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THE RESTRICTED COTANGENT ALGEBRA OF CURVES IN PROJECTIVE SPACES

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Abstract: Here we study the restriction of the cotangent bundle $\Omega_{\mathbf{P}^n}$ of $\mathbf{P}^n$ and its exterior powers to non-special smooth curves $C \subset \mathbf{P}^n$ and in particular the bigraded algebra $\oplus_{c\geq 0, t\in \mathbf{Z}} H^0(C, (\Omega_{\mathbf{P}^n}(2))^{\otimes c}(t)|C)$.

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1. Restricted Cotangent Algebra

For all integers n, g, d with $n \geq 3$, $g \geq 0$ and $d \geq g + n$ let $H(d, g, n)$ be the closure in the Hilbert scheme $\mathrm{Hilb}(\mathbf{P}^n)$ of $\mathbf{P}^n$ of the set of all smooth irreducible non-degenerate curves $X \subset \mathbf{P}^n$ with $h^1(X, \mathcal{O}_X(1)) = 0$, $\deg(X) = d$ and $p_a(X) = g$. It is known that $H(d, g, n)$ is an irreducible generically smooth projective scheme of dimension $(n+1)d + (3-n)(1-g)$. We work over an algebraically closed field $\mathbf{K}$ with $\mathrm{char}(\mathbf{K}) = 0$. Set $\Omega := \Omega_{\mathbf{P}^n}$. By the Euler sequence the vector bundle $\Omega(1)$ is a subbundle of a trivial rank $(n+1)$

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bundle and $h^0(\mathbf{P}^n, \Omega(1)) = 0$. The vector bundle $\Omega(2)$ is spanned by its global sections and $h^0(\mathbf{P}^n, \Omega(2)) = n(n+1)/2$; more precisely, if $\mathbf{P}^n = \mathbf{P}(V)$, then $H^0(\mathbf{P}^n, \Omega(2)) = \wedge^2(V)$. Fix homogeneous coordinates $z_0, \dots, z_n$ of $\mathbf{P}^n$ and let $R := \mathbf{K}[z_0, \dots, z_n]$ be the homogeneous coordinate ring of $\mathbf{P}^n$. For any closed subscheme T of $\mathbf{P}^n$ and all integers t, c with $c \geq 0$, set

$$A(T,c,t) := H^0(T, (\Omega(2))^{\otimes c}(t)|T),$$

$$A(T) := \oplus_{c \geq 0, t \in \mathbf{Z}} A(T,c,t),$$

$$A(T,c) := \oplus t \in \mathbf{Z} A(T,c,t), \quad W(T,i,t) := H^0(T, \wedge^i(\Omega)(t)|T)$$

and $W(T,i) := \oplus_{t \geq 0} W(T,i,t)$. $A(T,c)$ is a $\mathbf{Z}$-graded R-module, while $A(T)$ is a graded algebra (with respect to the integer $c \geq 0$) and a $\mathbf{Z}$-graded R-module, the two gradings being compatible. Here is our main result.

Theorem 1. *Fix integers n, g, d such that $n \geq 3$ and $d \geq n + n[g/(n-2)]$. Then for a general $X \in H(d,g,n)$ the bigraded R-algebra $A(X)$ is generated by its summands of bidegree (c,t) with $0 \leq c \leq 1$ and $0 \leq t \leq 1$.*

We, also, prove the following result.

Theorem 2. *Fix integers n, i, g, d with $n \geq 3$, $1 \leq i \leq n$ and $0 \leq g \leq d-n$. Set $c(i,n) := [(i(n+1)+n-1)/n]$ and $a(i,n) := nc(i,n) - i(n+1)$. Then for a general $C \in H(d,g,n)$ the graded R-module $W(C,i)$ is generated in degree at most $c(i,n)+2$. If $a(i,n) \geq 3$ and $0 \leq g \leq [d/n](a(i,n)-2)$, then $W(C,i)$ is generated in degree at most $c(i,n)+1$.*

Notice that we always have $0 \leq a(i,n) \leq n-1$.

