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SYMMETRIC CONFIGURATIONS ARISING FROM MIXED PARTITIONS OF PROJECTIVE GEOMETRIES

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Abstract: We give geometric constructions of symmetric tactical configurations of type 8_3 and 15_4. The constructions arise from some mixed partitions of projective geometries.

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1. Introduction

A set P of m *points* $p_1, \ldots, p_m$ together with a family B of certain subsets (called *blocks*) $B_1, \ldots, B_b$ of P is a *tactical configuration of type* (m_n, b_k) if:

- each block consists of k points;
- there are exactly n blocks trough each point.

A tactical configuration (P, B) is *symmetric* when $n = k$ and hence $m = b$. A tactical configuration (P, B) satisfying the condition that there is at most

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one block through each pair of points is called a *configuration,* and if (P, B) is symmetric then the term of an m_n-configuration is used. The incidence matrix associated to an m_n-configuration is an $m \times m$ matrix $A = (a_{ij})$, whose entries are zeros and ones according to the following rule: $a_{ij} = 1$ if p_i is in the block B_j, and $a_{ij} = 0$ if not.

In this paper, we describe a geometric construction that provides symmetric tactical configurations of type 8_3 and 15_4. Our configurations arise from a mixed partition of $\mathrm{PG}(3, q)$ into two lines and $q^2 - 1$ twisted cubics (see [5]) and from a generalization of a construction technique due to L. M. Abatangelo [1], who used a family of ovals of an affine plane π of even order q to obtain a $2-(q-1, q/2-1, q/4-1)$ Hadamard design. The 15_4-configuration is related to a recent work by A. Betten and D. Betten [2], who gave a complete classification of 15_4-configurations. There are only four 15_4-configurations up to isomorphisms, and they have automorphism groups of order 15, 30, 360 or 24. It should be noted that the first three were already known to Merlin [7]. Our contribution to this subject is a purely geometric construction of the 15_4-configuration with the largest automorphism group.

Our paper is also related to some previous results in [6], where the authors investigated further the above mentioned construction due to L. M. Abatangelo for odd $q \geq 9$. They were able to construct some $(q-1)_{(q-3)/2}$-configurations by a computer aided search. In Section 5, we give a geometric description for the smallest case $q = 9$.

2. Preliminary Results

By a previous result, see [5], the projective space $\mathrm{PG}(3, q)$ is partitioned into two lines and $q^2 - 1$ twisted cubics. Such a mixed partition depends on lifting of linear collineations of $\mathrm{PG}(1, q)$ to linear collineations of $\mathrm{PG}(3, q)$ with an invariant twisted cubic C. More precisely, for a linear collineation α of $\mathrm{PG}(1, q)$, given by $y \mapsto yA$ with

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

the lifting provides the linear collineation $\overline{\alpha}$ of $\mathrm{PG}(3, q)$ given by $x \mapsto x\overline{A}$, where

$$\overline{A} = \begin{pmatrix} a^3 & a^2c & ac^2 & c^3 \\ 3a^2b & a^2d + 2abc & bc^2 + 2acd & 3c^2d \\ 3ab^2 & b^2c + 2abd & ad^2 + 2bcd & 3cd^2 \\ b^3 & b^2d & bd^2 & d^3 \end{pmatrix}.$$

The twisted cubic C left invariant by $\overline{\alpha}$ has parametric equations $X_0 = u^3$, $X_1 = u^2 t, X_2 = ut^2, X_3 = t^3$. In such a way, the projective linear group is lifted to a subgroup $H \cong \mathrm{PGL}(2, q)$ of $PGL(4, q)$ that leaves C invariant.

For a primitive element ω of $GF(q^2)$, let $f = a_0 - a_1 X - X^2$ be its minimal polynomial over $GF(q)$. The companion matrix $C(f)$ of f is defined as

$$C(f) = \begin{pmatrix} 0 & 1 \\ a_0 & a_1 \end{pmatrix}.$$

The linear transformation associated to $C(f)$ is a Singer cycle of $\mathrm{GL}(2, q)$, and $C(f)$ is conjugate in $\mathrm{GL}(2, q^2)$ to the diagonal matrix

$$D = \begin{pmatrix} \omega & 0 \\ 0 & \omega^q \end{pmatrix}.$$

The lifting of the Singer linear collineation $y \mapsto yC(f)$, has matrix representation $\overline{C(f)}$, which turns out to be conjugate in $\mathrm{GL}(4, q^2)$ to the diagonal matrix

$$\overline{D} = \begin{pmatrix} \omega^3 & 0 & 0 & 0 \\ 0 & \omega^{q^2+2} & 0 & 0 \\ 0 & 0 & \omega^{2q+1} & 0 \\ 0 & 0 & 0 & \omega^{3q} \end{pmatrix}.$$

