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CONTROLLING THE DYNAMICS OF THE KURAMOTO-SIVASHINSKY EQUATION

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Abstract: Finite-dimensional feedback control schemes of the Kuramoto-Sivashinsky (K-S) partial differential equation (PDE) with periodic boundary conditions are presented. First, the dynamical behavior of the K-S equation at the bifurcation parameter $\alpha = 17.75$ representing a homoclinic connection in phase space is investigated, where a pseudo-spectral Galerkin method is used to solve the one-dimensional (1-d) K-S equation. Then, the Karhunen-Loéve (K-L) decomposition is applied on the numerical simulation data to extract the coherent structures or eigenfunctions of the dynamics of the equation. Projecting the one-dimensional K-S equation along the three most energetic K-L eigenfunctions, a system of ordinary differential equations (ODEs) is obtained. Three different control schemes are designed for the system of ODEs with the task of stabilizing the dynamics of the K-S equation to a stable solution. Theoretical development of the control schemes are illustrated through numerical simulations.

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1. Introduction

The 1-d Kuramoto-Sivashinsky equation

$$\frac{\partial u}{\partial t} + \nu \frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial x^2} + \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 = 0, \qquad (x,t) \in I\!R \mathrm{x} I\!R^+, \tag{1}$$

$$u(x,t) = u(x+2\pi, t), \tag{2}$$

$$u(x,0) = u_0(x), \tag{3}$$

with viscosity $\nu > 0$, has been used to model various physical and chemical systems. In 1976, Kuramoto and Tsuzuki [1] derived the above equation as a model for the interfacial instabilities for a system of a reaction-diffusion equation that models Belouzov-Zabotinskii reaction patterns. In 1977, Sivashinsky [2] independently used the above equations to model thermal diffusion instabilities observed in flame fronts. Among other applications, the K-S equation appeared in the area of falling liquid films [3], interfacial instabilities between two viscous fluids [4] and in the study of flames [5-7].

Recently, analytical and numerical studies of the dynamics of the K-S equation for different values of the viscosity have revealed the existence of a variety of interesting behaviors, including: steady states, periodic waves, traveling waves, beating waves, homoclinic and heteroclinic orbits, and chaos [8-12]. In addition, it has been shown that the dynamics of the K-S equation can be characterized by a small number of degrees of freedom [13], and therefore, the existence of an inertial manifold is established [14-15].

From a control viewpoint, the K-S equation with periodic boundary conditions was addressed by Liu and Krstić [16], Armaou and Christofides [17], Christofides and Armaou [18], and Yiming and Christofides [19]. In [16], a nonlinear boundary feedback controller was proposed to enhance the rate of convergence to the spatially uniform steady state of the K-S equation when the steady state is naturally stable. On the other hand in [17], linear finite-dimensional output feedback controllers are synthesized to achieve stabilization of the steady state solutions of the K-S equation. In [18], global exponential stabilization of the steady state solution of the K-S equation is addressed via distributed static output feedback control. In both studies of [17] and [18], a set of ordinary differential equations was derived using the Fourier-Galerkin method to mimic the dynamics of the K-S equation. In [19], optimal actuator/sensor placement for nonlinear control of the K-S equation was discussed.

In this paper, we address the control problem of the K-S equation via the Karhunen-Loéve-Galerkin method. First, we show that when the K-L Galerkin method is used, a set of ODEs is obtained with lower dimension than the set of ODEs obtained by using the Fourier-Galerkin approach. Then the reduced set of ODEs is controlled using three different control schemes. The controlled system is simulated, and the simulation results indicate that the proposed control schemes work well.

The remainder of this paper is organized as follows: In Section 2, we describe the K-L decomposition. Section 3 presents two numerical simulation results: the first is obtained by integrating the one dimensional K-S equation, and the second simulation is obtained by integrating the ODE system representing the one dimensional K-S equation extracted from a K-L Galerkin projection. Section 4 presents a feedback linearization control scheme, which is used to stabilize the K-S equation to a stable solution; simulations results are given to illustrate the developed theory. Section 5 gives an improved version of the feedback linearization controller developed in Section 4. Section 6 presents a control scheme for the K-S equation when a single actuator is used to control the system. Finally, some concluding remarks are given in Section 7.

2. The Karhunen-Loéve Decomposition

The Karhunen-Loéve decomposition has been widely used in many different applications. In the literature, the K-L decomposition is known by different names such as the principal component analysis (PCA) [20], the empirical orthogonal functions [21], the quasi-harmonic modes [22], the singular value decomposition (SVD) [23], the proper orthogonal decomposition (POD) [23], and the Hotteling transform [24]. In many applications, the K-L decomposition was mainly used for data compression and feature identification. In the present work, we exploit the K-L decomposition as an efficient mean of extracting the eigenfunctions of the 1-d Kuramoto-Sivashinsky partial differential equation with the task of representing this equation with a reduced number of ordinary differential equations. The following gives a detailed description of the K-L decomposition.

