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HOLOMORPHIC VECTOR BUNDLES ON ZARISKI OPEN SUBSETS OF INFINITE-DIMENSIONAL PROJECTIVE SPACES

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Abstract: Let V be an infinite-dimensional Banach space and U an open subset of $\mathbf{P}(V)$ such that there are finitely many continuous homogeneous polynomials $f_1, \dots, f_s$ on V such that $Z := \mathbf{P}(V) \backslash U = \{Q \in \mathbf{P}(V) : f_1(Q) = \cdots = f_s(Q) = 0\}$. Assume $Z \neq \emptyset$ and that Z has codimension at least three in $\mathbf{P}(V)$ at each of its points. Here we prove that for every large integer r there is an open subset Ω of $\mathbf{P}(V^*)$ and a holomorphic rank r vector bundle $\mathcal{E}$ on $U \times \Omega$ such that for all $(a, b) \in \Omega \times \Omega$ with $a \neq b$ the holomorphic vector bundles $E_a := \mathcal{E}|U \times \{a\}$ and $E_b := \mathcal{E}|U \times \{b\}$ are not isomorphic.

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1. Zariski Open Subsets of Projective Spaces

Let V be a nice infinite-dimensional Banach space (e.g. a separable Hilbert space) and $\mathbf{P}(V)$ the projective space of all one-dimensional linear subspaces on V. L. Lempert proved vanishing theorems for the cohomology of holomorphic line bundles on $\mathbf{P}(V)$ and that every holomorphic vector bundle on $\mathbf{P}(V)$ with finite rank is isomorphic to a direct sum of holomorphic line bundles ([1], Section 7 and Section 8). Here we will show that the latter property is not true if

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instead of $\mathbf{P}(V)$ we take a large, but not too large, Zariski open subset U of $\mathbf{P}(V)$. Here "large Zariski open subset" means that $Z := \mathbf{P}(V)\backslash U$ is a closed analytic subset of $\mathbf{P}(V)$ with finite codimension in $\mathbf{P}(V)$. By [2], Theorem III.2.3.1, this is equivalent to require that Z is the zero-locus of finitely many continuous homogeneous polynomials on V. Here "not too large" means that we require Z has codimension at least three in $\mathbf{P}(V)$ at each of its points. Of course, by Lempert results we also need to require $Z \neq \emptyset$. Our assumption implies that we need at least three continuous homogeneous polynomials to define Z. We will even give a lower bound for the rank of such non-trivial holomorphic vector bundles and the existence of infinite-dimensional families of mutually non-isomorphic holomorphic vector bundles on U depending from infinitely many parameters. More precisely, we will prove the following result.

Theorem 1. *Let V be an infinite-dimensional Banach space and U an open subset of $\mathbf{P}(V)$ such that $Z := \mathbf{P}(V)\backslash U$ is a closed anaytic subset of $\mathbf{P}(V)$ with finite codimension. Assume $Z \neq \emptyset$ and that Z has codimension at least three in $\mathbf{P}(V)$ at each of its points. Fix finitely many continuous homogeneous polynomials $f_1, \dots, f_s$ on V such that $Z = \{Q \in \mathbf{P}(V) : f_1(Q) = \cdots = f_s(Q) = 0\}$. Then:*

- *(i) For every integer $r \geq s - 1$ there are countably many mutually non-isomorphic rank r vector bundles on U, each of them not isomorphic to a direct sum of line bundles.*
- *(ii) For every integer $r \geq s$ there is an open subset Ω of $\mathbf{P}(V^*)$ and a holomorphic rank r vector bundle $\mathcal{E}$ on $U \times \Omega$ such that for all $(a, b) \in \Omega \times \Omega$ with $a \neq b$ the holomorphic vector bundles $E_a := \mathcal{E}|U \times \{a\}$ and $E_b := \mathcal{E}|U \times \{b\}$ are not isomorphic (after the obvious identification of $U \times \{a\}$ and $U \times \{b\}$ with U given by the projection $U \times \Omega \to U$).*

