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TURNPIKE PROPERTY FOR INFINITE DIMENSIONAL CONVEX DISCRETE-TIME CONTROL SYSTEMS IN A BANACH SPACE

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Abstract: In this work we study the structure of "approximate" solutions for an infinite dimensional discrete-time optimal control problem determined by a convex function $v : K \times K \to R^1$, where K is a convex closed bounded subset of a Banach space. We show that for a generic function v there exists $y_v \in K$ such that each "approximate" optimal solution $\{x_i\}_{i=0}^n \subset K$ is a contained in a small neighborhood of y_v for all $i \in \{N, \dots, n-N\}$, where N is a constant, which depends on the neighborhood and does not depend on n. This result is a generalization of the main result of Zaslavski [*Journal of Convex Analysis*, **5** (1998), 237-248], which was established for convex uniformly continuous functions.

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1. Introduction

Let $(X, ||\cdot||)$ be a Banach space and let $K \subset X$ be a nonempty closed convex bounded set. Denote by $\mathfrak{A}$ the set of all bounded convex functions $v : K \times K \to R^1$, which are continuous at a point (x, x) for any $x \in K$. Denote by $\mathfrak{A}_l$ the set of all lower semicontinuous functions $v \in \mathfrak{A}$, by $\mathfrak{A}_c$ the set of all continuous

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functions $v \in \mathfrak{A}$ and by $\mathfrak{A}_u$ the set of all functions $v \in \mathfrak{A}$ which satisfy the following assumption:

Uniform Continuity. For each $\epsilon > 0$ there exists $\delta > 0$ such that for each $x_1, x_2, y_1, y_2 \in K$ satisfying $||x_i - y_i|| \leq \delta$, $i = 1, 2$ the relation $|v(x_1, x_2) - v(y_1, y_2)| \leq \epsilon$ holds.

We equip the space $\mathfrak{A}$ with the metric

$$\rho(u, v) = \sup\{|v(x, y) - u(x, y)| : x, y \in K\}, \quad u, v \in \mathfrak{A}. \tag{1.1}$$

Evidently the metric space $(\mathfrak{A}, \rho)$ is complete and $\mathfrak{A}_l$, $\mathfrak{A}_c$ and $\mathfrak{A}_u$ are closed subsets of $(\mathfrak{A}, \rho)$. We equip the sets $\mathfrak{A}_l$, $\mathfrak{A}_c$ and $\mathfrak{A}_u$ with the metric ρ.

In this paper we investigate the structure of "approximate" solutions of optimization problems

$$\sum_{i=0}^{n-1} v(x_i, x_{i+1}) \to \min, \tag{P}$$

$$\{x_i\}_{i=0}^{n} \subset K, \quad x_0 = y, \; x_n = z\,,$$

where $v \in \mathfrak{A}$, $y, z \in K$ and an integer $n \geq 1$.

The interest in these discrete-time optimal problems stems from the study of various optimization problems, which can be reduced to this framework, e.g., continuous-time control systems, which are represented by ordinary differential equations, whose cost integrand contains a discounting factor (see Leizarowitz [3]), the infinite-horizon control problem of minimizing $\int_0^T L(z, z')dt$ as $T \to \infty$ in Leizarowitz [4] and the analysis of a long slender bar of a polymeric material under tension in Leizarowitz and Mizel [5], Marcus and Zaslavski [10], Marcus and Zaslavski [11], Zaslavski [18] and Zaslavski [19]. Similar optimization problems are also considered in mathematical economics (see Dzalilov et al [2], Makarov et al [6], Makarov and Rubinov [7], Mamedov and Pehlivan [8], Mamedov and Pehlivan [9], McKenzie [12], Radner [13], Rubinov [14], Rubinov [15] and Samuelson [16]).

In Zaslavski [20] we showed that for a generic function $v \in \mathfrak{A}_u$ the following property holds:

(TP) There is $y_v \in K$ such that for all large enough n and each $y, z \in K$ an "approximate" solution $\{x_i\}_{i=0}^{n}$ of problem (P) is contained in a small neighborhood of y_v for all $i \in \{N, \dots, n - N\}$, where N is a constant, which depends on the neighborhood and does not depend on n.