Let $u(x,t)$ describes data created from a numerical simulation of a partial differential equation such as the 1-d Kuramoto-Sivashinsky equation. Suppose that we take instantaneous measures of u at times t_n, $n = 1, 2, \ldots, N$. We will call these measures $\{u_n\}_{n=1}^{N}$ snapshots. The issue under consideration is how to obtain coherent structures or eigenfunctions $\psi_n' s$ from these snapshots $\{u_n\}_{n=1}^{N}$. These eigenfunctions will be chosen such that the projection of the data set onto all possible functions $\psi(x)$, given by

$$\lambda = \frac{\langle(\psi, u_n)^2\rangle}{(\psi, \psi)}, \tag{4}$$

is maximal.

In equation (4), $(.,.)$ denotes the scalar product given by

$$(\psi, u_n) = \int_\Omega \psi(x) u_n(x) dx, \tag{5}$$

where $\Omega = [0, 2\pi]$, and $\langle . \rangle$ denotes the ensemble average which is given by

$$\langle(\psi, u_n)^2\rangle = \frac{1}{N}\sum_{n=1}^{N}(\psi, u_n)^2. \tag{6}$$

It should be noted that if we normalize $(\psi, \psi) = 1$, then equation (4) will be written as:

$$\begin{aligned} \lambda &= \langle(\psi, u_n)^2\rangle & (7)\\ &= \langle \int_\Omega \psi(x) u_n(x) dx \int_\Omega \psi(x') u_n(x') dx' \rangle & (8)\\ &= \int_\Omega \left\{ \int_\Omega \langle u_n(x) u_n(x')\rangle \psi(x) dx \right\} \psi(x') dx'. & (9) \end{aligned}$$

Introducing the two-point correlation function defined as,

$$K(x, x') = \langle u_n(x) u_n(x') \rangle = \frac{1}{N}\sum_{n=1}^{N} u_n(x) u_n(x'). \tag{10}$$

Equation (9) can be expressed as,

$$\lambda = \int_\Omega \left\{ \int_\Omega K(x, x')\psi(x') dx' \right\} \psi(x) dx = \left(\int_\Omega K(x, x')\psi(x') dx', \psi(x)\right). \tag{11}$$

Equation (11) can be easily reduced to the following eigenvalue problem:

$$\int_\Omega K(x, x')\psi(x') dx' = \lambda \psi(x). \tag{12}$$

Thus, the functions that maximize λ of equation (4) are the eigenfunctions of the Fredholm-type integral given by equation (12) corresponding to the largest eigenvalue. The kernel, K, as represented by equation (10) is degenerate and as a result it has eigenfunctions of the form,

$$\psi(x) = \sum_{m=1}^{N} A_m u_m(x), \tag{13}$$

where the constants A_m remain to be found. Substituting equation (13) into equation (12) yields,

$$C_{nm} A_m = \lambda A_n, \tag{14}$$

where,

$$C_{nm} = \frac{1}{N} \int_{\Omega} u_n(x') u_m(x') dx'. \tag{15}$$

The matrix C_{nm} is symmetric, its eigenvalues λ_i, $(i = 1, \ldots, N)$ are real and its eigenvectors, ϕ_i, $(i = 1, \ldots, N)$ form a complete orthogonal set [25]. Thus, the orthogonal eigenfunctions of the data are defined as:

$$\psi_k = \sum_{m=1}^{N} \phi_m^{[k]} u_m, \qquad k = 1, \ldots, N, \tag{16}$$

where $\phi_m^{[k]} = A_m$ is the m-th component of the k-th eigenvector. Lumley [23] refers to these eigenfunctions as coherent structures of the data. A characterization of the eigenfunctions is that they form an optimal basis for the expansion of a spatio-temporal data set.

For each eigenfunction ψ_k, an energy percentage E_k based on the eigenfunctions associated eigenvalue λ_k is assigned, i.e.,

$$E_k = \frac{\lambda_k}{E}, \tag{17}$$

where,

$$E = \sum_{i=1}^{N} \lambda_i. \tag{18}$$

Assuming that the eigenvalues are sorted from the largest to the smallest, then we have an ordering of the eigenfunctions from the most energetic to the least energetic. Hence, we can reconstruct any sample snapshot using the eigenfunctions such that,

$$u(x,t) = \sum_{n=1}^{N} a_n(t)\psi_n(x), \tag{19}$$

where the coefficients a_n's are computed from the projection of the sample vector onto an eigenfunction:

$$a_n = \frac{(u, \psi_n)}{(\psi_n, \psi_n)}. \tag{20}$$

Using only the first M most energetic eigenfunctions ($M \ll N$), we can construct an approximation of the data such that,

$$u(x,t) \approx \sum_{n=1}^{M} a_n(t)\psi_n(x). \tag{21}$$

In the next section, the K-L decomposition described above will be applied to the 1-d Kuramoto-Sivashinsky equation.