Proof. Z is the zero-locus of finitely many continuous homogeneous polynomials on V by [2], Th. III.2.3.1. Hence there is an integer $s \geq 3$ and $f_1, \dots, f_s$ with Z as zero-locus. For any integer t let $\mathcal{O}_{\mathbf{P}(V)}(t)$ be the degree t holomorphic line bundle on $\mathbf{P}(V)$. Hence $f_i \in \mathcal{O}_{\mathbf{P}(V)}(n_i)$. For any locally closed analytic subset B of $\mathbf{P}(V)$ set $\mathcal{O}_B(t) := \mathcal{O}_{\mathbf{P}(V)}(t)|B$. Fix an integer $k \geq 1$. Let $\alpha_k : \mathcal{O}_{\mathbf{P}(V)} \to \oplus_{i=1}^{s} \mathcal{O}_{\mathbf{P}(V)}(n_i)$ be defined by $\alpha_k(u) := (f_1^k u, \dots, f_s^k u)$ for any local section u of $\mathcal{O}_{\mathbf{P}(V)}$. Set $F_k := \mathrm{Coker}(\alpha_k)$ and $E_k := F_k|U$. By construction F_k is a finitely presented $\mathcal{O}_{\mathbf{P}(V)}$-sheaf on $\mathbf{P}(V)$. Since for every $Q \in U$ there is an index i such that $f_i(Q) \neq \emptyset$, E_k is locally free, i.e. it is a rank $s - 1$ homolorphic vector bundle on U. Since Z has everywhere codimension at least three, there is a plane $M \subset \mathbf{P}(V)$ with $M \cap Z = \emptyset$. We have

