\tau|^2}{1-|z|^2} < \kappa, \quad \kappa > 0 \right\},$$

internally tangent to $\partial\Delta$ at τ, the mapping F_t takes $D(\tau,\kappa)$ into itself.

Moreover, it was shown in [9] (see also [25]) that in this case there exists the angular derivative of f at the point $\tau \in \partial\Delta$:

$$\angle f'(\tau) := \angle \lim_{z\to\tau} \frac{f(z)}{z-\tau} \left(= \angle \lim_{z\to\tau} f'(z) \right) = \beta\,, \tag{1.6}$$

with $\beta \geq 0$. By symbol $\angle \lim_{z\to\tau} g\,(z)$ we denote the limit of a function $g \in \mathrm{Hol}\,(\Delta, \mathbb{C})$ at the point $\tau \in \partial\Delta$, where z approaches τ in any Stolz angle in vertex at τ (see, for example, [24] and [17]).

In addition, for each $t \geq 0$ the angular derivative of the self-mapping $F_t \in S$ at the point $\tau \in \partial\Delta$ is

$$\angle \lim_{z\to\tau} \frac{F_t(z)-\tau}{z-\tau} \left(= \angle \lim_{z\to\tau} \frac{\partial F_t(z)}{\partial z} \right) = e^{-t\beta}.$$

Hence, by the classical Julia-Carathéodory Theorem (see, for example, [5], [24] and [17]) the mapping $F_t \in S$ satisfies the condition

$$\sup_{z\in\Delta} \frac{\varphi_\tau(F_t(z))}{\varphi_\tau(z)} \quad = \quad e^{-t\beta}, \qquad t \geq 0\,, \tag{1.7}$$

where $\varphi_\tau(z) = \frac{|z-\tau|^2}{1-|z|^2}$ is the "distance" from $z \in \Delta$ to $\tau \in \partial\Delta$. Thus, condition (1.7) gives a more precise (than (1.5)) estimate:

$$\frac{|F_t(z) - \tau|^2}{1 - |F_t(z)|^2} \le e^{-t\beta} \frac{|z - \tau|^2}{1 - |z|^2}, \quad t \ge 0, \quad z \in \Delta . \tag{1.8}$$

Obviously, the subclass of G of all holomorphic generators with a given null point $\tau \in \partial\Delta$ such that their angular derivatives at this point are nonnegative is also a real cone in $\mathrm{Hol}\,(\Delta, \mathbb{C})$.

Our purpose in this section is to study a more general class $G[\tau] \subset G$, which consists of all generators $f \in G$ vanished at a point $\tau \in \partial\Delta$ with the *finite angular derivative at this point,* i.e.,

$$G\,[\tau] = \left\{ f \in G,\ \angle f'(\tau) = \angle \lim_{z\to\tau} \frac{f(z)}{z-\tau} \left(= \angle \lim_{z\to\tau} f'\,(z)\right) \right.$$
$$\left. \text{exists finitely} \right\}$$

Note that we do not assume that $f \in G[\tau]$ has no other null points in $\overline{\Delta}$.

By $G^+\,[\tau]$ we denote the class of all holomorphic generators on Δ which vanish at $\tau \in \partial\Delta$ and have a *nonnegative angular derivative at* τ :

$$G^+\,[\tau] = \left\{ f \in G,\ \angle f'(\tau) = \angle \lim_{z\to\tau} \frac{f(z)}{z-\tau} \ge 0 \right\} .$$

By $\mathcal{F}\,[\tau]$ we denote the class of holomorphic self-mappings of Δ with a boundary fixed point $\tau \in \partial\Delta$ such that each $F \in \mathcal{F}\,[\tau]$ *has the finite angular derivative* at τ.

The class of holomorphic self-mappimgs of Δ with a *boundary sink point* $\tau \in \partial\Delta$ will be denoted by $\mathcal{F}^+\,[\tau]$. So, by the result in [9] mentioned above $f \in G^+[\tau]$ if and only if the semigroup $S = \{F_t\}_{t\ge0}$, generated by f belongs to the class $\mathcal{F}^+[\tau]$, i.e., fixes $\tau \in \partial\Delta$ and this point is a sink point for $\{F_t\}_{t\ge0}$. In this case f has no null point in Δ, but may have other null points on the boundary $\partial\Delta$.

Without loss of generality we can set $\tau = 1$. Note that the class, $G^+\,[1]$, can be described also by using the Berkson-Porta representation. Namely, $f \in G^+[1]$ *if and only if it has the form:*

$$f(z) = -(1-z)^2 p(z), \tag{1.9}$$

with $\mathrm{Re}\,p(z) \ge 0,\ z \in \Delta$.

If for $f \in G^{+}[1]$ the angular derivative $\angle f'(1) = \angle \lim_{z\to 1} \frac{f(z)}{z-1} = 0$ (or which is the same $\angle \lim_{z\to 1}(1-z)p(z) = 0$ (due to (1.9)) then f is said to be a generator of *parabolic type. The* subcone of $G^{+}[1]$ of parabolic type generators will be denoted by $G_p[1]$ i.e.,

$$G_p[1] = \left\{ f \in G,\ \angle f'(1) = \angle \lim_{z\to 1} \frac{f(z)}{z-1} = 0 \right\}.$$

If for $f \in G^{+}[1]$ the angular derivartive $\angle f'(1)$ is positive then f is said to be a generator of *hyperbolic type*. The subcone of $G^{+}[1]$ of hyperbolic type generators will be denoted by $G_h[1]$, i.e.,

$$G_h[1] = \left\{ f \in G,\ \angle f'(1) = \angle \lim_{z\to 1} \frac{f(z)}{z-1} > 0 \right\}.$$

The problem of describing of these subclasses of $G\,(= G\,(\Delta))$ goes back to sources of the probability branching processes theory (see, for example, [14]). According to the terminology of this theory the classes $G_p[1]$ and $G_h[1]$ correspond to the so-called critical and subcritical cases of homogeneous continuous branching processes, respectively, whilst the class $G[1]$ includes also the supercritical case.

V.V.Goryinov in [12] has considered an important class $\mathcal{F}^{+}[1,-1] = \mathcal{F}^{+}[1] \cap \mathcal{F}[-1]$ of self-mappings of Δ, which fixes two boundary points 1 and -1 on $\partial\Delta$ in the sense:

$$\angle \lim_{z\to \pm 1} F(z) = \pm 1, \tag{1.10}$$

and such that $\tau = 1$ is a sink point for F.

The corresponding class of all generators of flows in $\mathcal{F}^{+}[1,-1]$ is denoted by $G^{+}[1,-1]$.