This phenomenon, which is called the turnpike property (TP) is well known in mathematical economics. The term was first coined by Samuelson in 1958 (see Samuelson [16]), where he showed that an efficient expanding economy would spend most of the time in the vicinity of a balanced equilibrium path (also called a von Neumann path). This property was further investigated in in mathematical economics (see Dzalilov et al [1], Dzalilov et al [2], Makarov et al [6], Makarov and Rubinov [7], Mamedov and Pehlivan [8], Mamedov and Pehlivan [9], McKenzie [12], Radner [13], Rubinov [14] and Rubinov [15]) for optimal trajectories of models of economic dynamics. A related weak version of the turnpike property was studied in Zaslavski [17] with a nonconvex function $v : K \times K \to R^1$ and a compact metric space K.

When we say that a certain property holds for a generic element of a complete metric space Y we mean that the set of points, which have this property contains a G_δ everywhere dense subset of Y. Such an approach, when a certain property is investigated for the whole space Y and not just for a single point in Y, has already been successfully applied in many areas of analysis. In this paper we generalize the main result of Zaslavski [20] and show that the turnpike property holds for a generic function $v \in \mathfrak{A}$.

In almost all studies of discrete time control systems the turnpike property was considered for a single cost function v and a space of states K, which was a compact convex set in a finite dimensional space. In these studies the compactness of K playes an important role. Specifically for the optimization problems considered in this paper and in Zaslavski [20] if a function v has the turnpike property then its "turnpike" y_v is a unique solution of the following optimization problem

$$v(x, x) \to \min, \quad x \in K.$$

The existence of a solution of this problem is guaranteed only if K satisfies some compactness assumptions. To obtain the uniqueness of the solution we need additional assumptions on v such as its strict convexity.

In Zaslavski [20] and here, instead of considering the turnpike property for a single cost function v, we investigate it for spaces of all such functions equipped with some natural metric, and show that this property holds for most of these functions. In Zaslavski [20] we established the turnpike property without compactness assumption on the space of states for a generic convex uniform continuous cost function. Note that the uniform continuity assumption is essential for the proof of the main result of Zaslavski [20]. In the present paper we establish the turnpike property for a generic convex cost function without this assumption.

For each $v \in \mathfrak{A}$, each pair of integers $m_1, m_2 > m_1$ and each $y_1, y_2 \in K$ we define

$$\sigma(v, m_1, m_2) = \inf\{\sum_{i=m_1}^{m_2-1} v(z_i, z_{i+1}) : \{z_i\}_{i=m_1}^{m_2} \subset K\}, \tag{1.2}$$

$$\sigma(v, m_1, m_2, y_1, y_2) = \inf\{\sum_{i=m_1}^{m_2-1} v(z_i, z_{i+1}) : \{z_i\}_{i=m_1}^{m_2} \subset K, \\ z_{m_1} = y_1, z_{m_2} = y_2\}, \tag{1.3}$$

and the minimal growth rate

$$\mu(v) = \inf\{\liminf_{N\to\infty} N^{-1} \sum_{i=0}^{N-1} v(z_i, z_{i+1}) : \{z_i\}_{i=0}^{\infty} \subset K\}. \tag{1.4}$$

We will show (see Proposition 2.1) that for any $v \in \mathfrak{A}$,

$$\mu(v) = \inf\{v(z, z) : \ z \in K\}.$$

In this paper we will construct a set $\mathfrak{F}$ ($\mathfrak{F}_l, \mathfrak{F}_c, \mathfrak{F}_u$, respectively), which is a countable intersection of open everywhere dense subsets of $\mathfrak{A}$ ($\mathfrak{A}_l, \mathfrak{A}_c, \mathfrak{A}_u$, respectively) and such that

$$\mathfrak{F}_l \subset \mathfrak{A}_l \cap \mathfrak{F}, \ \mathfrak{F}_c \subset \mathfrak{A}_c \cap \mathfrak{F}, \ \mathfrak{F}_u \subset \mathfrak{A}_u \cap \mathfrak{F}.$$

We will establish the following two theorems.