As (ω^3, ω^{3q}) and $(\omega^{q+2}, \omega^{2q+1})$ are two pairs of elements of $GF(q^2)$, which are conjugate over $GF(q)$, we have that $\overline{C(f)}$ has a rational canonical form given by the matrix

$$M = \begin{pmatrix} S & 0 \\ 0 & T \end{pmatrix},$$

where each non–zero block is the companion matrix of a quadratic irreducible polynomial over $GF(q)$. Hence, the linear collineation of $\mathrm{PG}(3, q)$ induced by $\overline{C(f)}$ fixes two lines that can be chosen - without loss of generality - as the lines ℓ_1 and ℓ_2 with equations $X_0 = X_1 = 0$ and $X_2 = X_3 = 0$, respectively.

Proposition 1. *M induces a linear collineation Γ of $\mathrm{PG}(3, q)$ of order $q + 1$.*

Clearly, M induces S on ℓ_1 and T on ℓ_2. If $\gcd(q + 1, 3) = 1$, then S and T act on ℓ_1 and ℓ_2 respectively as a Singer cycle and so both ℓ_1 and ℓ_2 are full point-orbits under M. Otherwise, S induces on ℓ_1 a group having three orbits of length $(q + 1)/3$ on ℓ_1.

Proposition 2. *Let G be the cyclic group generated by Γ. If $\gcd(q + 1, 3) = 1$ then the orbits of G on $\mathrm{PG}(3, q)$ are two lines and $q^2 - 1$ twisted cubics.*

3. The Construction

Let Γ be the linear collineation of $\mathrm{PG}(3,q)$ described in the previous section. Let $B = \{C_1, \cdots, C_{q^2-1}\}$ be the set of the twisted cubics which are G–invariant. Now consider the action of G on the dual of $\mathrm{PG}(3,q)$. We get two pencils of planes through ℓ_1 and ℓ_2, respectively and q^2-1 dual cubics. We denote the set of these dual cubics by $P = \{D_1, \cdots, D_{q^2-1}\}$.

A new incidence structure (P, B) is defined as follows:

- *thick-points* of P are the dual cubics $D_1, \cdots, D_{q^2-1}$;
- *blocks* of B are the twisted cubics $C_1, \cdots, C_{q^2-1}$;
- a point $D_i \in P$ and a block $C_j \in B$ are said to be *incident* if they are disjoint as sets of points of the projective space $\mathrm{PG}(3,q)$.

Theorem 1. *The incidence structure (P, B) is a symmetric tactical configuration of type $(q^2-1)_k$ with $k = q(q-1)/3$ and $q \in \{3, 4\}$.*

Proof. We divide the proof in two parts.

(P_1) *Each dual cubic contains*

$$N = (q+1)(q^2+q+1) - \binom{q+1}{2}(q+1) + \binom{q+1}{3}$$

points of $\mathrm{PG}(3,q)$.

Taking into account that a dual cubic is a set of $q+1$ planes of $\mathrm{PG}(3,q)$ such that no four of them meet in a point, the result follows immediately by the inclusion-exclusion principle (see [3] for instance).

Now we are going to show that:

(P_2) *each dual cubic contains exactly* $(q-1)(2q+3)/3$ *(resp.* $(q+1)(2q-1)/3$*) twisted cubics of B if* $\gcd(q+1,3) = 1$ *(resp.* $\gcd(q+1,3) = 3$*).*

Let π be a plane of $\mathrm{PG}(3,q)$, which passes neither through ℓ_1, nor through ℓ_2. Let A and B be the intersection points of π with ℓ_1 and ℓ_2 respectively. The orbit of π under the action of G on the dual of $\mathrm{PG}(3,q)$ must clearly contain the orbits of A and B under the action of G on $\mathrm{PG}(3,q)$. The orbit of B is ℓ_2, while the orbit of B is ℓ_1 if $\gcd(q+1,3) = 1$, otherwise it is a set of $(q+1)/3$ points of ℓ_1. Therefore, in the former case we have to subtract from N the number $2(q+1)$. It turns out that each dual cubic D_i contains $(q^2-1)(2q+3)/3$ points of $PG(3, q) \setminus \{\ell_1 \cup \ell_2\}$, which are split in $(q-1)(2q+3)/3$ twisted cubics. In the latter case we have to compute the difference $N - 4(q+1)/3$, which is equal to $(q+1)^2(2q-1)/3$ and - as before - we obtain the result dividing the number $(q+1)^2(2q-1)/3$ by $q+1$. □