3. The 1-d Kuramuto-Sivashinsky Equation

The 1-d Kuramoto-Sivashinsky (K-S) equation (1) can be transformed into the following equation:

$$\frac{\partial u}{\partial t} + 4\frac{\partial^4 u}{\partial x^4} + \alpha\left(\frac{\partial^2 u}{\partial x^2} + \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2\right) = 0; \qquad 0 \le x \le 2\pi, \tag{22}$$

by setting $\tilde{t} = \frac{\nu t}{4}$ and $\alpha = \frac{4}{\nu}$. Equation (22) has been the subject of recent research studies [8-15]. Hyman et al [8] analyzed numerically the dynamics of the K-S equation for different values of the bifurcation parameter α. Kirby and Armbruster [9] used the K-L decomposition as a tool to analyze the complex spatio-temporal structures of the K-S equation for four different values of α (i.e., $\alpha = 17.75$, 68, 84.25 and 87). For $\alpha = 17.75$, the dynamics of the K-S equation consists of a heteroclinic cycle between fixed points, which are almost pure Fourier modes of the form Ae^{2ix} [10, 15]. At $\alpha = 68$, a complicated regime described by bursting events was observed. However, when $\alpha = 84.25$, periodic solutions corresponding to a limit cycle in phase space appeared to bifurcate to

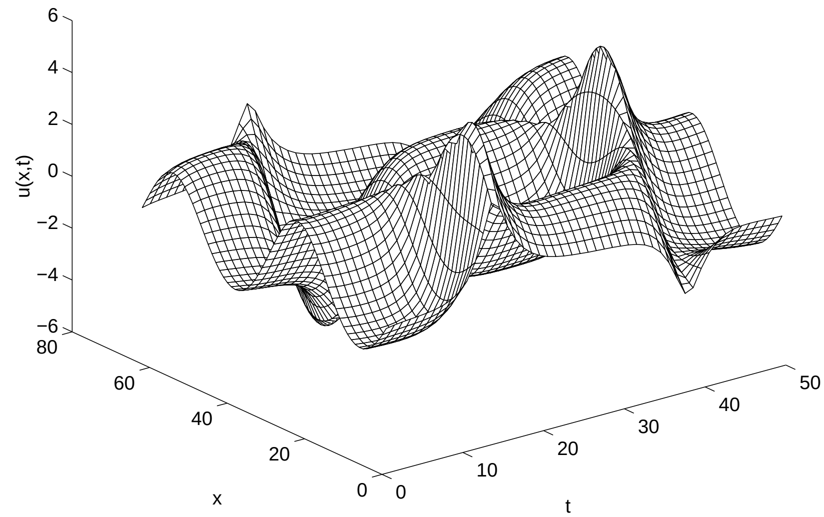

Figure 1: Numerical simulations of the K-S equation at $\alpha = 17.75$

modulated traveling wave at $\alpha = 87$ [9]. A nonlinear dimensionality reduction of the K-S equation was investigated by Smaoui [11]; Smaoui and Al-Enzi [12] proposed a neural network model of the K-S equation for $\alpha = 17.75$.

Since the objective of this work is to control the dynamics of the regime corresponding to $\alpha = 17.75$, we numerically compute the time series of the 1-d spatial solution $u(x,t)$ of the K-S equation using a pseudo-spectral Galerkin method with the "slaved-frog" scheme [26] used as a temporal scheme (see Figure 1). Looking carefully at Figure 1, one can see that it represents two states: one state between the two bursts, and the other state is outside the two bursts.

The numerical simulations data described in Figure 1 was decomposed using the KLTOOL software [27]. Three K-L eigenfunctions capturing 99.2 % of the total energy were extracted (see Figure 2). The first eigenfunction ψ_1 captures 57.6% of the energy, the second and the third eigenfunctions capture 26.2% and 15.4% of the energy, respectively. In phase space concept, the dynamics of this regime is represented by a heteroclinic orbit connecting the two unstable fixed points; this heteroclinic orbit is called a homoclinic cycle [15].

The data coefficients a_n, as defined by equation (20), are computed for this regime (i.e., $\alpha = 17.75$). Figure 3 shows the amplitudes of the data coefficients

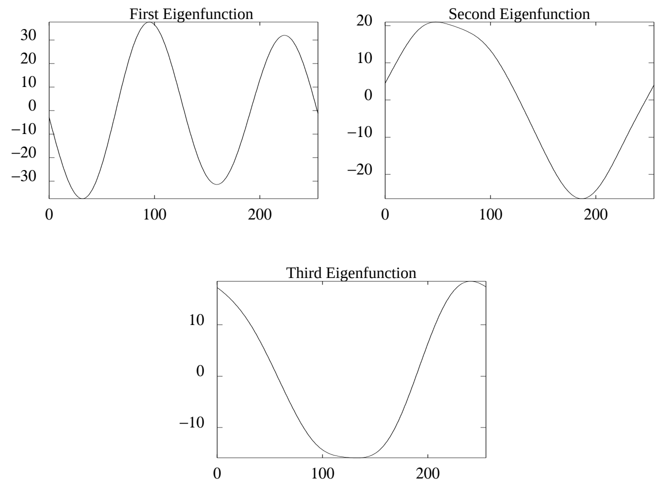

Figure 2: The three most energetic K-L eigenfunctions of the numerical simulations presented in Figure 1