A result in [12] asserts that $f \in G^{+}[1,-1]$ *if and only if it admits the representation*

$$f(z) = \alpha(1+z)(1-z)^2 \frac{1+g(z)}{zg(z)-1}, \tag{1.11}$$

in which $\alpha \geq 0$ *and* $g \in \operatorname{Hol}(\Delta)$ *or* g *is a constant of modulus less or equals to 1.*

However, it is not clear whether $G^{+}[1,-1] = G^{+}[1] \cap G[-1]$. The answer should be affirmative if we show that $f \in G[\tau]$ if and only if $S = \{F_t\}_{t\geq 0}$ generated by f belongs to $\mathcal{F}[\tau]$ for a boundary point $\tau \in \partial\Delta$.

Obviously, it is possible to use conjugation of the semigroup $S \subset \mathcal{F}^{+}[1,-1]$ by a Möbius transformation of the unit disk Δ onto itself to get a characterization of the class $G^{+}[-1,1]$ of all generators f on Δ such that the semigroup $S=\{F_t\}_{t\geq 0}$, generated by f belongs to the class $\mathcal{F}^{+}[-1,1]$, i.e., fixes -1 and 1 and such that $\tau=-1$ is a sink point for $\{F_t\}_{t\geq 0}$.

It is clear that $G^{+}[-1,1]\cap G^{+}[1,-1]=\{0\}$ and for each pair $f_{+}\in G^{+}[1,-1]$ and $f_{-}\in G^{+}[-1,1]$ its sum $f=f_{+}+f_{-}\in G\,[1,-1]:=G^{+}[1,-1]\cup G^{+}[-1,1]$. Then the question is how to recognize to which class $G^{+}[1,-1]$ or $G^{+}[-1,1]$ belongs this element f. Also note that the class $G\,[1,-1]$ contains the subclass

$$G_{aut}\,[1,-1]=\left\{f\,(z)=\alpha\left(z^2-1\right),\quad \alpha\in(-\infty,\infty)\right\},$$

of all (quadratic) generators vanished at the points 1 and -1. In fact, an element $f\in G_{aut}\,[1,-1]$ generates a group of *hyperbolic automorphisms* fixed these points.

It is clear, that mentioned above classes $G_{aut}\,[1,-1]$, $G^{+}[1,-1]$, $G_h\,[1]$, $G_p\,[1]$ and $G^{+}[1]$ belong to $G[1]$.

So, the question is: *whether each element* $f\in G[1]$ *generates a semigroup of the class* $\mathcal{F}\,[1]$.

Note that, in general, the only fact that $f\in G$ vanishes at a boundary point $\tau\in\partial\Delta$ does not imply that τ is necessarily a fixed point of the semigroup $S=\{F_t\}_{t\geq 0}\subset \mathrm{Hol}\,(\Delta)$ generated by f. Indeed, the example 5.1 in [18], shows that the function $f\,(z)=z-1+\sqrt{1-z}$ is of the class G and clearly vanishes at the boundary point $\tau=1$, while all elements of the semigroup $S=\{F_t\}_{t\geq 0}$ generated by f are strictly less than 1 at this point, i.e., $F_t\,(1)<1$ for all $t\geq 0$.

The point is actualy that, the angular derivative of this function at $\tau=1$ does not exists finitely, so f does not belong to $G[1]$.

In this section we first present a decomposition theorem for the class $G[1]$ by its subclasses $G_{aut}\,[1,-1]$ and $G_p\,[1]$. Namely, we show that

$$G[1]=G_p\,[1]\oplus G_{aut}\,[1,-1]\,.$$

As a consequence we will derive at that fact that $f\in G[1]$ if and only if $S=\{F_t\}_{t\geq 0}$ generated by f belongs to $\mathcal{F}\,[1]$.

Theorem 1.1. *A function* $f\in G[1]$ *if and only if it admits the representation:*

$$f(z)=-(1-z)^2p(z)+\frac{\beta}{2}(z^2-1)\,, \tag{1.12}$$

where $p\in\mathrm{Hol}\,(\Delta)$ *with* $\mathrm{Re}\,p\geq 0$ *and*

$$\angle\lim_{z\to 1}(1-z)\;p(z)=0, \tag{1.13}$$

and $\beta \in \mathbb{R} = (-\infty, \infty)$.

Moreover, $\tau = 1$ *is the sink point of the semigroup* $S = \{F_t\}_{t\geq 0}$ *generated by* f *if and only if* $\beta \geq 0$. *In this case* f *has no null point in* Δ.

Proof. Sufficiency. Let $f \in \mathrm{Hol}\,(\Delta, \mathbb{C})$ admit representation (1.12). Then the first term

$$g(z) = -(1-z)^2 p(z)$$

is an element of $G[1]$ due to the Berkson-Porta representation (see formulas (1.2) and (1.9)). The second term

$$h(z) = \frac{\beta}{2}(z^2 - 1)$$

is a generator of a group of automorphisms on Δ.

Since G (hence, $G[1]$) is a real cone in $\mathrm{Hol}\,(\Delta, \mathbb{C})$ the function $f = g + h$ belongs to $G[1]$.

Note that due to (1.12) and (1.13) we have

$$\angle \lim_{z \to 1} \frac{f(z)}{z-1} \left(= \angle \lim_{z \to 1} f'(z) \right) = \beta.$$

Hence, $z = 1$ is the sink point of $S = \{F_t\}_{t \geq 0}$ if and only if $\beta \geq 0$ by a result in [9].

Necessity. Let $f \in G[1]$. For z, w in Δ we denote by $\rho(z, w)$ the Poincaré metric on Δ, i.e.,

$$\rho(z, w) = \frac{1}{2} \log \frac{1 + |z|}{1 - |z|}\,.$$

Observe, that each element of the class G is a so-called ρ-monotone function on Δ, i.e.,

$$\rho(z + rf(z), \quad w + rf(w)) \leq \rho(z, w), \tag{1.14}$$

$z, w \in \Delta$, $r \geq 0$, whenever $z + rf(z)$ and $w + rf(w)$ are in Δ (see, [19]). Differentiating inequality (1.14) at $r = 0^+$ we obtain

$$\mathrm{Re} \left[\frac{f(z)\overline{z}}{1 - |z|^2} - \frac{f(w)\overline{w}}{1 - |w|^2} \right] \geq \mathrm{Re}\, \frac{\overline{z} f(w) + w\overline{f(z)}}{1 - \overline{z} w}\,, \tag{1.15}$$

for all $z, w \in \Delta$.