Theorem 1.1. *Let $v \in \mathfrak{F}$. Then there exists a unique $y_v \in K$ such that $v(y_v, y_v) = \mu(v)$ and the following assertion holds:*

For each $\epsilon > 0$ there exist a neighborhood $\mathfrak{U}$ of v in $\mathfrak{A}$ and $\delta > 0$ such that for each $u \in \mathfrak{U}$ and each $y \in K$ satisfying $u(y, y) \le \mu(u) + \delta$ the relation $||y - y_v|| \le \epsilon$ holds.

Theorem 1.2. *Let $w \in \mathfrak{F}$ and $\epsilon > 0$. Then there exist $\delta \in (0, \epsilon)$, a neighborhood $\mathfrak{U}$ of w in $\mathfrak{A}$ and an integer $N \ge 1$ such that for each $u \in \mathfrak{U}$, each integer $n \ge 2N$ and each sequence $\{x_i\}_{i=0}^{n} \subset K$ satisfying*

$$\sum_{i=0}^{n-1} u(x_i, x_{i+1}) \le \sigma(u, 0, n, x_0, x_n) + \delta$$

there exist $\tau_1 \in \{0, \ldots, N\}$ and $\tau_2 \in \{n - N, \ldots, n\}$ such that

$$||x_t - y_w|| \le \epsilon, \quad t = \tau_1, \ldots, \tau_2.$$

Moreover, if $||x_0 - y_w|| \le \delta$, then $\tau_1 = 0$, and if $||y_w - x_n|| \le \delta$, then $\tau_2 = n$.

2. Preliminary Results

Set

$$D_0 = \sup\{||x|| : x \in K\}. \tag{2.1}$$

For each bounded function $v : K \times K \to R^1$ we set

$$||v|| = \sup\{|v(x,y)| : x, y \in K\}. \tag{2.2}$$

Proposition 2.1. *Let $v \in \mathfrak{A}$. Then*

$$\mu(v) = \inf\{v(z,z) : z \in K\}.$$

Proof. It is not difficult to see that

$$\inf\{v(x,x) : x \in K\} \geq \mu(v) \tag{2.3}$$

and that for each integer $m \geq 1$ and each $y_1, y_2 \in K$,

$$\sigma(v, 0, m, y_1, y_2) - 4||v|| \leq \sigma(v, 0, m) \leq m\mu(v). \tag{2.4}$$

Let $\epsilon > 0$. Choose a natural number $m \geq 4$ for which

$$(4||v|| + 1)m^{-1} < \epsilon. \tag{2.5}$$

There exists a sequence $\{y_i\}_{i=0}^{m} \subset K$ such that

$$y_0 = y_m \text{ and } \sigma(v, 0, m, y_0, y_0) + 1 \geq \sum_{i=0}^{m-1} v(y_i, y_{i+1}). \tag{2.6}$$

By (2.6) and (2.4),

$$\sum_{i=0}^{m-1} v(y_i, y_{i+1}) \leq m\mu(v) + 4||v|| + 1. \tag{2.7}$$

Set

$$z_0 = m^{-1} \sum_{i=0}^{m-1} y_i. \tag{2.8}$$

By (2.5-2.8) and the convexity of v,

$$v(z_0, z_0) = v(m^{-1} \sum_{i=0}^{m-1} (y_i, y_{i+1})) \leq m^{-1} \sum_{i=0}^{m-1} v(y_i, y_{i+1})$$

$$\leq m^{-1}(m\mu(v) + 4||v|| + 1) = \mu(v) + (4||v|| + 1)m^{-1} < \mu(v) + \epsilon$$

and

$$v(z_0, z_0) < \mu(v) + \epsilon.$$

Therefore

$$\inf\{v(z, z) :\ z \in K\} < \mu(v) + \epsilon.$$

Since ϵ is an arbitrary positive number we conclude that

$$\inf\{v(z, z) :\ z \in K\} \leq \mu(v).$$

Together with (2.3) this completes the proof of Proposition 2.1. □

In Zaslavski [20] Proposition 2.1 was proved for $v \in \mathfrak{A}_u$. In the proof we used uniform continuity of v.