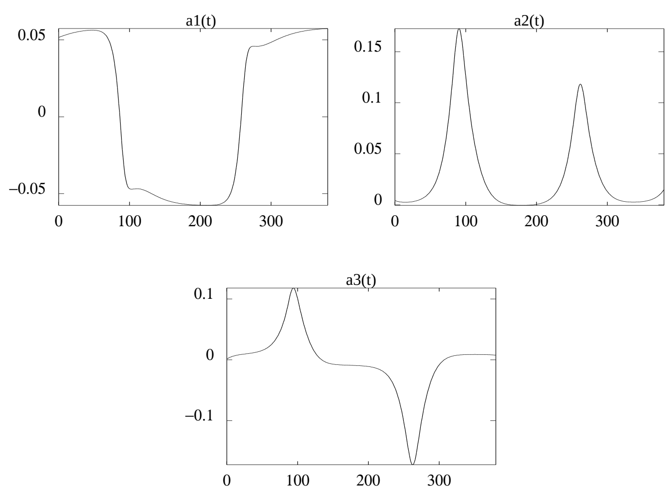

Figure 3: The first three K-L data coefficients associated with the most energetic eigenfunctions at $\alpha = 17.75$

versus time. Using the obtained eigenfunctions and the data coefficients, an approximation of the original simulation can be easily obtained using equation (21) (see Figure 4).

The exact solution $u(x,t)$ of the K-S equation can be expanded in terms of the K-L eigenfunctions ψ_n, as follows:

$$u(x,t) = \sum_{n=1}^{N} a_n(t)\psi_n(x), \tag{23}$$

where

$$\psi_n(x) = \sum_{k=-H}^{H} c_{k,n}e^{ikx} \tag{24}$$

are the K-L eigenfunctions, and H is an integer, which depends on the spatial discretization of the $\psi_n's$.

The derivation of the system of ordinary differential equations based on the K-L analysis proceeds by substituting equation (23) into equation (22) and differentiating, we obtain

$$\sum_{n=1}^{N} \dot{a}_n(t)\psi_n(x) + 4\sum_{n=1}^{N} a_n(t)\psi_n^{(4)}(x) + \alpha\left(\sum_{n=1}^{N} a_n(t)\psi_n''(x) + \frac{1}{2}\left(\sum_{n=1}^{N} a_n(t)\psi_n'(x)\right)^2\right) = 0, \tag{25}$$

where $\dot{a}_n(t)$ denotes differentiation of $a_n(t)$ with respect to t, and $\psi_n'(x)$ represents the differentiation of $\psi_n(x)$ with respect to x.

Multiplying equation (25) by ψ_m, and integrating from 0 to 2π, and applying the orthogonality condition of the ψ_n,

$$(\psi_n, \psi_m) = \begin{cases} 0 & n \neq m, \\ c_n & n = m, \ (c_n \text{ is a constant}), \end{cases} \tag{26}$$

results in

$$\dot{a}_m(t) = -4\sum_{n=1}^{N} a_n(\psi_m, \psi_n^{(4)}) - \alpha\sum_{n=1}^{N} a_n(\psi_m, \psi_n'') - \frac{\alpha}{2}\sum_{n,k=1}^{N} a_n a_k(\psi_m, \psi_k'\psi_n'). \tag{27}$$

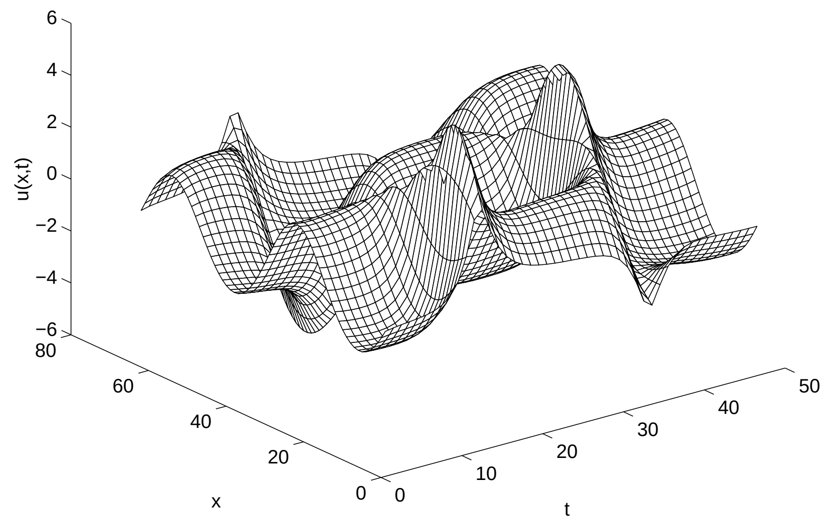

Figure 4: The approximation of the numerical simulation data of the K-S equation at $\alpha = 17.75$ using the three most energetic eigenfunctions

Now, using the representation of the K-L eigenfunctions in terms of their Fourier coefficients as given in equation (24) results in the following system of ODEs:

$$\dot{a}_m(t) = \sum_{n=1}^{N} \sum_{k=-H}^{H} (\alpha k^2 - 4k^4) c_{k,m} \overline{c}_{k,n} a_n + \frac{1}{2}\alpha \sum_{n,l=1}^{N} \sum_{k=-H}^{H} k k' c_{k+k',m} \overline{c}_{k,l} \overline{c}_{k',n} a_n a_l, \quad (28)$$

where $|k + k'| \leq N$ and $H = 32$.