We do not know what is the best possible lower bound for d as a function of n and g to obtain Theorem 1 for the triple (n,d,g). We do not even have any guess. We would be interested in the corresponding result for components of $\mathrm{Hilb}(\mathbf{P}^n)$ formed by curves with special hyperplane bundle.

Remark 1. Since $\Omega(2)$ is spanned by its global sections (i.e. it is a quotient of a trivial vector bundle), we have $h^1(T, (\Omega(2)^{\otimes c}|T)(t)) = 0$ for all integers $c \geq 0$ and $t > 0$ and for every closed subscheme T of $\mathbf{P}^n$ with $\dim(T) \leq 1$ and $h^1(T, \mathcal{O}_T(1)) = 0$. Hence $h^0(T, (\Omega(2)^{\otimes c}|T)(t)) = \deg((\Omega(2)^{\otimes c}|T)(t)) + n^c(1-g)$ for every $T \in H(d,g,n)$ with $h^1(T, \mathcal{O}_T(1)) = 0$ and in particular for a general $T \in H(d,g,n)$.

Remark 2. Set $\mathbf{P}^n = \mathbf{P}(V)$. By the dual of the Euler sequence for $T\mathbf{P}^n$ for all integers $c \geq 0$ and $t > 0$ we have the exact sequence

$$0 \to \Omega(2)^{\otimes(c+1)}(t-1) \to \Omega(2)^{\otimes c} \otimes V(t) \to \Omega^{\otimes c}(t+1) \to 0\,. \tag{1}$$

Hence for all integers $c \geq 0$ and $t > 0$ and any $T \in H(d, g, n)$ with $h^1(T, \mathcal{O}_T(1)) = 0$ the multiplication map $H^0(T, (\Omega(2)^{\otimes c}|T)(t)) \otimes V \to H^0(T, (\Omega(2)^{\otimes c}|T)(t+1))$ is surjective if and only if $H^1(T, \Omega(2)^{\otimes(c+1)}(t-1)) = 0$.

Notice that the assumptions of Remark 2 are always satisfied if $t \geq 0$, i.e. we have the following result.

Proposition 1. *Let $X \subset \mathbf{P}^n$ be a curve with $h^1(X, \mathcal{O}_X(1)) = 0$. Then for all integers $c \geq 0$ the graded R-module $A(X, c)$ is generated in degrees at most two.*

Remark 3. Fix integers n, d with $d \geq n \geq 3$. For a general $D \in H(d, 0, n)$ the vector bundle $\Omega|D$ is rigid, i.e. its splitting type $a_1 \geq \cdots \geq a_n$ satisfies $a_n \geq a_1 - 1$ (see [3]). Thus, the splitting type of $\Omega|D$ depends only on the integers n and d. Since $\deg(\Omega|D) = -(n+1)d$, we have $a_i = [-(n+1)d/n] + \epsilon_i$ with $\epsilon_1 = 0$, $-1 \leq \epsilon_i \leq 0$, $\epsilon_i \geq \epsilon_{i+1}$ for $1 \leq i \leq n-1$ and $\epsilon_n = -1$ if and only if n does not divide $(n+1)d$, i.e. if and only if n does not divides d. The splitting type of $\Omega(2)|D$ is $b_1, \ldots . b_n$ with $b_i = a_i + 2d$ for every i.

Remark 4. Fix integers n, m with $n \geq m \geq 1$ and a linear m-dimensional subspace $M \subset \mathbf{P}^n$. Let $D \subseteq M$ be a rational normal curve of M, i.e. a smooth rational curve of degree m spanning M. The vector bundle $\Omega(2)|D$ is the direct sum of $n-m$ line bundles of degree m and m line bundles of degree $m-1$. Fix an integer x with $1 \leq x \leq m-1$. Let $T \subset \mathbf{P}^n$ be a reduced curve intersecting quasi-transversally D at x points. Assume $h^1(T, \Omega(2)|T) = 0$. From the Mayer-Vietoris exact sequence

$$0 \to \Omega(2)|T \cup D \to \Omega(2)|T \oplus \Omega(2)|D \to \Omega(2)|(T \cap D) \to 0\,, \tag{2}$$

we obtain $h^1(T \cup D, \Omega(2)|T \cup D) = 0$.