Now rewriting (1.15) in the form

$$\mathrm{Re}\ f(z) \left(\frac{\overline{z}}{1 - |z|^2} - \frac{\overline{w}}{1 - z\overline{w}} \right) \geq \mathrm{Re}\ f(w) \left[\frac{\overline{z}}{1 - \overline{z} w} - \frac{\overline{w}}{1 - |w|^2} \right],$$

and setting $w = r \in (0,1)$ we have

$$\mathrm{Re}\ f(z)\left(\frac{\overline{z}}{1-|z|^2} - \frac{r}{1-zr}\right) \geq \mathrm{Re}\,\frac{f(r)}{r-1}\cdot\frac{r-\overline{z}}{(1-r\overline{z})(1+r)}.$$

Letting now r to 1^- and taking into account that $\lim_{r\to 1^-} f(r) = 0$ and

$$\lim_{r\to 1^-}\frac{f\,'(r)}{r-1} = \angle f\,'(1)$$

we get

$$\mathrm{Re}\, f(z)\left(\frac{\overline{z}}{1-|z|^2} - \frac{1}{1-z}\right) \geq \mathrm{Re}\,\frac{\angle f\,'(1)}{2}, \tag{1.16}$$

or (after some manipulations):

$$-\,\mathrm{Re}\,\frac{f(z)}{(z-1)^2} \geq \frac{1}{2}\,\mathrm{Re}\,\angle f\,'(1)\frac{1-|z|^2}{|1-z|^2}. \tag{1.17}$$

Setting

$$q(z) = -\frac{f(z)}{(1-z)^2}\,, \tag{1.18}$$

and

$$p(z) = q(z) - \frac{1}{2}\frac{1+z}{1-z}\,\mathrm{Re}\,\angle f\,'(1)\,, \tag{1.19}$$

we obtain from (1.17)-(1.19)

$$f(z) = -(1-z)^2\left[p(z) + \frac{1}{2}\frac{1+z}{1-z}\,\mathrm{Re}\,\angle f\,'(1)\right]\,, \tag{1.20}$$

with

$$\begin{aligned}\mathrm{Re}\, p(z) &= \mathrm{Re}\, q(z) - \frac{1}{2}\,\mathrm{Re}\,\angle f\,'(1)\cdot\mathrm{Re}\,\frac{1+z}{1-z}\\ &\geq \frac{1}{2}\,\mathrm{Re}\,\angle f\,'(1)\left[\frac{1-|z|^2}{|1-z|^2} - \mathrm{Re}\,\frac{1+z}{1-z}\right] = 0.\end{aligned}$$

It is clear that (1.20) is equivalent to (1.12) with $\beta = \mathrm{Re}\,\angle f\,'(1)$.

Now we want to show that $\angle f\,'(1)$ is, in fact, a real number, or which is the same, that $\angle f\,'(1) = \beta$. To this end we observe, that it follows by the Riesz-Herglotz formula (see formula (4.1) below) that for each $p \in \mathrm{Hol}\,(\Delta)$ with $\mathrm{Re}\ p \geq 0$ the limit

$$\angle\lim_{z\to 1}(1-z)\ p(z)$$

exists and is a nonnegative real number (see, for example,[25]). Therefore,

$$\angle f\,'(1) = \angle \lim_{z\to 1} \frac{f(z)}{z-1} = \beta \left(= \operatorname{Re} \angle f\,'(1)\right) .$$

by (1.12). This immediately implies (1.13). The theorem is proved. □

Remark 1.1. It can be seen easily from the proof of Theorem 1.1 that actually, the condition:

$$\angle \lim_{z\to 1} \frac{f(z)}{z-1} \quad \text{exists}$$

can be replaced by a formally weaker condition:

$$\liminf_{r\to 1^-} \operatorname{Re} \frac{f(r)}{r-1} > -\infty. \tag{1.21}$$

Indeed, repeating the necessary part of the proof of the theorem we get that (1.21) implies representation (1.12) with $\beta = \liminf_{r\to 1^-} \operatorname{Re} \frac{f(r)}{r-1}$. In turn, this representation and the Riesz-Herglotz formula (4.1) applied to $p(z)(\operatorname{Re}\ p \geq 0)$ show that, in fact,

$$\lim_{r\to 1^-} (1-r)\ p(r) \left(= \lim_{r\to 1^-} \left[\frac{f(r)}{r-1} - \frac{\beta}{2}(r^2-1)\right]\right)$$

exists and equals to zero (cf.[9]). Finally, using Harnack's inequality (see formula (4.2) below) for the function p ($\operatorname{Re}\ p \geq 0$), we get that the function

$$s(z) = (1-z)\ p(z)$$

is bounded on each nontangential approach region at the point $z = 1$. Thus, condition (1.13) follows from the Lindelöff principle (see [17]) and we are done.

This observation leads to the following infinitesimal version of the Julia-Carathéodory Theorem.

Theorem 1.2. *Let $f \in G(\Delta)$ and let $S = \{F_t\}_{t\geq 0}$ be the semigroup generated by f. Then $f \in G[1]$ if and only if $S \in \mathcal{F}[1]$. Moreover, if*

$$\beta = \angle f'(1) := \angle \lim_{z\to 1} f'(z)\,, \tag{1.22}$$

then for each $t \geq 0$

$$\angle (F_t)'(1) := \lim_{z\to 1} (F_t)'(z) = e^{-t\beta}\,. \tag{1.23}$$

Proof. Sufficiency. Let $S = \{F_t\}_{t\geq 0}$ be the semigroup generated by $f \in G(\Delta)$. Suppose that $S \in \mathcal{F}[1]$, that is for each $t \geq 0$ there exists the angular limit

$$\alpha_t := \angle (F_t)'(1) \left(= \angle \lim_{t\to 1} \frac{F_t(z)-1}{z-1} \right) < \infty\,. \tag{1.24}$$

Then it follows again by the classical Julia-Carathéodory Theorem that

$$\frac{|1-F_t(z)|^2}{1-|F_t(z)|^2} \leq \alpha_t \frac{|1-z|^2}{1-|z|^2}\,.$$

Differentiating this inequality at $t = 0^+$ and using the semigroup property we obtain that

$$\liminf_{t\to 0^+} \frac{1-\alpha_t}{t} = \beta > -\infty\,,$$

and

$$\operatorname{Re} f(z) \left(\frac{\overline{z}}{1-|z|^2} - \frac{1}{1-z} \right) \geq \beta \geq 2\,.$$

Setting here $z = r$ and letting r tend to 1^- we get the inequality

$$\liminf_{r\to 1^-} \frac{f(r)}{r-1} \geq \frac{\beta}{2}\,,$$

which is equivalent to (1.21). Then Remark 1.1 implies the required inclusion: $f \in G[1]$.

Necessity. Suppose now that $f \in G[1]$. First, we show that $z = 1$ is a common fixed point for the semigroup $S = \{F_t\}_{t\geq 0}$.