For the proofs of the following two propositions see Proposition 2.2 and Proposition 2.3 of Zaslavski [20].

Proposition 2.2. *Let $v \in \mathfrak{A}$, $\epsilon \in (0, 1)$. Then there exist $\delta \in (0, \epsilon)$, $u \in \mathfrak{A}$ and $z_0 \in K$ such that*

$$0 \leq u(x, y) - v(x, y) \leq \epsilon, \quad x, y \in K, \quad \mu(v) + \delta \geq v(z_0, z_0)\,,$$

and for each $y \in K$ satisfying $u(y, y) \leq \mu(u) + \delta$ the inequality $||y - z_0|| \leq \epsilon$ holds.

Moreover, if $v \in \mathfrak{A}_l$ ($\mathfrak{A}_c$, $\mathfrak{A}_u$, respectively), then $u \in \mathfrak{A}_l$ ($\mathfrak{A}_c$, $\mathfrak{A}_u$, respectively).

Proposition 2.3. *There exist sets $\mathfrak{F}^{(0)} \subset \mathfrak{A}$, $\mathfrak{F}_l^{(0)} \subset \mathfrak{A}_l \cap \mathfrak{F}^{(0)}$, $\mathfrak{F}_c^{(0)} \subset \mathfrak{A}_c \cap \mathfrak{F}^{(0)}$ and $\mathfrak{F}_u^{(0)} \subset \mathfrak{A}_u \cap \mathfrak{F}^{(0)}$ such that $\mathfrak{F}^{(0)}$ ($\mathfrak{F}_l^{(0)}, \mathfrak{F}_c^{(0)}$ and $\mathfrak{F}_u^{(0)}$, respectively) is a countable intersection of open everywhere dense subsets of $\mathfrak{A}$ ($\mathfrak{A}_l$, $\mathfrak{A}_c$ and $\mathfrak{A}_u$, respectively) and such that for each $v \in \mathfrak{F}^{(0)}$ the following assertions hold:*

(i) there exists a unique $y_v \in K$ such that $v(y_v, y_v) = \mu(v)$.

(ii) for each $\epsilon > 0$ there exist a neighborhood $\mathfrak{U}$ of v in $\mathfrak{A}$ and $\delta > 0$ such that for each $u \in \mathfrak{U}$ and each $y \in K$ satisfying $u(y, y) \leq \mu(u) + \delta$ the relation $||y - y_v|| \leq \epsilon$ holds.

In Zaslavski [20] we proved these propositions for $v \in \mathfrak{A}_u$ but in the proofs we did not use uniform continuity. The proof of Proposition 2.3 is based on Proposition 2.2.

3. Proofs of Theorem 1.1 and Theorem 1.2

Let the sets $\mathfrak{F}^{(0)}$, $\mathfrak{F}_l^{(0)}$, $\mathfrak{F}_c^{(0)}$ and $\mathfrak{F}_u^{(0)}$ be as guaranteed in Proposition 2.3. For each $w \in \mathfrak{F}^{(0)}$ there exists a unique $y_w \in K$ such that

$$w(y_w, y_w) = \mu(w). \tag{3.1}$$

Let $v \in \mathfrak{F}^{(0)}$ and $\gamma \in (0,1)$. Define

$$v_\gamma(x,y) = v(x,y) + \gamma(||x - y_v|| + ||y - y_v||), \quad x, y \in K. \tag{3.2}$$

Clearly $v_\gamma \in \mathfrak{A}$, if $v \in \mathfrak{A}_l$ ($\mathfrak{A}_c$, $\mathfrak{A}_u$, respectively), then $v_\gamma \in \mathfrak{A}_l$ ($\mathfrak{A}_c$, $\mathfrak{A}_u$, respectively).