Since the three most energetic eigenfunctions of the K-S equation capture 99.2% of the total energy, M in equation (21) is taken to be 3. The system of three ODEs obtained using the three most energetic eigenfunctions for $\alpha = 17.75$ is given in Appendix A. This system of ODEs was numerically integrated using the *DSTOOL* software [28]. To prevent numerical instabilities, we added cubic damping terms to each equation of the ODE system, as suggested by Kirby and Armbruster [9]. The time series solutions obtained are presented in Figure 5. Comparing the time series solutions obtained from the system of ODEs to the original time series solutions of the PDE given by equation (22), one can see that their dynamical behaviors are very similar (see Figure 1 and Figure 5).

The rest of the paper deals with the control of the K-S equation given by equation (22). The objective is to stabilize the dynamics of the K-S equation to a stable solution.

4. Feedback Linearization Control Scheme for the K-S Equation

The 1-d Kuramoto-Sivashinsky equation given by equation (22) with control can be written as,

$$\frac{\partial u}{\partial t} = -4\frac{\partial^4 u}{\partial x^4} - \alpha \left(\frac{\partial^2 u}{\partial x^2} + \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 \right) + \sum_{j=1}^{n_a} b_j v_j(t); \quad 0 \leq x \leq 2\pi, \quad (29)$$

where $v_j(t)$ is the jth input, b_j is the actuator distribution function, and n_a is the number of actuators.

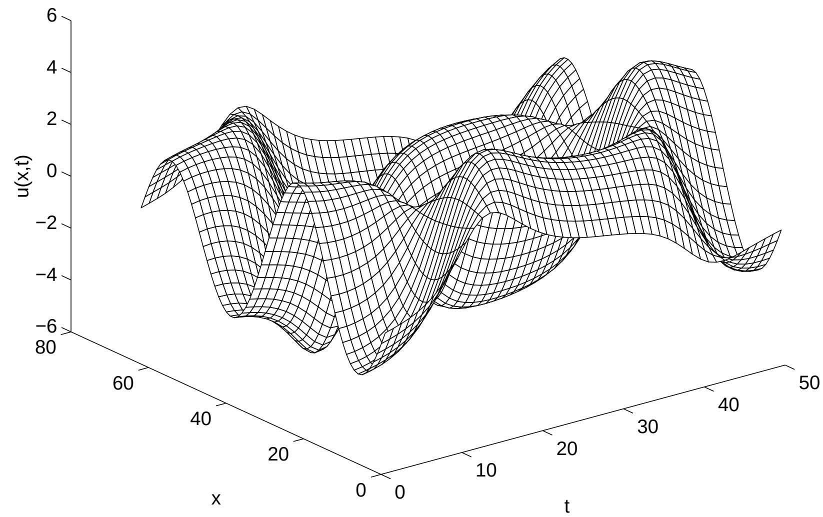

Figure 5: The time series solutions of the system of ODEs obtained via K-L Galerkin projection of the K-S equation

As discussed in the previous sections, the solutions of equation (29) $u(x,t)$ can be expanded in terms of the K-L eigenfunctions ψ_n as follows:

$$u(x,t) = \sum_{n=1}^{N} a_n(t)\psi_n(x)\,. \tag{30}$$

The derivation of the system of ordinary differential equations results in

$$\dot{a}_m(t) = \sum_{n=1}^{N} \sum_{k=-H}^{H} (\alpha k^2 - 4k^4) c_{k,m}\overline{c}_{k,n} a_n + \frac{1}{2}\alpha \sum_{n,l=1}^{N} \sum_{k=-H}^{H} kk' c_{k+k',m}\overline{c}_{k,l}\overline{c}_{k',n} a_n a_l + \sum_{j=1}^{n_a} \beta_j^m v_j(t), \tag{31}$$

where $|k+k'| \leq N$, $H = 32$ and

$$\beta_j^m = \int_0^{2\pi} \psi_m(x) b_j dx. \tag{32}$$

Assuming that the number of actuators $n_a = 3$ and approximating the dynamics of the K-S equation with three eigenfunctions, we obtain the following three ODEs:

$$\begin{aligned} \dot{a}_1(t) &= f_1(t) + w_1(t)\,, \\ \dot{a}_2(t) &= f_2(t) + w_2(t)\,, \\ \dot{a}_3(t) &= f_3(t) + w_3(t)\,, \end{aligned} \tag{33}$$

or

$$\dot{a}_j(t) = f_j(t) + w_j(t) \qquad (j = 1, 2, 3), \tag{34}$$

where

$$f_j = \sum_{n=1}^{3} \sum_{k=-H}^{H} (\alpha k^2 - 4k^4) c_{k,j} \overline{c}_{k,n} a_n + \frac{1}{2}\alpha \sum_{n,l=1}^{3} \sum_{k=-H}^{H} kk' c_{k+k',j} \overline{c}_{k,l} \overline{c}_{k',n} a_n a_l \tag{35}$$

and

$$w_j(t) = \beta_1^j v_1(t) + \beta_2^j v_2(t) + \beta_3^j v_3(t)\,, \tag{36}$$

with $j = 1, \cdots, 3$.

Proposition 1. *Let γ_1, γ_2 and γ_3 be three positive scalars. The feedback linearization control scheme,*

$$w_j(t) = -f_j(t) - \gamma_j a_j(t) \quad (j = 1, 2, 3) \tag{37}$$

renders the ODE system in equation (34) exponentially stable.