Now we are going to compute how many blocks pass through a thick-point D_i of P. From property (P_2) it follows that there are exactly either

$$q^2 - 1 - (q-1)(2q+3)/3 = q(q-1)/3,$$

or

$$q^2 - 1 - (q+1)^2(2q-1)/3 = (q+1)(q-2)/3,$$

twisted cubics disjoint from D_i, according as $\gcd(q+1, 3) = 1$ or $\gcd(q+1, 3) = 3$. This means that if $\gcd(q+1, 3) = 1$ there are exactly $q(q-1)/3$ blocks through a thick-point D_i, otherwise there are $(q+1)(q-2)/3$. By duality, it turns out that on each block there are either $q(q-1)/3$ or $(q+1)(q-2)/3$ points, according as $\gcd(q+1, 3) = 1$ or $\gcd(q+1, 3) = 3$, respectively. A necessary condition for the existence of a m_n-configuration is $m > n(n-1)+1$. However, the parameters of our configuration satisfy this condition only for $q = 3, 4$.

4. A Configuration 15_4

Henceforth, we deal with the case $q = 4$. Let Γ be the linear collineation of $PG(3, 4)$ of order 5, as described in Section 2. Let $1, 2, 3, \ldots, 84, 85$ be the points of $PG(3, 4)$. Using the software package *MAGMA* [4], we found out that Γ corresponds to the following permutation of $PGL(4, 4)$:

$$\begin{gathered}(1,47,76,39,4)(2,60,38,23,44)(3,73,85,7,84)(5,15,40,82,69)\\(6,57,58,48,41)(8,19,65,32,77)(9,81,34,50,10)(11,56,33,22,79)\\(12,20,45,55,59)(13,54,68,43,74)(14,67,30,27,29)\\(16,52,83,75,26)(17,62,80,24,18)(21,66,49,72,78)\\(25,31,64,46,35)(28,42,61,51,70)(36,63,53,37,71).\end{gathered}$$

There are exactly 17 orbits under the action of the group $G = <\Gamma>$ on $PG(3, 4)$: Two are lines ℓ_1, ℓ_2 and 15 are disjoint twisted cubics $C_1, \cdots, C_{15}$:

$$\ell_1 = \{9, 10, 34, 50, 81\}; \quad \ell_2 = \{25, 31, 35, 46, 64\};$$

$$C_1 = \{1, 4, 39, 47, 76\}; \quad C_2 = \{2, 23, 38, 44, 60\};$$
$$C_3 = \{3, 7, 73, 84, 85\}; \quad C_4 = \{5, 15, 40, 69, 82\};$$
$$C_5 = \{6, 41, 48, 57, 58\}; \quad C_6 = \{8, 19, 32, 65, 77\};$$
$$C_7 = \{11, 22, 33, 56, 79\}; \quad C_8 = \{12, 20, 45, 55, 59\};$$
$$C_9 = \{13, 43, 54, 68, 74\}; \quad C_{10} = \{14, 27, 29, 30, 67\};$$
$$C_{11} = \{16, 26, 52, 75, 83\}; \quad C_{12} = \{17, 18, 24, 62, 80\};$$
$$C_{13} = \{21, 49, 66, 72, 78\}; \quad C_{14} = \{28, 42, 51, 61, 70\};$$
$$C_{15} = \{36, 37, 53, 63, 71\}.$$

The action of G on the dual of PG(3,4) gives the following orbits: two pencils of planes through ℓ_1 and ℓ_2 respectively and 15 dual cubics $D_1, \cdots, D_{15}$. In particular,

$D_1 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 15, 16, 19, 20, 21, 23, 25, 26, 27, 28, 29, 30, 31, 32, 34, 35, 38, 39, 40, 41, 42, 44, 45, 46, 47, 48, 49, 50, 51, 52, 55, 57, 58, 59, 60, 61, 64, 65, 66, 67, 69, 70, 72, 73, 75, 76, 77, 78, 81, 82, 83, 84, 85\}$,

$D_2 = \{2, 3, 5, 7, 8, 9, 10, 12, 13, 14, 15, 17, 18, 19, 20, 21, 23, 24, 25, 27, 28, 29, 30, 31, 32, 34, 35, 36, 37, 38, 40, 42, 43, 44, 45, 46, 49, 50, 51, 53, 54, 55, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 77, 78, 80, 81, 82, 84, 85\}$,

$D_3 = \{2, 5, 6, 8, 9, 10, 11, 13, 14, 15, 16, 19, 21, 22, 23, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 40, 41, 42, 43, 44, 46, 48, 49, 50, 51, 52, 53, 54, 56, 57, 58, 60, 61, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 74, 75, 77, 78, 79, 81, 82, 83\}$,