Indeed, if $\beta = \angle f'(1)$ then

$$f = g + h\,,$$

where $g(z) = -(1-z)^2 p(z)$ and $h(z) = \frac{\beta}{2}(z^2-1)$ are elements of $G[1]$.

In addition, as we have mentioned above if $S_g = \{F_t^{(g)}\}_{t\geq 0}$ is the semigroup generated by g , then $z = 1$ is a sink point of this semigroup, hence,

$$\varphi(F_t^{(g)}(z)) \leq \varphi(z), \tag{1.25}$$

where

$$\varphi(z) = \frac{|1-z|^2}{1-|z|^2}\,.$$

Also, if $S_h = \{F_t^{(h)}\}_{t\geq 0}$ is the semigroup (actually group) generated by h, then it follows by direct calculations that

$$\varphi\left(F_t^{(h)}(z)\right) \leq e^{\beta t}\varphi(z)\,. \tag{1.26}$$

Now, it follows by the semigroup product formula (see, for example [21]) that

$$F_t = \lim_{n\to\infty}\left(F_{\frac{t}{n}}^{(h)}\circ F_{\frac{t}{n}}^{(g)}\right)^n, \tag{1.27}$$

where $(F)^n$ denotes the $n-$ fold iterate of a mapping $F:\Delta\to\Delta$. Then, applying (1.25), (1.26) and (1.27) we get by induction:

$$\varphi(F_t(z)) \leq e^{-t\beta}\varphi(z),$$

or, explicitly,

$$\frac{|1-F_t(z)|^2}{1-|F_t(z)|^2} \leq e^{-t\beta}\frac{|1-z|^2}{1-|z|^2}.$$

The latter inequality immediately implies that for each $t \geq 0$

$$\lim_{r\to 1^-} F_t(r) = 1$$

or, which is the same because of boundedness of F_t,

$$\angle\lim_{z\to 1} F_t(z) = 1. \tag{1.28}$$

It remains to show that

$$\angle(F_t)'(1) = \angle\lim_{z\to 1}\frac{F_t(z)-1}{z-1} = e^{-t\beta}. \tag{1.29}$$

To this end we note that it follows by representation (1.12) and (1.1) that

$$\frac{\partial F_t(z)}{\partial t} = (1-F_t(z))^2\ p(F_t(z)) - \frac{\beta}{2}((F_t(z))^2-1)$$

This implies that

$$\log\frac{F_t(z)-1}{z-1} = -\int_0^t (1-F_s(z)\ p(F_s(z))ds - \beta\int_0^t (F_s(z)+1)ds$$

Taking into account (1.13) we obtain (1.29). The theorem is proved. □

Remark 1.2. It turns out, that subclasses $G_h[1]$ and $G^+[1,-1]$ of $G[1]$ play a crucial role in the study of spirallike and starlike functions with respect to the boundary point $\tau = 1$(see Section 3). It is clear now that $f \in G[1]$ belongs to the class $G_h[1]$ of the generators of hyperbolic type, if and only if in its representation (1.12) $\beta > 0$. In its turn, $f \in G^+[1,-1]$ if and only if $\beta \geq 0$ and $\angle \lim_{z\to -1}\ p(z) = 0$.

Another useful characterization of these classes can be given as follows.

Theorem 1.3. (cf. [11]) *Let $f \in G_h[1]$, i.e., $f \in G[1]$ with $f'(1) = \angle \lim\limits_{z\to 1} f'(z) = \beta_+ > 0$. Then for each $c \in (0, \frac{\beta_+}{2}]$ there is a unique $F = F_c \in \mathcal{F}^+[1]$ with*

$$\angle \lim_{z\to 1} F\,'(z) =: F\,'(1) = \frac{2c}{\beta_+} \leq 1\,,$$

and such that

$$f(z) = -c(1-z)^2 \frac{1+F(z)}{1-F(z)}\,. \tag{1.30}$$

Moreover, if $f \in G^+[1,-1] \cap G_h[1]$ with $\beta_- = \angle \lim_{z\to 1} f'(z)$, then $F \in \mathcal{F}^+[1,-1]$, with

$$\angle \lim_{z\to -1} F\,'(z) = -\frac{\beta_-}{2c}.$$

Conversely, if $F \in \mathcal{F}^+[1]$ (respectively, $F \in \mathcal{F}^+[1,-1]$), then for each $c > 0$ the function f defined by (1.30) belongs to $G_h[1]$ (respectively, f belongs to $G^+[1,-1] \cap G_h[1]$, i.e., $\angle \lim\limits_{z\to\pm 1} f(z) = 0$ with $f'(1) > 0$ and $f'(-1) < 0$).

Proof. Let $f \in G_h[1] \subset G^+[1]$. Then one can present f in the form

$$f(z) = -(1-z)^2 p(z)\,,$$

with $\operatorname{Re} p(z) \geq 0$. For each $c > 0$ there is a unique holomorphic function $F = F_c$ on Δ such that $|F(z)| < 1,\ z \in \Delta$ and

$$p(z) = c\frac{1+F(z)}{1-F(z)}\,. \tag{1.31}$$

Furthermore, since

$$\beta_+ = \angle f'(1) = \angle \lim_{z\to 1} \frac{f(z)}{z-1} > 0\,,$$

we get

$$\beta_+ = c \lim_{r\to 1^-} \frac{1-r}{1-F(r)}(1+F(r))\,. \tag{1.32}$$

This relation shows that

$$\lim_{r\to 1^-} F(r) = 1$$

and

$$\lim_{r\to 1^-} \frac{1-r}{1-F(r)} = \frac{\beta_+}{2c} > 0\,.$$

Now, it follows by the Julia-Carathéodory Theorem that

$$\angle \lim_{z\to 1} F\,'(z) = \angle \lim_{z\to 1} \frac{1-F(z)}{1-z} = \frac{2c}{\beta_+} =: \alpha_+ > 0.$$

If $c \in (0, \frac{\beta_+}{2}]$ then $\alpha_+ \le 1$ and it follows by the Wolff Lemma (see, for example, [5], [24] and [17]) that $\zeta = 1$ is a sink point for F.