Proof. Substituting equation (37) into equation (34) leads to the following closed loop system:

$$\dot{a}_j(t) = -\gamma_j a_j(t) \quad (j = 1, 2, 3). \tag{38}$$

The solution of equation (38) is:

$$a_j(t) = e^{(-\gamma_j\, t)} a_j(0) \quad (j = 1, 2, 3). \tag{39}$$

Since $\gamma_j > 0$, then $a_j(t)$ $(j = 1, 2, 3)$ converges exponentially to zero as $t \to \infty$. Hence the ODE system (34) with controller equation (37) is exponentially stable. □

Since the goal of the control is to stabilize the dynamics of the K-S regime for $\alpha = 17.75$ at one of the fixed points, we apply the control in such a way that the long term dynamics of this regime is stable at one of the fixed points. This is achieved by making the following change of variables,

$$\tilde{a}_j(t) = a_j(t) + c_j \quad (j = 1, 2, 3)\,, \tag{40}$$

where c_j are the data coefficients corresponding to one of the two solutions.

Let the matrix M_β be such,

$$M_\beta = \begin{bmatrix} \beta_1^1 & \beta_2^1 & \beta_3^1 \\ \beta_1^2 & \beta_2^2 & \beta_3^2 \\ \beta_1^3 & \beta_2^3 & \beta_3^3 \end{bmatrix}\,. \tag{41}$$

Assuming that M_β is not a singular matrix, then the controllers $v_1(t)$, $v_2(t)$ and $v_3(t)$ in equation (29) are such:

$$\begin{bmatrix} v_1(t) \\ v_2(t) \\ v_3(t) \end{bmatrix} = M_\beta^{-1} \begin{bmatrix} -f_1(t) - \gamma_1 a_1(t) \\ -f_2(t) - \gamma_2 a_2(t) \\ -f_3(t) - \gamma_3 a_3(t) \end{bmatrix}\,. \tag{42}$$

The K-S equation given by equation (29) with the controller given by equation (37) was simulated. The simulation results are shown in Figure 6. It can be seen from the simulation results that the trajectories of $a_j(t)$ $(j = 1, 2, 3)$ converge to the fixed point $c = [c_1\ c_2\ c_3]^T$.

5. An Improved Control Scheme for the K-S Equation

We will again assume that the number of actuators $n_a = 3$ and $M = 3$. The control scheme given in equation (37) cancels all the terms of $f_j(t)$ $\quad (j = 1, 2, 3)$ in equation (34). However, some of the elements of $f_j(t)$ are stable and hence there is no need to cancel them. In the following, we will design an improved version of the feedback controller given in equation (37).

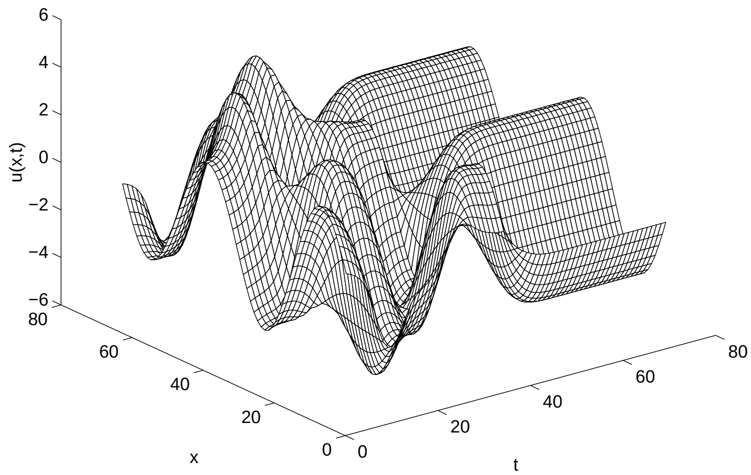

Figure 6: The time series solutions of the K-S equation at $\alpha = 17.75$ when the feedback linearization control scheme is applied

Let,

$$A_o = \begin{bmatrix} \sum_{k=-H}^{H} -4k^4 c_{k,1}\bar{c}_{k,1} & \sum_{k=-H}^{H} -4k^4 c_{k,1}\bar{c}_{k,2} & \sum_{k=-H}^{H} -4k^4 c_{k,1}\bar{c}_{k,3} \\ \sum_{k=-H}^{H} -4k^4 c_{k,2}\bar{c}_{k,1} & \sum_{k=-H}^{H} -4k^4 c_{k,2}\bar{c}_{k,2} & \sum_{k=-H}^{H} -4k^4 c_{k,2}\bar{c}_{k,3} \\ \sum_{k=-H}^{H} -4k^4 c_{k,3}\bar{c}_{k,1} & \sum_{k=-H}^{H} -4k^4 c_{k,3}\bar{c}_{k,2} & \sum_{k=-H}^{H} -4k^4 c_{k,3}\bar{c}_{k,3} \end{bmatrix} \quad (43)$$