$\det(F_k|M) \cong \mathcal{O}_M(kn_1 + \cdots + kn_s)$ for every x. Hence $F_k|M \not\cong G_x|M$ if $x \neq k$. Hence $F_k \not\cong G_x$ if $x \neq k$. In order to obtain a contradiction we assume the existence of $k \geq 1$ such that E_k is isomorphic to a direct sum of line bundles. There is a finite-dimensional linear subspace N of $\mathbf{P}(V)$ such that $\dim(N) \geq 3$, $Z \cap N \neq \emptyset$ and $Z \cap N$ is finite. Look at $(F_k|N)^{**}$, where * denote the dual $\mathrm{Hom}(-, \mathcal{O}_N)$. Hence $(F_k|N)^{**}$ is a reflexive sheaf on N extending $E_k|(N \cap U)$. Since $N \cap Z$ is finite and $\dim(N) \geq 2$, $(F_k|N)^{**}$ is the only reflexive extension of $E_k|(N \cap U)$ to N. By Serre Gaga it is the same to work on N with the algebraic Zariski topology and algebraic coherent sheaves or with the Euclidean topology and coherent analytic sheaves. The finiteness of $N \backslash (U \cap N)$ implies that any holomorphic line bundle R on $U \cap N$ extends to a holomorphic line bundle on N ([3], Theorem 1) and $R \cong \mathcal{O}_N(x)$ for some integer x. Assume $E_k|(U \cap N) \cong \oplus_{i=1}^{s-1} \mathcal{O}_{U \cap N}(x_i)$ for some integers x_i. Since $\dim(N) \geq 3$, it is easy to check using the corresponding Chern classes that $F|N$ is not isomorphic to a direct sum of line bundles and hence $E_k|(E \cap U)$ cannot be isomorphic to a direct sum of line bundles. To check part (i) when $r > s-1$, just take $E_k \oplus \mathcal{O}_U^{\oplus(r-s-1)}$, $k \geq 1$. Since $\det(F_k) \cong \det(F_k \oplus \mathcal{O}_U^{\oplus(r-s+1)})$ for any $k \geq 1$, we conclude as in the case $r = s-1$. Now we will prove part (ii) when $r = s$. Set $A := \{h \in V^* :$ every irreducible component of $\{h = 0\} \cap Z$ has positive codimension in $Z\}$. It is easy to see that if V is separable, then A is the intersection of a countable family of non-empty Zariski open subsets of V^* and that in the general case A contains a non-empty open (for the strong topology) subset of V^*. For any $a \in A$ define $\beta_a : \mathcal{O}_{\mathbf{P}(V)} \to \oplus_{i=1}^{s} \mathcal{O}_{\mathbf{P}(V)}(n_i) \oplus \mathcal{O}_{\mathbf{P}(V)}(1)$ by the formula $\beta_a(u) := (f_1^u, \ldots, f_s^u, au)$ for any local section u of $\mathcal{O}_{\mathbf{P}(V)}$. Set $F'_a := \mathrm{Coker}(\beta_a)$ and $E_a := F_a|U$. Thus, F_a is a sheaf with a finite presentation, while E_a is a rank s holomorphic vector bundle. Even more: we may define a rank s holomorphic vector bundle $\mathcal{E}$ on $U \times A$ using the same formula as β_a but on $\mathbf{P}(V) \times A$, i.e. just defining $\beta : \mathcal{O}_{\mathbf{P}(V) \times A} \to \oplus_{i=1}^{s} \mathcal{O}_{\mathbf{P}(V) \times A}(n_i) \oplus \mathcal{O}_{\mathbf{P}(V) \times A}(1)$ and the setting $\mathcal{F} := \mathrm{Coker}(\beta)$ and $\mathcal{E} := \mathcal{U}'|W \times A$. Since $Z \neq \emptyset$, V is infinite-dimensional and each germ of Z is of finite definition in the sense of [2], there is a non-empty open subset of Z which is an infinite dimensional Banach manifold. Hence it is easy to check the existence of a non-empty open subset Δ of A such that $Z \cap \{a = 0\} \neq Z \cap \{b = 0\}$ (set-theoretically) for all $(a, b) \in \Delta \times \Delta$ with a and b not proportional. Call Ω the open subset of $\mathbf{P}(V^*)$ associated to Δ. To prove part (ii) for $r = s$ it is sufficient to show that if $Z \cap \{a = 0\} \neq Z \cap \{b = 0\}$, then $E_a \not\cong E_b$. Fix any such pair (a, b). There is a finite-dimensional subspace G of $\mathbf{P}(V)$ such that $Z \cap \{a = 0\} \cap G \neq Z \cap \{b = 0\} \cap G$, $Z \cap \{a = 0\} \cap G$ and $Z \cap \{b = 0\} \cap G$ are finite, and $\dim(G) \geq 3$. It is sufficient to show that $E_a|(U \cap G) \not\cong E_b|(U \cap G)$. Assume $E_a|(U \cap G) \cong E_b|(U \cap G)$. As in the proof of part (i) we see that $E_a|(U \cap G)$ and $E_b|(U \cap G)$ extend to reflexive sheaves A_a and A_b on G and that any isomorphism between $E_a|(U \cap G) \ncong E_b|(U \cap G)$ extends to an isomorphism between A_a and A_b. However, it is easy to check just using the definition of the map β that $Z \cap \{a = 0\} \cap G$ (resp. $Z \cap \{b = 0\} \cap G$) is exactly the set of all points of G at which A_a (resp. A_b) is not locally free. Thus, A_a and A_b cannot be isomorphic, which is a contradiction. The same proof, just adding a trivial factor $\mathcal{O}_U^{(r-s)}$ gives part (ii) when $r > s$. $\square$

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References

[1] L. Lempert, L. Lempert, The Dolbeaut complex in infinite dimension I, *J. Amer. Math. Soc.*, **11** (1998), 485-520.

[2] J.-P. Ramis, *Sous-ensembles Analytique D'une Variété Banachique Complexe*, Springer-Verlag, Berlin-Heidelberg-New York (1970).

[3] J.-P. Serre, Prolongement de faisceaux analytiques cohérent, *Ann. Inst. Fourier*, Grenoble, **16** (1966), 363-374; Re-printed in: J.-P. Serre, Œvres — Collected Papers, Vol. **II**, Springer-Verlag, Berlin (1986), 277–288.