Let now $f \in G^+[1,-1]$ and set

$$\beta_- = f\,'(-1) = \angle \lim_{z\to -1} f\,'(z)\,.$$

Then

$$\begin{aligned} \angle \lim_{z\to -1} \frac{f(z)}{z+1} &= \beta_- = \angle \lim_{z\to -1} -(1-z)^2 c \frac{1+F(z)}{1-F(z)}\,\frac{1}{1+z} \qquad (1.33)\\ &= -4c \cdot \angle \lim_{z\to -1} \frac{1+F(z)}{1+z} \cdot \angle \lim_{z\to -1} \frac{1}{1-F(z)} \end{aligned}$$

Again, it follows by the Julia-Carathéodory Theorem and (1.33) that

$$\alpha_- = \angle \lim_{z\to -1} \frac{1+F(z)}{1+z}$$

exists finitely and

$$\alpha_- = -\frac{1}{2c}\beta_-$$

Reverse considerations complete our proof. □

Corollary 1.1. *Let $f \in G^+[1,-1]$ with $\angle f'(1) = \beta_+ \ge 0$ and $\angle f'(-1) = \beta_-$. Then $-\beta_- \ge \beta_+$. Moreover, the equality $-\beta_- = \beta_+$ is possible if and only if f is a generator of a group of hyperbolic automorphisms of Δ, i.e.,*

$$f\,(z) = \frac{\beta_+}{2}(z^2 - 1).$$

Proof. The assertion is obvious if $\beta_+ = 0$ since $\beta_- < 0$. Let us assume that $\beta_+ > 0$ and $-\beta_- \in (0, \beta_+)$. Then the function g defined by

$$g(z) = f(z) + \frac{\beta_-}{2}\,(z^2 - 1)$$

belongs to the class $G^+[1, -1]$, because this class is a real cone. In addition, we have

$$g'(1) = \beta_+ + \beta_- > 0\,,$$

while

$$g'(-1) = \beta_- - \beta_- = 0.$$

Then $g \neq 0$ and both points 1 and -1 are sink points of the semigroup generated by g which is impossible. This contradiction implies that $\beta_+ \leq -\beta_-$. Moreover, the same considerations show that $\beta_+ = -\beta_-$ if and only if $g(z) = 0$. Hence, $f(z)$ has the form

$$f(z) = \frac{\beta_-}{2}(1 - z^2) = \frac{\beta_+}{2}(z^2 - 1). \tag{1.34}$$

Thus, f belongs to $G_{aut}[1, -1]$. This complete our proof. □

Corollary 1.2. (cf. [12]) *If* $F \in \mathcal{F}^+[1, -1]$ *then*

$$F'(1) \cdot F'(-1) \geq 1\,. \tag{1.35}$$

Moreover, the equality $F'(1) \cdot F'(-1) = 1$ *is possible if and only if* F *is a hyperbolic automorphism of* Δ.

Proof. Indeed, if $F \in \mathcal{F}^+[1, -1]$ then it follows by Theorem 1.3 that f defined by (1.30) belongs to $G^+[1, -1]$. In addition, (1.35) and (1.30) imply that

$$F'(1) \cdot F'(-1) = \alpha_+ \cdot \alpha_- = -\frac{2c}{\beta_+} \cdot \frac{\beta_-}{2c} = -\frac{\beta_-}{\beta_+} \geq 1$$

(see previous corollary).

If now $\alpha_+ \cdot \alpha_- = 1$, then $-\beta_- = \beta_+$ and by the same corollary we have

$$f(z) = -c(1 - z)^2 \frac{1 + F(z)}{1 - F(z)} = \frac{\beta_+}{2}(z^2 - 1)\,.$$

Hereby, F must be of the form

$$F(z) = \frac{(z + 1) - \alpha_+(z - 1)}{(z + 1) + \alpha_+(z - 1)}, \qquad \text{where } \alpha_+ = \frac{2c}{\beta_+}.$$

The assertion is proved. □

2. Disrortion Theorems for the Class $\mathbf{G^+[1,-1]}$

Let $f(z) \in G^+[1]$ with $f\,'(1) = \beta_+ > 0$. It follows by (1.30) and Julia's Lemma that

$$\begin{aligned} |f(z)| &= c\,|1-z|^2 \left|\frac{1+F(z)}{1-F(z)}\right| \qquad (2.1)\\ &= c\,|1-z|^2 \frac{\left|1-F^2(z)\right|}{|1-F(z)|^2} \geq c\,|1-z|^2 \frac{1-|F(z)|^2}{|1-F(z)|^2} \\ &\geq \frac{c\,|1-z|^2}{\alpha_+} \cdot \frac{(1-|z|^2)}{|1-z|^2} = \frac{\beta_+}{2}(1-|z|^2), \qquad z \in \Delta. \end{aligned}$$

Similarly, for $x \in (-1,1)$ we obtain

$$\begin{aligned} \operatorname{Re} f(x) &= -c(1-x)^2 \operatorname{Re} \frac{1+F(x)}{1-F(x)} \qquad (2.2)\\ &= -c(1-x)^2 \frac{1-|F(x)|^2}{|1-F(x)|^2} \leq -\frac{\beta_+}{2}(1-x^2)\,. \end{aligned}$$

On the other hand, if $f \in G^+[1,-1]$

$$\begin{aligned} \left|\frac{1}{f(z)}\right| &= \frac{1}{c\,|1-z|^2} \left|\frac{1-F(z)}{1+F(z)}\right| \\ &\geq \frac{1}{c\,|1-z|^2} \frac{1}{\alpha_-} \frac{1-|z|^2}{|1+z|^2} = -\frac{2}{\beta_-} \frac{1-|z|^2}{|1-z^2|^2}\,, \end{aligned}$$

or

$$|f(z)| \leq -\frac{\beta_-}{2} \frac{\left|1-z^2\right|^2}{1-|z|^2}. \qquad (2.3)$$

Similarly, for $x \in (-1,1)$

$$\begin{aligned} \operatorname{Re} \frac{1}{f(x)} &= -\frac{1}{c(1-x)^2}\ \operatorname{Re} \frac{1-F(x)}{1+F(x)} = -\frac{1}{c(1-x)^2} \frac{1-|F(x)|^2}{|1+F(x)|^2} \\ &\leq +\frac{2}{|\beta_-|} \frac{(1-x^2)}{(1-x^2)^2} = \frac{2}{|\beta_-|} \frac{1}{1-x^2} \qquad (2.4) \end{aligned}$$

Combining (2.1) and (2.3) we get the following assertion.