Lemma 3.1. *Let $\epsilon \in (0,1)$. Then there exists an integer $n \geq 1$ such that for each sequence $\{x_i\}_{i=0}^n \subset K$, which satisfies*

$$\sum_{i=0}^{n-1} v_\gamma(x_i, x_{i+1}) \leq \sigma(v_\gamma, 0, n, x_0, x_n) + 4\,,$$

there is $j \in \{0, \ldots, n-1\}$ such that

$$||x_j - y_v||, \quad ||x_{j+1} - y_v|| \leq \epsilon.$$

For the proof of this lemma see Lemma 3.1 of Zaslavski [20]. Lemma 3.1 implies the following auxiliary result.

Lemma 3.2. *Let $\epsilon \in (0,1)$. Then there exist a neighborhood $\mathfrak{U}$ of v_γ in $\mathfrak{A}$ and an integer $n \geq 1$ such that for each $u \in \mathfrak{U}$ and each sequence $\{x_i\}_{i=0}^n \subset K$, which satisfies*

$$\sum_{i=0}^{n-1} u(x_i, x_{i+1}) \leq \sigma(u, 0, n, x_0, x_n) + 3\,,$$

there is $j \in \{0, \ldots, n-1\}$ such that $||x_j - y_v||, ||x_{j+1} - y_v|| \leq \epsilon$.

It is not difficult to see that

$$\sigma(v, 0, m, y_v, y_v) = m\mu(v) \text{ for all natural numbers } m, \tag{3.3}$$

$$\mu(v_\gamma) = \mu(v) = v(y_v, y_v) = v_\gamma(y_v, y_v) \tag{3.4}$$

and

$$\sigma(v_\gamma, 0, m, y_v, y_v) = m\mu(v_\gamma) \text{ for all natural numbers } m. \tag{3.5}$$

Lemma 3.3. *Let $\epsilon \in (0,1)$. Then there exists $\delta \in (0,\epsilon)$ such that for each integer $n \geq 1$ and each sequence $\{x_i\}_{i=0}^n \subset K$, which satisfies*

$$x_0 = x_n = y_v, \quad \sum_{i=0}^{n-1} v_\gamma(x_i, x_{i+1}) \leq \sigma(v_\gamma, 0, n, y_v, y_v) + \delta, \tag{3.6}$$

the following relation holds:

$$||x_i - y_v|| \leq \epsilon, \quad i = 0, \dots, n. \tag{3.7}$$

Proof. Choose a positive number

$$\delta < \epsilon\gamma. \tag{3.8}$$

Assume that n is a natural number and a sequence $\{x_i\}_{i=0}^n \subset K$ satisfies (3.6). Then by (3.4-3.6), the definition of v_γ (see (3.2)) and (3.3),

$$n\mu(v) = n\mu(v_\gamma) = \sigma(v_\gamma, 0, n, y_v, y_v)$$

$$\geq \sum_{i=0}^{n-1} v_\gamma(x_i, x_{i+1}) - \delta = -\delta + \sum_{i=0}^{n-1} v(x_i, x_{i+1})$$

$$+\gamma \sum_{i=0}^{n-1} (||x_i - y_v|| + ||x_{i+1} - y_v||)$$

$$\geq -\delta + \sigma(v, 0, n, y_v, y_v) + \gamma \sum_{i=0}^{n} ||x_i - y_v|| = -\delta + \gamma \sum_{i=0}^{n} ||x_i - v_v|| + n\mu(v)\,,$$

and

$$\delta \geq \gamma \sum_{i=0}^{n} ||x_i - y_v||.$$

Combined with (3.8) this implies that

$$||x_i - y_v|| \leq \delta\gamma^{-1} < \epsilon, \ i = 0, \dots, n.$$

Lemma 3.3 is proved. □

Lemma 3.4. *Let $\epsilon \in (0,1)$. Then there exists $\delta \in (0,\epsilon)$ such that for each integer $n \geq 1$ and each sequence $\{x_i\}_{i=0}^n \subset K$, which satisfies*

$$||x_i - y_v|| \leq \delta, \quad i = 0, n, \quad \sum_{i=0}^{n-1} v_\gamma(x_i, x_{i+1}) \leq \sigma(v_\gamma, 0, n, x_0, x_n) + \delta, \tag{3.9}$$

the following relation holds:

$$||x_i - y_v|| \le \epsilon, \quad i = 0, \dots, n. \tag{3.10}$$

Proof. By Lemma 3.3 there exists $\delta_1 \in (0, 2^{-1}\epsilon)$ such that for each natural number q and each sequence $\{x_i\}_{i=0}^q \subset K$, which satisfies

$$x_0 = x_q = y_v, \ \sum_{i=0}^{q-1} v_\gamma(x_i, x_{i+1}) \le \sigma(v_\gamma, 0, q, y_v, y_v) + 2\delta_1\,, \tag{3.11}$$

the following inequality holds:

$$||x_i - y_v|| \le \epsilon, \ i = 0, \dots, q. \tag{3.12}$$

Since v_γ is continuous at the point (y_v, y_v) there exists

$$\delta \in (0, \delta_1/4)\,, \tag{3.13}$$

such that

$$|v_\gamma(z_1, z_2) - v_\gamma(y_v, y_v)| \le \delta_1/4\,, \tag{3.14}$$

for any $(z_1, z_2) \in K \times K$ satisfying

$$||z_i - y_v|| \le \delta, \ i = 1, 2. \tag{3.15}$$

Assume that n is a natural number and a sequence $\{x_i\}_{i=0}^n \subset K$ satisfies (3.9). We will show that (3.10) is valid. Define

$$y_0 = x_0, \ y_n = x_n, \ y_i = y_v \text{ for all } i \in \{0, \dots, n\} \setminus \{0, n\}, \tag{3.16}$$

$$z_i = y_v, \ i = 0, \dots, n. \tag{3.17}$$

By (3.17), (3.1), (3.4), (3.5) and (3.3),

$$\sum_{i=0}^{n-1} v_\gamma(z_i, z_{i+1}) = n\mu(v_\gamma) = n\mu(v) \tag{3.18}$$

$$= \sigma(v_\gamma, 0, n, y_v, y_v) = \sigma(v, 0, n, y_v, y_v).$$

It follows from (3.16), (3.17), (3.9) and the definition of δ (see (3.13), (3.15)) that

$$|\sum_{i=0}^{n-1} v_\gamma(z_i, z_{i+1}) - \sum_{i=0}^{n-1} v_\gamma(y_i, y_{i+1})| \le 2^{-1}\delta_1.$$

Together with (3.18) this implies that

$$\sum_{i=0}^{n-1} v_\gamma(y_i, y_{i+1}) \le 2^{-1}\delta_1 + n\mu(v). \tag{3.19}$$

By (3.9), (3.16) and (3.19),

$$\sum_{i=0}^{n-1} v_\gamma(x_i, x_{i+1}) \le \sigma(v_\gamma, 0, n, x_0, x_n) + \delta$$

$$\le \delta + \sum_{i=0}^{n-1} v_\gamma(y_i, y_{i+1}) \le 2^{-1}\delta_1 + \delta + n\mu(v),$$

$$\sum_{i=0}^{n-1} v_\gamma(x_i, x_{i+1}) \le 2^{-1}\delta_1 + \delta + n\mu(v). \tag{3.20}$$

Define

$$\bar{y}_0 = y_v,\ \bar{y}_i = x_{i-1},\ i = 1, \dots, n+1,\ \bar{y}_{n+2} = y_v. \tag{3.21}$$

It follows from (3.21), (3.9) and the definition of δ (see (3.13), (3.15)) that

$$|v_\gamma(\bar{y}_0, \bar{y}_1) - v_\gamma(y_v, y_v)|,\ |v_\gamma(\bar{y}_{n+1}, \bar{y}_{n+2}) - v_\gamma(y_v, y_v)| \le \delta_1/4. \tag{3.22}$$