Lemma 1. *Fix integers n, g, d with $n \geq 3$, $g \geq 0$, $d \geq n + [g/(n-2)]$. Then for a general $X \in H(d, g, n)$ we have $h^1(X, \Omega(2)|X) = 0$.*

Proof. By semicontinuity it is sufficient to find $Y \in H(d, g, n)$ (even reducible) with $h^1(Y, \Omega(2)|Y) = 0$. Set $c := [d/n]$ and $e := [g/(n-2)]$. Thus, $cn \leq d < (c+1)n$ and $c > e$. First, assume $c \geq 2$. We will take as Y the union of c smooth rational curves $D_1, \ldots, D_c$ with $\deg(D_i) = n$ for $1 \leq i \leq c-1$, $\deg(D_c) = d - (c-1)n$, Y nodal and connected, $D_i \cap D_j \neq \emptyset$ if and only if $|i - j| \leq 1$ and $p_a(Y) = g$. Thus, $\mathrm{card}(\mathrm{Sing}(Y)) = g + c - 1$. We will assume $\mathrm{card}(D_i \cap D_{i+1}) = n - 1$ if $1 \leq i \leq e$, $\mathrm{card}(D_{e+1} \cap D_{e+2}) = 1 + g - (n-2)e$ and $\mathrm{card}(D_i \cap D_{i+1}) = 1$ if $e + 1 \leq i \leq c - 1$. Taking $\mathcal{O}_{\mathbf{P}^n}(1)$ instead of $\Omega(2)$ in c Mayer-Vietoris exact sequences like (2) we obtain $h^1(Y, \mathcal{O}_Y(1)) = 0$. By [1], Corollary 1.2, or [4] we have $Y \in H(d,g,n)$. Applying $c-2$ times the last part of Remark 4 we obtain $h^1(D_1 \cup \cdots \cup D_{c-1}, \Omega(2)|D_1 \cup \cdots \cup D_{c-1}) = 0$. Then we use Remark 3 for the integer $\deg(D_c) = d-(c-1)n$ if n does not divide d. Any two subsets E, F of $\mathbf{P}^n$ with $\mathrm{card}(E) = \mathrm{card}(F) \le n$ and spanning a linear space of dimension $\mathrm{card}(E)-1$ are projectively equivalent. Thus, for any such E and any integer $b \ge n$ there is a smooth non-degenerate rational curve $T \subset \mathbf{P}^n$ such that $E \subset T$ and $\deg(T) = b$. Since $\mathrm{card}(D_{c-1} \cap D_c) \le n-1$, we may assume that $D_{c-1} \cap D_c$ is a general subset with that cardinality of $\mathbf{P}^n$ and that D_c is a general element of $H(\deg(D_c), 0, n)$. Thus, Remark 3 and the Mayer-Vietoris exact sequences used in Remark 4 give $h^1(Y, \Omega(2)|Y) = 0$. Now, assume $c = 1$, i.e. $d \le 2n-1$. If $d = n$ we have $g = 0$. In this case we use that for any rational normal curve $T \subset \mathbf{P}^n$ the vector bundle $\Omega(2)|T$ is the direct sum of n line bundles of degree $n-1$ (Remark 3). Now, assume $n+1 \le d \le 2n-1$. We have $d \ge n+g+1$. Fix a $(d-n)$-dimensional linear subspace $M \subset \mathbf{P}^n$ and a set $S \subset M$ with $\mathrm{card}(S) = g+1$ and S spanning a g-dimensional linear subspace of M. There is a rational normal curve $T \subset \mathbf{P}^n$ and a rational normal curve D of M (i.e. a smooth rational curve of degree $d-n$ spanning M) such that $T \cap D = S$ and T intersects quasi-transversally D at each point of S. Set $Y := T \cup D$. By [1], Corollary 1.2, or [4] we have $Y \in H(d,g,n)$. We have $h^1(D, \Omega(2)|D) = 0$ by the case $m = d-n$ of Remark 3. Since $\Omega(2)|T$ is the direct sum of n line bundles of degree $n-1$ (Remark 3) and $\mathrm{card}(S) = g+1 \le n-2$, the Mayer-Vietoris exact sequence (2) gives $h^1(Y, \Omega(2)|Y) = 0$. □

Remark 5. By Riemann-Roch for every $X \in H(d,g,n)$ we have $\deg(\Omega(1)|X) < 0$ and hence $h^1(X, \Omega(1)|X) \ne 0$ if $g \ge 1$.