$D_4 = \{1, 2, 4, 5, 8, 9, 10, 11, 12, 13, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 43, 44, 45, 46, 47, 49, 50, 52, 53, 54, 55, 56, 59, 60, 62, 63, 64, 65, 66, 68, 69, 71, 72, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83\}$,

$D_5 = \{1, 4, 5, 6, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 19, 20, 24, 25, 26, 27, 28, 29, 30, 31, 32, 34, 35, 36, 37, 39, 40, 41, 42, 43, 45, 46, 47, 48, 50, 51, 52, 53, 54, 55, 57, 58, 59, 61, 62, 63, 64, 65, 67, 68, 69, 70, 71, 74, 75, 76, 77, 80, 81, 82, 83\}$,

$D_6 = \{1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 13, 14, 15, 20, 21, 22, 23, 25, 27, 29, 30, 31, 33, 34, 35, 36, 37, 38, 39, 40, 41, 43, 44, 45, 46, 47, 48, 49, 50, 53, 54, 55, 56, 57, 58, 59, 60, 63, 64, 66, 67, 68, 69, 71, 72, 73, 74, 76, 78, 79, 81, 82, 84, 85\}$,

$D_7 = \{1, 2, 3, 4, 5, 6, 7, 9, 10, 13, 15, 16, 17, 18, 21, 23, 24, 25, 26, 28, 31, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 46, 47, 48, 49, 50, 51, 52, 53, 54, 57, 58, 60, 61, 62, 63, 64, 66, 68, 69, 70, 71, 72, 73, 74, 75, 76, 78, 80, 81, 82, 83, 84, 85\}$,

$D_8 = \{1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 12, 13, 16, 19, 20, 22, 23, 25, 26, 28, 31, 32, 33, 34, 35, 36, 37, 38, 39, 41, 42, 43, 44, 45, 46, 47, 48, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 63, 64, 65, 68, 70, 71, 73, 74, 75, 76, 77, 79, 81, 83, 84, 85\}$,

$D_9 = \{2, 3, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 22, 23, 24, 25, 26, 27, 29, 30, 31, 32, 33, 34, 35, 38, 40, 41, 43, 44, 45, 46, 48, 50, 52, 54, 55, 56, 57, 58, 59, 60, 62, 64, 65, 67, 68, 69, 73, 74, 75, 77, 79, 80, 81, 82, 83, 84, 85\}$,

$D_{10} = \{1, 2, 3, 4, 5, 7, 9, 10, 11, 12, 14, 15, 16, 17, 18, 20, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 33, 34, 35, 36, 37, 38, 39, 40, 42, 44, 45, 46, 47, 50, 51, 52, 53, 55, 56, 59, 60, 61, 62, 63, 64, 67, 69, 70, 71, 73, 75, 76, 79, 80, 81, 82, 83, 84, 85\}$,

$D_{11} = \{3, 6, 7, 9, 10, 11, 12, 13, 14, 16, 17, 18, 20, 21, 22, 24, 25, 26, 27, 28, 29, 30, 31, 33, 34, 35, 36, 37, 41, 42, 43, 45, 46, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 61, 62, 63, 64, 66, 67, 68, 70, 71, 72, 73, 74, 75, 78, 79, 80, 81, 83, 84, 85\}$,

$D_{12} = \{1, 2, 3, 4, 7, 8, 9, 10, 11, 13, 14, 16, 17, 18, 19, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 38, 39, 42, 43, 44, 46, 47, 49, 50, 51, 52, 54, 56, 60, 61, 62, 64, 65, 66, 67, 68, 70, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 83, 84, 85\}$,

$D_{13} = \{1, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 15, 17, 18, 19, 20, 21, 22, 24, 25, 28, 31, 32, 33, 34, 35, 39, 40, 41, 42, 43, 45, 46, 47, 48, 49, 50, 51, 54, 55, 56, 57, 58, 59, 61, 62, 64, 65, 66, 68, 69, 70, 72, 73, 74, 76, 77, 78, 79, 80, 81, 82, 84, 85\}$,

$D_{14} = \{1, 2, 4, 6, 8, 9, 10, 11, 12, 14, 17, 18, 19, 20, 21, 22, 23, 24, 25, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 41, 42, 44, 45, 46, 47, 48, 49, 50, 51, 53, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 70, 71, 72, 76, 77, 78, 79, 80, 81\}$,

$D_{15} = \{1, 3, 4, 5, 6, 7, 8, 9, 10, 11, 14, 15, 16, 17, 18, 19, 21, 22, 24, 25, 26, 27, 29, 30, 31, 32, 33, 34, 35, 36, 37, 39, 40, 41, 46, 47, 48, 49, 50, 52, 53, 56, 57, 58, 62, 63, 64, 65, 66, 67, 69, 71, 72, 73, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85\}$.