and

$$a(t) = \begin{bmatrix} a_1(t) \\ a_2(t) \\ a_3(t) \end{bmatrix}, \quad \tilde{f}(t) = \begin{bmatrix} \tilde{f}_1(t) \\ \tilde{f}_2(t) \\ \tilde{f}_3(t) \end{bmatrix}, \quad w(t) = \begin{bmatrix} w_1(t) \\ w_2(t) \\ w_3(t) \end{bmatrix}, \quad (44)$$

where

$$\tilde{f}_j(t) = \alpha \sum_{n=1}^{M} \sum_{k=-H}^{H} k^2 c_{k,j}\bar{c}_{k,n} a_n + \frac{1}{2}\alpha \sum_{n,l=1}^{M} \sum_{k=-H}^{H} kk' c_{k+k',j}\bar{c}_{k,l}\bar{c}_{k',n} a_n a_l \,, \quad (45)$$

with $j = 1, \cdots, 3$. The system of ODEs given in equation (34) can be written as

$$\dot{a}(t) = A_o a(t) + \tilde{f}(t) + w(t)\,. \tag{46}$$

The matrix A_o can be easily computed from the system of ODEs given in Appendix A.

Remark. It can be easily checked that the matrix A_o is negative definite.

Proposition 2. *The controller,*

$$w(t) = -\tilde{f}(t) \tag{47}$$

renders the ODE system in equation (46) exponentially stable.

Proof. Substituting the controller given by equation (47) into equation (46) leads to:

$$\dot{a}(t) = A_o a(t)\,. \tag{48}$$

The solution of the above equation is:

$$a(t) = e^{A_o t} a(0)\,. \tag{49}$$

Since the matrix A_o is negative definite, then $a(t)$ converges exponentially to zero as $t \to \infty$. Hence the ODE system with the controller given by equation (47) is exponentially stable. □

Assuming that the matrix M_β in equation (41) is not a singular matrix, then the control signals $v_1(t)$, $v_2(t)$ and $v_3(t)$ in equation (29) are:

$$\begin{bmatrix} v_1(t) \\ v_2(t) \\ v_3(t) \end{bmatrix} = M_\beta^{-1} \begin{bmatrix} -\tilde{f}_1(t) \\ -\tilde{f}_2(t) \\ -\tilde{f}_3(t) \end{bmatrix}. \tag{50}$$

The K-S equation (29) when the controller equation (47) is used was simulated. The results are similar to those shown in Figure 6.

6. A Control Scheme for the K-S Equation Using a Single Actuator

Section 4 and Section 5 addressed the control problem of the K-S equation when the system uses three actuators. This section discusses the control of the K-S equation when only one actuator is used.

The 1-d Kuramoto-Sivashinsky equation with one actuator can be written as,

$$\frac{\partial u}{\partial t} = -4\frac{\partial^4 u}{\partial x^4} - \alpha\left(\frac{\partial^2 u}{\partial x^2} + \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2\right) + bv(t)\,, \quad 0 \le x \le 2\pi, \tag{51}$$

where $v(t)$ is the input.

The derivation of the system of ordinary differential equations (ODEs) results in

$$\begin{aligned}\dot{a}_m(t) = &\sum_{n=1}^{N}\sum_{k=-H}^{H}(\alpha k^2 - 4k^4)c_{k,m}\overline{c}_{k,n}a_n \\ &+ \frac{1}{2}\alpha\sum_{n,l=1}^{N}\sum_{k=-H}^{H}kk'c_{k+k',m}\overline{c}_{k,l}\overline{c}_{k',n}a_n a_l + \beta^m v(t),\end{aligned} \tag{52}$$

where

$$\beta^m = \int_0^{2\pi}\psi_m(x)bdx. \tag{53}$$

Hence the behavior of the K-S equation can be approximated by the following three ODEs:

$$\begin{aligned}\dot{a}_1(t) &= f_1(t) + \beta^1 v(t)\,,\\ \dot{a}_2(t) &= f_2(t) + \beta^2 v(t)\,,\\ \dot{a}_3(t) &= f_3(t) + \beta^3 v(t)\,,\end{aligned} \tag{54}$$

where f_j $(j = 1, \ldots, 3)$ is given by equation (35). Then the ODE system (54) can be written as

$$\dot{a}(t) = Aa(t) + F(t) + Bv(t)\,, \tag{55}$$

and

$$a(t) = \begin{bmatrix} a_1(t)\\ a_2(t)\\ a_3(t)\end{bmatrix}, \quad F(t) = \begin{bmatrix} a^T(t)M_1 a(t)\\ a^T(t)M_2 a(t)\\ a^T(t)M_3 a(t)\end{bmatrix}, \quad B = \begin{bmatrix}\beta^1\\ \beta^2\\ \beta^3\end{bmatrix}, \tag{56}$$

where the matrices A, M_1, M_2 and M_3 can be found from the ODE system given in Appendix A.

Remark. The quadratic term $F(t)$ in equation (56) is bounded such that

$$||F(t)|| \leq \delta ||a||^2 \,, \tag{57}$$

where $\delta \geq \sqrt{||M_1||^2 + ||M_1||^2 + ||M_1||^2}$. Note that $||.||$ denotes the Euclidian norm.