Theorem 2.1. (A Distortion Theorem) *Let $f \in G^{+}[1,-1]$ with $\beta_{+} = f'(1) \geq 0$ and $f'(-1) = \beta_{-} < 0$. Then for $z \in \Delta$*

$$\frac{\beta_{+}}{2}\left(1-|z|^{2}\right) \leq |f(z)| \leq \frac{|\beta_{-}|}{2}\frac{\left|1-z^{2}\right|^{2}}{1-|z|^{2}}. \tag{2.5}$$

In particular, if z is in the circular lune

$$\Omega = \left\{z \in \Delta : \frac{\left|1-z^{2}\right|}{1-|z|^{2}} \leq M, \quad M \geq 1\right\},$$

then

$$\frac{\beta_{+}}{2M}\left|1-z^{2}\right| \leq |f(z)| \leq \frac{M\,|\beta_{-}|}{2}\left|1-z^{2}\right|. \tag{2.6}$$

Similarly, by using (2.2)–(2.4) we obtain the following

Theorem 2.2. *Let $f \in G^{+}[1,-1] \cap G_{h}[1]$ with $f'(1) = \beta_{+} > 0$ and $f'(-1) = \beta_{-} < 0$. Then for all $x \in (-1,1)$*

$$\frac{\beta_{-}}{2}(1-x^{2}) \leq Re\ f(x) \leq -\frac{\beta_{+}}{2}(1-x^{2}) < 0,$$

and

$$-\frac{2}{\beta_{+}}\frac{1}{1-x^{2}} \leq Re\ \frac{1}{f(x)} \leq \frac{2}{\beta_{-}}\frac{1}{1-x^{2}} < 0.$$

Finally, we note that regarding the asymptotic behavior of the flow generated by $f \in G^{+}[1,-1]$, one can derive now by using Theorem 1.2 at the following rates of convergence.

Corollary 2.1. *Let $f \in G^{+}[1,-1]$ with $\beta_{+} = f\,'(1) \geq 0$ and $f\,'(-1) = \beta_{-} < 0$. Then for $z \in \Delta$*

$$e^{t\beta_{-}}\left[\varphi_{-}(z)\right]^{-1} \leq \varphi_{+}(F_{t}(z)) \leq e^{-t\beta_{+}}\varphi_{+}(z),$$

where

$$\varphi_{+}(z) = \frac{|1-z|^{2}}{1-|z|^{2}} \quad \text{and} \quad \varphi_{-}(z) = \frac{|1+z|^{2}}{1-|z|^{2}}.$$

In particular, if $z = x \in (-1,1)$ then

$$e^{t\beta_{-}}\frac{1-x}{1+x} \leq \frac{|1-F_{t}(x)|^{2}}{1-|F_{t}(x)|^{2}} \leq e^{-t\beta_{+}}\frac{1-x}{1+x}.$$

3. Distortion Theorems for Starlike and Spirallike Functions

In turn, the above assertions enable us to get distortion theorems for starlike and spirallike functions with respect to a boundary point (see [23], [13], [16], [26], [15], [6], [7] and [8]).

Let h be a univalent function on Δ. *We say that h is a spirallike (respectively, starlike) function on Δ if for some $\alpha \in \mathbb{C}$ with $\operatorname{Re}\alpha > 0$ (respectively, $\alpha \in \mathbb{R}$ with $\alpha > 0$) and for each $t \geq 0$ the element $e^{-\alpha t} h(z)$ belongs to $h(\Delta)$, whenever $z \in \Delta$.*

Suppose now that h is a spirallike(respectively, starlike) function on Δ normalized by the conditions $h(0) = 1,\ h(1) := \angle \lim_{z\to 1} h(z) = 0$. In this case h is said to be spirallike (respectively, starlike) with respect to the boundary point $h(1) = 0$.

By $Q_h(z, 1)$ we denote the Visser-Ostrowski quotient

$$Q_h(z,1) = \frac{h'(z)(z-1)}{h(z)}$$

(see, for example, [17] and [25]).

It is known (see, [3] and [11]) that

$$\angle \lim_{z\to 1} Q_h(z,1) = \mu$$

exists and satisfies the condition

$$|\mu - 1| \leq 1, \qquad \mu \neq 0\,.$$

In addition, the function $f \in \operatorname{Hol}(\Delta, \mathbb{C})$ defined by

$$f(z) = \frac{\mu h(z)}{h'(z)}\,, \tag{3.1}$$

is of class $G^{+}[1]$ with $f'(1) = 1$ (see also [25]) and

$$h(F_t(z)) = e^{-\mu t} h(z)\,,$$

where $\{F_t\}_{t\geq 0}$ is the semigroup generated by f.

In this case h is called $\mu-$spirallike. The class of $\mu-$spirallike functions with respect to the boundary point $\tau = 1$ is denoted by $\operatorname{Spiral}_\mu[1]$.

Now for $h \in \operatorname{Spiral}_\mu[1]$ we obtain from (2.1) and (3.1)

$$\left|\frac{h'(z)}{h(z)}\right| = |\mu| \frac{1}{|f(z)|} \leq \frac{2\,|\mu|}{(1-|z|^2)}\,. \tag{3.2}$$

Integrating (3.2) we get

$$|\log h(z)| \leq \log \left(\frac{1+|z|}{1-|z|}\right)^{2|\mu|}. \tag{3.3}$$

Assume now that

$$\angle \lim_{z\to -1} h(z) = \infty.$$

Then for the univalent function

$$g(z) = \frac{1}{h(z)},$$

we have $\angle \lim_{z\to -1} g(z) = 0$ and

$$\frac{g'(z)}{g(z)} = -\frac{h'(z)}{h(z)}.$$

Then we define $Q_h(z,-1) = -Q_g(z,-1)$. Consequently,

$$\nu := \angle \lim_{z\to -1} \frac{(z+1)h'(z)}{h(z)} = -\angle \lim_{z\to -1} Q_g(z,-1),$$

if it exists. Let us assume that ν is finite and is not zero. Then again from (3.1) we have :

$$\angle \lim_{z\to -1} \frac{f(z)}{z+1} = \mu \lim_{z\to -1} \frac{h(z)}{(z+1)h'(z)} = \frac{\mu}{\nu}. \tag{3.4}$$

Hence, $\angle \lim_{z\to -1} f(z) = 0$ and

$$\angle \lim_{z\to -1} f'(z) =: \beta_- \tag{3.5}$$

is finite, i.e., $f \in G^+[1,-1]$. Moreover, it follows by (3.4) and (3.5) that

$$\nu\beta_- = \mu \tag{3.6}$$

and

$$Re\,,\nu < 0. \tag{3.7}$$

Then we get by (2.5)

$$\left|\frac{h'(z)}{h(z)}\right| \geq 2\,|\nu|\,\frac{1-|z|^2}{|1-z^2|^2}, \tag{3.8}$$

for all $z \in \Delta$.

Definition 3.1. We say that a univalent function h belongs to the class $\mathrm{Spiral}_{(\mu,v)}[1,-1]$ if it is μ-spirallike with respect to the boundary point $h(1)=0$, where

$$\mu = \angle \lim_{z\to 1} \frac{h'(z)(z-1)}{h(z)},$$

and satisfies the conditions:

(i) $\angle \lim_{z\to -1} h(z) = \infty$;

and

(ii) $\angle \lim_{z\to -1} \frac{h'(z)(z+1)}{h(z)} = \nu$.

Thus, we have obtained the following distortion theorem for a function of the class $\mathrm{Spiral}_{(\mu,v)}[1,-1]$.