Recalling that a point D_i is incident to the block C_k if and only if the point-sets of the corresponding dual cubic and twisted cubic are disjoint in $\mathrm{PG}(3,4)$ we obtain the following incidence matrix:

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}	C_{12}	C_{13}	C_{14}	C_{15}
D_1	0	0	0	0	0	0	1	0	1	0	0	1	0	0	1
D_2	1	0	0	0	1	0	1	0	0	0	1	0	0	0	0
D_3	1	0	1	0	0	0	0	1	0	0	0	1	0	0	0
D_4	0	0	1	0	1	0	0	0	0	1	0	0	0	1	0
D_5	0	1	1	0	0	0	1	0	0	0	0	0	1	0	0
D_6	0	0	0	0	0	1	0	0	0	0	1	1	0	1	0
D_7	0	0	0	0	0	1	1	1	0	1	0	0	0	0	0
D_8	0	0	0	1	0	0	0	0	0	1	0	1	1	0	0
D_9	1	0	0	0	0	0	0	0	0	0	0	0	1	1	1
D_{10}	0	0	0	0	1	1	0	0	1	0	0	0	1	0	0
D_{11}	1	1	0	1	0	1	0	0	0	0	0	0	0	0	0
D_{12}	0	0	0	1	1	0	0	1	0	0	0	0	0	0	1
D_{13}	0	1	0	0	0	0	0	0	0	1	1	0	0	0	1
D_{14}	0	0	1	1	0	0	0	0	1	0	1	0	0	0	0
D_{15}	0	1	0	0	0	0	0	1	1	0	0	0	0	1	0

This gives a configuration 15_4 whose automorphism group is isomorphic to $C_3 \rtimes S_5$ of size 360.

5. On the Configuration 8_3

In [5] it is showed that a Singer cycle of the projective line $\mathrm{PG}(1,q)$ can be lifted to a collineation γ of the projective plane $\mathrm{PG}(2,q)$ leaving a conic invariant. Using arguments similar to those described in Section 2, we observe that the plane $\mathrm{PG}(2,q)$ can be partitioned into one point S, one line ℓ, and $q-1$ conics $\Omega_1, \cdots, \Omega_{q-1}$, which are the orbits under $G =< \gamma >$. If q is odd, then the line ℓ splits into two orbits of length $(q+1)/2$. In [6] it is showed that the technique used by L. M. Abatangelo to construct a Hadamard design when

q is even , can be used also when q is odd and greater than 7, to obtain a symmetric configuration of type $(q-1)_{(q-3)/2}$. In particular, for $q = 9$ it yields a configuration 8_3.

We are going to prove this result. Recall that we are dealing with an incidence structure defined as follows:

- *thick-points* are line conics $\Sigma_1, \cdots, \Sigma_{q-1}$;
- *blocks* are the conics $\Omega_1, \cdots, \Omega_{q-1}$;
- a thick-point Σ_i and a block Ω_j are said to be *incident* if they are disjoint as sets of points of the projective plane $\mathrm{PG}(2, q)$.

We have the following result.

Proposition 3. *Each line conic contains*

$$N = (q+1)^2 - \binom{q+1}{2}$$

points of $\mathrm{PG}(2, q)$

Proof. A line conic is a set of $q + 1$ lines no three of which are concurrent in a point. Hence the result follows immediately by the inclusion-exclusion principle, see [3]. ☐

Proposition 4. *Let q be an odd prime power greater than 7. Then a line conic contains $(q+1)/2$ conics of* $\mathrm{PG}(2, q)$.

Proof. Let ℓ' be a line of π different from ℓ, and which does not pass through S, and let A be the intersection of ℓ and ℓ'. The orbit of ℓ' under the action of G on the dual of $\mathrm{PG}(2, q)$ must clearly contain the orbits of A under the action of G on $\mathrm{PG}(2, q)$. The orbit of A is a set of $(q+1)/2$ points of ℓ. Therefore, we have to subtract $(q+1)/2$ from N, and hence each line conic contains $(q+1)^2/2$ points of $\mathrm{PG}(2, q) \setminus \{\ell \cup S\}$, which are split into $(q+1)/2$ conics. ☐

It turns out that there are exactly $q - 1 - (q+1)/2 = (q-3)/2$ blocks through a thick-point, and by duality it follows that on each block there are $(q-3)/2$ thick-points.

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