It follows from (3.21), (3.22), (3.4), (3.20) and (3.13) that

$$\sum_{i=0}^{n+1} v_\gamma(\bar{y}_i, \bar{y}_{i+1}) = v_\gamma(\bar{y}_0, \bar{y}_1) + v_\gamma(\bar{y}_{n+1}, \bar{y}_{n+2})$$

$$+ \sum_{i=0}^{n-1} v_\gamma(x_i, x_{i+1}) \le 2(\delta_1/4 + \mu(v)) + \sum_{i=0}^{n-1} v_\gamma(x_i, x_{i+1})$$

$$\le n\mu(v) + \delta_1/2 + \delta + \delta_1/2 + 2\mu(v)$$

$$= (n+2)\mu(v) + \delta + \delta_1 < 2\delta_1 + (n+2)\mu(v)\,,$$

and

$$\sum_{i=0}^{n+1} v_\gamma(\bar{y}_i, \bar{y}_{i+1}) \le (n+2)\mu(v) + 2\delta_1. \tag{3.23}$$

By (3.23), (3.21), (3.4) and (3.5),

$$\bar{y}_i = y_v,\ i = 0, n+2,$$

$$\sum_{i=0}^{n+1} v_\gamma(\bar{y}_i, \bar{y}_{i+1}) \le 2\delta_1 + \sigma(v_\gamma, 0, n+2, y_v, y_v).$$

It follows from these relations and the definition of δ_1 (see (3.11), (3.12)) that

$$||\bar{y}_i - y_v|| \le \epsilon,\ i = 0, \dots, n+2.$$

Together with (3.21) this implies that

$$||x_i - y_v|| \le \epsilon,\ i = 0, \dots, n.$$

Lemma 3.4 is proved. □

Lemma 3.5. *Let $\epsilon \in (0,1)$. Then, there exist $\delta \in (0,\epsilon)$, a neighborhood $\mathfrak{U}$ of v_γ in $\mathfrak{A}$ and an integer $N \ge 1$ such that for each $u \in \mathfrak{U}$, each integer $n \ge 2N$ and each sequence $\{x_i\}_{i=0}^n \subset K$ satisfying*

$$\sum_{i=0}^{n-1} u(x_i, x_{i+1}) \le \sigma(u, 0, n, x_0, x_n) + \delta\,,$$

there exist $\tau_1 \in \{0, \dots, N\}$, $\tau_2 \in \{-N+n, n\}$ such that

$$||x_i - y_v|| \le \epsilon, \quad t = \tau_1, \dots \tau_2,$$

and moreover, if $||x_0 - y_v|| \le \delta$, then $\tau_1 = 0$, and if $||x_n - y_v|| \le \delta$, then $\tau_2 = n$.

For the proof of this lemma see Lemma 3.4 of Zaslavski [20]. It is based on Lemma 3.4.

Let $v \in \mathfrak{F}^{(0)}$, $\gamma \in (0,1)$ and let $j \ge 1$ be an integer. There exist an integer $N(v,\gamma,j) \ge 1$, an open neighborhood $\mathfrak{U}_0(v,\gamma,j)$ of v_γ in $\mathfrak{A}$ and a number $\delta(v,\gamma,j) \in (0, 2^{-j})$ such that Lemma 3.5 holds with v,γ and $\epsilon = 2^{-j}$, $\delta = \delta(v,\gamma,j)$, $\mathfrak{U} = \mathfrak{U}_0(v,\gamma,j)$, $N = N(v,\gamma,j)$.

There are an open neighborhood $\mathfrak{U}(v,\gamma,j)$ of v_γ in $\mathfrak{A}$ and an integer $N_1(v,\gamma,j) \ge 1$ such that $\mathfrak{U}(v,\gamma,j) \subset \mathfrak{U}_0(v,\gamma,j)$ and Lemma 3.2 holds with v,γ and $\mathfrak{U} = \mathfrak{U}(v,\gamma,j)$, $n = N_1(v,\gamma,j)$, $\epsilon = 4^{-j}\delta(v,\gamma,j)$.