Theorem 3.1. *Let $h \in \mathrm{Spiral}_{(\mu,v)}[1,-1]$. Then for all $z \in \Delta$ the following estimate holds:*

$$\frac{(1-|z|^2)\, 2\,|\nu|}{|1-z^2|^2} \le \left|\frac{h'(z)}{h(z)}\right| \le \frac{2\,|\mu|}{1-|z|^2}.$$

In particular, if z is in the circular lune

$$\Omega = \{z \in \Delta : \left|1-z^2\right| \le M(1-|z|^2),\ M \ge 1\},$$

then

$$\frac{1}{M}\frac{2\,|\nu|}{|1-z^2|} \le \left|\frac{h'(z)}{h(z)}\right| \le \frac{2\,|\mu|\, M}{1-|z|^2}.$$

In fact, we have shown also that if $h \in \mathrm{Spiral}_{(\mu,v)}[1,-1]$, then f defined by (3.1) belongs to $G^+[1,-1] \cap G_h[1]$.

Conversely, let now $f \in G^+[1,-1] \cap G_h[1]$ be given with $f'(1) = \beta_+ = 1$. Again by a result in [11] it follows that for each μ such that $|\mu-1| \le 1,\ \mu \ne 0$, equation (3.1) has a unique univalent solution h satisfying conditions $h(0)=1$ and $\lim_{r\to 1^-} h(r) = 0$ which is μ−spirallike. We want to show that actually $h \in \mathrm{Spiral}_{(\mu,v)}[1,-1]$ for some $\nu \in \mathbb{C}$ with $\mathrm{Re}\,\nu < 0$.

Indeed, because of relations (3.4)-(3.6) we have that such ν exists and condition (ii) of Definition 3.1 holds. Thus, it rests to show that condition (i) of this definition holds too, i.e., that

$$\lim_{x\to -1} h(x) = \infty.$$

To this end we return to formula (1.30) of a representation of $f \in G^+[1,-1]$:

$$f(z) = -c(1-z)^2\, \frac{1+F(z)}{1-F(z)}$$

and consider equation

$$h_1(z) = h\,'_1(z) \cdot f(z)\,. \tag{3.9}$$

This equation has a unique univalent solution $h_1(z)$ satisfying conditions $h_1(0) = 1,\ h_1(1) = 0$. Also, it is easy to see (comparing (3.1) and (3.9)) that, in fact,

$$h(z) = h_1(z)^\mu$$

(cf, [3] and [11]). Therefore, it is sufficient to prove our assertion for the function h_1.

By (1.30), and (3.9) we have for $x \in (-1, 1)$

$$\begin{aligned} -\operatorname{Re}\frac{h\,'_1(x)}{h_1(x)} &= \frac{1}{c(1-x)^2}\ \operatorname{Re}\frac{1-F(x)}{1+F(x)} \\ &= \frac{1}{c(1-x)^2}\frac{1-|F(x)|^2}{|1+F(x)|^2} \geq \frac{2}{|\beta_-|}\frac{1}{1-x^2}. \end{aligned} \tag{3.10}$$

Now, if $x \in (-1, 0)$ we have

$$\begin{aligned} -\ln|h_1(x)| &= -\operatorname{Re}\ln h_1(x) = -\int_0^x \operatorname{Re}\frac{h_1'(x)}{h_1(x)}dx \\ &\leq \frac{2}{|\beta_-|}\int_0^x \frac{dx}{1-x^2} = \frac{2}{|\beta_-|}\cdot\frac{1}{2}\ln\frac{1+x}{1-x}, \end{aligned} \tag{3.11}$$

and we have $\ln|h_1(x)| \to \infty$ as $x \to -1$.

Note that inequality (3.11) is equivalent

$$|h_1(x)| \geq \left(\frac{1-x}{1+x}\right)^{\frac{1}{|\beta_-|}}.$$

If $x \in (0, 1)$ then (3.10) implies

$$-\ln|h_1(x)| \geq \frac{1}{|\beta_-|}\ln\frac{1-x}{1+x},$$

or

$$|h_1(x)| \leq \left(\frac{1-x}{1+x}\right)^{\frac{1}{|\beta_-|}}.$$

Finally, we observe that if $\nu = -\mu$, then $\beta_- = -1 = -\beta_+$, hence, $f(z) = \frac{\mu h(z)}{h'(z)}$ must be an element of $G_{aut}[1, -1]$, i.e., $f(z) = \frac{\beta_+}{2}\left(z^2 - 1\right)$. Thus, we have proved the following assertion.

Theorem 3.2. *A univalent function h on Δ belongs to the class* $\text{Spiral}_{(\mu,v)}[1,-1]$ *if and only if the function*

$$f(z) = \mu h(z)/h'(z)$$

belongs to the class $G^+[1,-1]$ with $f'(1) > 0$. Moreover, if μ is real, ($\mu \in (0,2]$) then so is ν ($\nu \le -\mu < 0$) and h is, in fact, a starlike function on Δ satisfying the estimates:

$$|h(x)| \le \left(\frac{1-x}{1+x}\right)^{-\nu}, \ x \in [0,1) \ ,$$

and

$$|h(x)| \ge \left(\frac{1-x}{1+x}\right)^{-\nu}, \ x \in (-1,0] \ .$$

The equality $\nu = -\mu$ is possible if and only if

$$h(z) = \left(\frac{1-z}{1+z}\right)^{\mu}.$$

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4. Appendix

The well known Riesz—Herglots formula

$$p(z) = \int\limits_{\partial\Delta} \frac{1 + z\bar{\zeta}}{1 - z\bar{\zeta}}\, d\mu_p(\zeta) + i \operatorname{Im} p\,(0)\,, \tag{4.1}$$

establishes a linear one-to-one correspondence between the set of all positive measure functions $\mu\,(= \mu_p)$ and Carathéodory's class C of holomorphic functions on Δ with the nonnegative real part:

$$C = \{p \in \text{Hol}\ (\Delta, \mathbb{C}) : \operatorname{Re} p\,(z) \geq 0,\ z \in \Delta\}\,.$$

A consequence of this formula is the classical Harnack's inequality:

$$\operatorname{Re} p\,(0)\, \frac{1 + |z|}{1 - |z|} \geq \operatorname{Re} p(z) \geq \operatorname{Re} p\,(0)\, \frac{1 - |z|}{1 + |z|}. \tag{4.2}$$

At the same time, it follows by (4.1) that for each $\tau \in \partial\Delta$ the angular limit

$$\sigma = (\sigma_p\,(\tau)) = \angle \lim_{z \to \tau}\ (1 - z\overline{\tau})\ p(z) = 2\mu(\tau)$$

exists finitely and is a real nonnegative number.