Proposition 2. *Fix integers n, g, d with $n \ge 3$, $g \ge 0$, $d \ge n+(n-2)[g/n]$, and a general $X \in H(d,g,n)$. Then for all integers $c \ge 0$ the graded R-module $A(X,c)$ is generated in degrees at most one.*

Proof. Use Remark 4, Lemma 1 and its proof. □

Remark 6. Fix integers d, n with $d \le n \le 3$ and let $b_1 \ge \cdots \ge b_n$ be the splitting type of $T\mathbf{P}^n|C$ for a general $C \in H(d,0,n)$. By Remark 3 $b_n \ge b_1 - 1$ and the integers $b_1, \ldots, b_n$ are uniquely determined by the integers d and n. Fix an integer w with $1 \le w \le b_b + 1$ and subspace $A \subset \mathbf{P}^n$ with $\dim(A) \le n-2$. Let $S \subset \mathbf{P}^n$ be a general subset of $\mathbf{P}^n$ with $\mathrm{card}(S) = w$. By [2], 1.5, there is $Y \in H(d,0,n)$ such that $S \subset Y$. Moving S inside the symmetric product of w copies of $\mathbf{P}^n$ we may even assume that Y is a general element of $H(d,0,n)$. Fix an integer z such that $0 \le z \le w$, a general union B of z points of A and a smooth curve C such that $B \subset C \subset A$. We take as S a general union of B and of $w-z$ points of $\mathbf{P}^n$. We may find (Y,S) as above such that $T\mathbf{P}^n|Y$ has splitting type $b_1,\dots,b_n$, Y intersects quasi-transversally C, $S \subset Y$, $Y \cap C = B$.

Remark 7. Let E, F be vector bundles on $\mathbf{P}^n$ and $T, C \subset \mathbf{P}^n$ curves with C smooth and rational. Set $S := (C \cap T)_{red}$ and assume that T and C intersects quasi-transversally, i.e. assume that S is the scheme-theoretic intersection of T and C. Set $w := \mathrm{card}(S)$. Let $a_1 \ge \cdots \ge a_x$ (resp. $b_1 \ge \cdots \ge b_y$) be the splitting type of $E|C$ (resp. $F|C$). Assume $w \le b_x$, $w \le b_y$, $h^1(T,E|T) = h^1(T,F|T) = 0$, $E|T$ and $F|T$ spanned by their global sections and that the multiplication map $H^0(T,E|T) \otimes H^0(T,F|T) \to H^0(T,E \otimes F|T)$ is surjective. We obtain $h^1(T \cup C, E|T \cup C) = h^1(T \cup C, F|T \cup C) = 0$ and then that $E|T \cup C$ and $F|T \cup C$ are spanned by their global sections and then the surjectivity of the multiplication map $H^0(T \cup C, E|T \cup C) \otimes H^0(T \cup C, F|T \cup C) \to H^0(T \cup C, E \otimes F|T \cup C)$ (use a Mayer-Vietoris exact sequence as in Remark 4).

Proof of Theorem 1. Use Remarks 6 and 7 to copy the proof of Lemma 1. □

Proof of Theorem 2. Fix an integer $a \ge 1$. For a general $C \in H(an,0,n)$ the vector bundle $\Omega|Y$ is the ditect sum of n line bundles with the same degree. Hence for every integer i with $1 \le i \le n-1$ the vector bundle $\wedge^i(\Omega)|Y$ is the direst sum of $\binom{n}{i}$ line bundles with the same degree. Hence we may use semicontinuity, reducible curves and $[d/n]$ Mayer-Vietoris exact sequences as in the proof of Proposition 2. □

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