Define:

$$\mathfrak{F} = [\cap_{q=1}^{\infty} \cup \{\mathfrak{U}(v,\gamma,j) : v \in \mathfrak{F}^{(0)}, \gamma \in (0,1), j = q, q+1, \dots\}] \cap \mathfrak{F}^{(0)},$$

$$\mathfrak{F}_l = [\cap_{q=1}^{\infty} \cup \{\mathfrak{U}(v,\gamma,j) : v \in \mathfrak{F}_l^{(0)}, \gamma \in (0,1), j = q, q+1, \dots\}] \cap \mathfrak{F}_l^{(0)},$$

$$\mathfrak{F}_c = [\cap_{q=1}^{\infty} \cup \{\mathfrak{U}(v,\gamma,j) : v \in \mathfrak{F}_c^{(0)}, \gamma \in (0,1), j = q, q+1, \dots\}] \cap \mathfrak{F}_c^{(0)},$$

and

$$\mathfrak{F}_u = [\cap_{q=1}^{\infty} \cup \{\mathfrak{U}(v,\gamma,j) : v \in \mathfrak{F}_u^{(0)}, \gamma \in (0,1), j = q, q+1, \dots\}] \cap \mathfrak{F}_u^{(0)}.$$

It is not difficult to see that $\mathfrak{F}$ ($\mathfrak{F}_l$, $\mathfrak{F}_c$, $\mathfrak{F}_u$, respectively) is a countable intersection of open everywhere dense subsets of $\mathfrak{A}$ ($\mathfrak{A}_l$, $\mathfrak{A}_c$, $\mathfrak{A}_u$, respectively).

It is easy to see that Theorem 1.1 follows from Proposition 2.3 and the definition of $\mathfrak{F}$.

Proof of Theorem 1.2. Let $w \in \mathfrak{F}$, $\epsilon > 0$. We may assume that $\epsilon < 1$. Choose an integer $q \geq 1$ such that

$$64 \cdot 2^{-q} < \epsilon. \tag{3.24}$$

There exist $v \in \mathfrak{F}^{(0)}$, $\gamma \in (0,1)$ and an integer $j \geq q$ such that

$$w \in \mathfrak{U}(v,\gamma,j). \tag{3.25}$$

By (3.25), Lemma 3.2, which holds with $\mathfrak{U} = \mathfrak{U}(v,\gamma,j)$, $n = N_1(v,\gamma,j)$, $\epsilon = 4^{-j}\delta(v,\gamma,j)$, v, γ, and the equality $\sigma(w,0,N_1(v,\gamma,j),y_w,y_w) = N_1(v,\gamma,j)\mu(w)$,

$$||y_w - y_v|| \leq 4^{-j}\delta(v,\gamma,j). \tag{3.26}$$

Set

$$\mathfrak{U} = \mathfrak{U}(v,\gamma,j), \quad N = N(v,\gamma,j), \quad \delta = 4^{-j}\delta(v,\gamma,j). \tag{3.27}$$

Assume that $u \in \mathfrak{U}$, an integer $n \geq 2N$ and a sequence $\{x_i\}_{i=0}^{n} \subset K$ satisfies

$$\sum_{i=0}^{n-1} u(x_i, x_{i+1}) \leq \sigma(u,0,n,x_0,x_n) + \delta. \tag{3.28}$$

It follows from (3.28), (3.27), the definition of $\mathfrak{U}_0(v,\gamma,j)$, $N(v,\gamma,j)$, $\delta(v,\gamma,j)$ and Lemma 3.5 that there exist $\tau_1 \in \{0,\dots,N\}$, $\tau_2 \in \{n-N,\dots,n\}$ such that

$$||x_i - y_v|| \leq 2^{-j}, \quad t = \tau_1,\dots,\tau_2.$$

Moreover, if $||x_0 - y_v|| \leq \delta(v,\gamma,j)$, then $\tau_1 = 0$, and if $||x_n - y_v|| \leq \delta(v,\gamma,j)$, then $\tau_2 = n$. Together with (3.26), (3.24) and (3.27) this implies that

$$||x_i - y_w|| \leq 2^{1-j} < \epsilon, \quad i = \tau_1,\dots,\tau_2;$$

if $||x_0 - y_w|| \leq \delta$, then $||x_0 - y_v|| \leq \delta(v,\gamma,j)$ and $\tau_1 = 0$; if $||x_n - y_w|| \leq \delta$, then $||x_n - y_v|| \leq \delta(v,\gamma,j)$ and $\tau_2 = n$. This completes the proof of Theorem 1.2. □

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