We call this number the *charge* of p at the boundary point $\tau \in \partial\Delta$.

Consider the class

$$C^{+}[\tau] = \{p \in C : \sigma_p(\tau) > 0\}$$

of Carathéodory's functions with the strictly positive charge at a point $\tau \in \partial\Delta$.

Once again, by using the Julia-Carathéodory theorem (as in the proof of the Theorem 2.1) one can establish another inequality, which gives a somewhat more precise information of the boundary behavior for a function of the class $C^{+}[\tau]$ than the right-hand Harnack's ineguality (4.2).

Without loss of generality we assume again that $\tau = 1$.

Lemma 4.1. *A function p belongs to the class C if and only if the following inequality holds*

$$\operatorname{Re} p(z) \geq a\, \frac{1-|z|^2}{|1-z|^2},$$

for some $a \geq 0$. Moreover, in this case the number a can be chosen $a = \frac{1}{2}\sigma_p(1) = \mu_p(1)$.

Thus, $p \in C^{+}[1]$ with $\sigma_p(1) > 0$ if and only if $a > 0$.

In particular, if $z = x \in (-1,1)$ we have

$$|\mu_p|\, \frac{1+|x|}{1-|x|} \geq \operatorname{Re} p(x) \geq \mu_p(1)\frac{1+x}{1-x},$$

where μ_p is the Riesz-Herglotz measure function for p and $|\mu_p| = \int_{\partial\Delta} d\mu_p(\zeta)$.

Now consider the class

$$C^{+}[1,-1] = \left\{p \in C : \sigma_{+} = \sigma_p(1) > 0 \text{ and } \sigma_{-} = \sigma_{\frac{1}{p}}(-1) > 0\right\},$$

where

$$\sigma_{+} = \angle \lim_{z\to 1} (1-z)\, p(z), \quad \sigma_{-} = \angle \lim_{z\to -1} \frac{z+1}{p(z)}.$$

First, we note that it follows again from (4.1) that

$$\sigma_{+} = 2\mu_p(1) \leq 2\,|\mu_p| = 2\operatorname{Re} p(0)$$

and

$$\sigma_- = 2\mu_{\frac{1}{p}}(-1) \le 2\,\mathrm{Re}\,\frac{1}{p(0)}\,.$$

Since

$$Re\ p(0) \cdot Re\frac{1}{p(0)} \le 1\,,$$

we get that

$$\sigma_+ \cdot \sigma_- \le 4.$$

In addition, as in the above lemma we have

$$\left|\frac{1}{p\,(z)}\right| \ge \mathrm{Re}\,\frac{1}{p(z)} \ge \frac{\sigma_-}{2}\frac{1-|z|^2}{|1+z|}\,.$$

Again, since

$$\left|\frac{1}{p(z)}\right| \cdot \mathrm{Re}\ \ p(z) \le 1$$

we obtain

$$|p(z)| \le \frac{2}{\sigma_-}\ \frac{|1+z|^2}{1-|z|^2}.$$

Thus, we have the following assertion.

Lemma 4.2. *A function $p \in C$ belongs to the class $C^+[1,-1]$ if and only if*

$$b\ \frac{|1+z|^2}{1-|z|^2} \ge |p(z)| \ge \mathrm{Re}\ \ p(z) \ge a\ \frac{1-|z|^2}{|1-z|^2}\,,$$

for some positive a and b. In this case the numbers a and b can be chosen $a = \frac{1}{2}\sigma_+$ and $b = \frac{2}{\sigma_-}$. Moreover,

$$\sigma_+ \cdot \sigma_- \le 4\,,$$

and equality holds if and only if $p\,(z) = \alpha\frac{1+z}{1-z}$ for some $\alpha > 0$.

In particular, if $z = x \in (-1,1)$ we have for $p \in C^+[1,-1]$

$$\frac{2}{\sigma_-}\ \frac{1+x}{1-x} \ge \mathrm{Re}\ \ p(x) \ge \frac{\sigma_+}{2}\ \frac{1+x}{1-x}\,.$$

Finally, observe that it follows by the Berkson-Porta representation (1.9) of generators with the boundary sink point $\tau = 1$ that *a holomorphic function on Δ belongs to the* class $G^+\,[1]$ *with $f\,'(1) = \beta_+ > 0$ (respectively, $G^+[1,-1]$ with $f\,'(1) = \beta_+ > 0$ and $f\,'(-1) = \beta_- < 0$) if and only if the function*

$p\,(z) = -\frac{f(z)}{(1-z)^2}$ *is of the class* $C^+[1]$ *(respectively,* $C^+[1,-1]$*).* In these cases, $\beta_+ = \sigma_+$ while $\beta_- = -\frac{4}{\sigma_-}$.

Thus Theorem 2.1 and Theorem 2.2, as well as Corollary 1.4 can be derived from the last lemma. Moreover, comparing now (1.9) and (1.11) we have the following representation of the class $C^+[1,-1]$:

Proposition 4.1. *A function* $p \in C$ *belongs to the class* $C^+[1,-1]$ *if and only if it admits the representation*

$$p(z) = \alpha(1+z)\frac{1+g(z)}{1-zg(z)}\,,$$

in which $\alpha = \frac{1}{\sigma_-} > 0$ *and* g *is either a holomorphic self-mapping of* Δ *such that* $g \in \mathcal{F}^+[1]$ *with* $g'(1) = \frac{4}{\sigma_+ \cdot \sigma_-} - 1 > 0$ *or* $g \equiv 1$.

Finally, we observe that Lemma 3.4 and Theorem 1.1 show that the boundary behavior of generators of the class $G^+[1,-1]$ "near" their null points 1 and -1 is close enough to behavior of certain generators of hyperbolic groups vanished at these points. More precisely, combining the mentioned assertions we have the following quantitative complement of Corollary 1.4.

Proposition 4.2. *Let* $f \in G^+[1,-1]$ *with* $f\,'(1) = \beta_+ > 0$ *and* $f\,'(-1) = \beta_- < 0$ *. Then the following estimate holds:*

$$\left| f\,(z) - \frac{\beta_+}{2}\left(z^2-1\right)\right| \le \frac{|\beta_+ + \beta_-|}{2}\frac{\left|1-z^2\right|^2}{1-|z|^2}\,.$$

In particular, if z *is in the circular lune*

$$\Omega = \left\{ z \in \Delta : \frac{\left|1-z^2\right|}{1-|z|^2} \le M, \quad M \ge 1 \right\},$$

then

$$\left| f(z) - \frac{\beta_+}{2}\left(z^2-1\right)\right| \le \frac{M\,|\beta_+ + \beta_-|}{2}\left|1-z^2\right|\,.$$