Remark. It can be checked that the pair (A, B) is controllable [29] because the matrix $\mathcal{C} = [B \;\; AB \;\;\; A^2B]$ has full rank. Therefore, the poles (i.e., eigenvalues) of the closed loop system can be selected such that the response of the system is as desired.

Let, the gain matrix $\mathcal{K}$ be such that the matrix $A_c = A - B\mathcal{K}$ has the desired eigenvalues. Let the symmetric positive definite matrix P be the solution of the following algebraic Riccati equation,

$$A_c^T P + P A_c = -Q \,, \tag{58}$$

where Q is a symmetric positive definite matrix.

Proposition 3. *The controller,*

$$v(t) = -\mathcal{K} a(t) + v_N(t) \,, \tag{59}$$

with

$$v_N(t) = -\frac{\delta ||a(t)||^3 \; ||P|| \; B^T P a(t)}{|B^T P a(t)|^2} \,, \tag{60}$$

renders the ODE system in equation (55) asymptotically stable.

Proof. The closed loop system can be written as,

$$\begin{aligned} \dot{a}(t) &= Aa(t) + F(t) + Bv(t) \\ &= Aa(t) + F(t) + B(-\mathcal{K}a(t) + v_N(t)) \\ &= (A - B\mathcal{K})a(t) + F(t) + Bv_N(t) \\ &= A_c a(t) + F(t) + Bv_N(t) \,. \end{aligned} \tag{61}$$

Consider the following Lyapunov function candidate

$$V = a^T(t) P a(t) \,. \tag{62}$$

Note that $V > 0$ for $a(t) \neq 0$ and $V = 0$ for $a(t) = 0$. Taking the derivative of V with respect to time and using equation (61) and equations (58)-(60), it follows that:

$$\begin{aligned}
\dot{V} &= \dot{a}^T P a + a^T P \dot{a} \\
&= (A_c a + F + B v_N)^T P a + a^T P (A_c a + F + B v_N) \\
&= a^T (A_c^T P + P A_c) a + 2 a^T P B v_N + 2 F^T P a \\
&= -a^T Q a + 2 a^T P B v_N + 2 F^T P a \\
&= -a^T Q a - 2 a^T P B \frac{\delta ||a||^3 \; ||P|| \; B^T P a}{|B^T P a|^2} + 2 F^T P a \\
&= -a^T Q a - 2\delta ||a||^3 \; ||P|| + 2 F^T P a \\
&\leq -a^T Q a \\
&\leq -\lambda_{min}(Q) ||a||^2,
\end{aligned} \tag{63}$$

where λ_{min} denotes the minimum eigenvalue of the matrix Q. Hence the controller scheme given by equations (59)-(60) guarantees the asymptotic stability of the K-S equation given by equation (55). □

7. Concluding Remarks

In this paper, we addressed the control problem of the one dimensional Kuramoto-Sivashinsky equation. First, the Karhunen-Loéve decomposition is used to extract the coherent structures of the dynamical behavior of the K-S equation represented in phase space by a heteroclinic connection at $\alpha = 17.75$. Then a set of ordinary differential equations is obtained through the K-L Galerkin projection using the three most energetic eigenfunctions of the numerical simulation of the K-S equation. The reduced set of ODEs is controlled using three different control schemes. Simulation results are presented to support the analytical results. Controlling the dynamics of the K-S equation for different values of α will be the subject of future studies.

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Appendix A

The following system of ODEs was derived by applying the K-L Galerkin projection on the three most energetic eigenfunctions of the numerical simulation of the K-S equation at $\alpha = 17.75$.

$$\begin{aligned}\dot{a}_1 = &-64.0458a_1 + 4.00002\alpha a_1 - 0.0217628\alpha a_1^2 - 0.0968048a_2 \\ &+ 0.00135795\alpha a_2 - 0.251959\alpha a_1 a_2 - 0.00365152\alpha a_2^2 + 1.04431a_3 \\ &- 0.0542265\alpha a_3 + 0.00325813\alpha a_1 a_3 + 0.883613\alpha a_2 a_3 \\ &+ 0.00396569\alpha a_3^2 - d a_1^3\end{aligned}$$

$$\begin{aligned}\dot{a}_2 = &-0.0968048a_1 + 0.00135795\alpha a_1 - 0.0597984\alpha a_1^2 \\ &- 4.90446a_2 + 1.01447\alpha a_2 + 0.0168235\alpha a_1 a_2 \\ &- 0.00916079\alpha a_2^2 + 0.011195a_3 - 0.000163096\alpha a_3 \\ &- 2.06671\alpha a_1 a_3 - 0.000149962\alpha a_2 a_3 + 0.0396351\alpha a_3^2 - d a_2^3\end{aligned}$$

$$\begin{aligned}\dot{a}_3 = &\ 1.04431a_1 - 0.0542265\alpha a_1 - 0.00134954\alpha a_1^2 \\ &+ 0.011195a_2 - 0.000163096\alpha a_2 - 2.08204\alpha a_1 a_2 + 0.000115257\alpha a_2^2 \\ &- 4.16438a_3 + 1.00408\alpha a_3 - 0.0194354\alpha a_1 a_3 + 0.012863\alpha a_2 a_3 \\ &+ 0.000397914\alpha a_3^2 - d a_3^3\end{aligned}$$