International Journal of Pure and Applied Mathematics
Volume 7 No. 3 2003, 315-325

UNREDUCED TRIGONAL CURVES: RIBBONS, ROPES AND BRILL-NOETHER THEORY

E. Ballico
Department of Mathematics
University of Trento
380 50 Povo (Trento) - Via Sommarive, 14, ITALY
e-mail: ballico@science.unitn.it

Abstract: Here we study locally Cohen-Macaulay projective curves from the point of view of Brill-Noether theory of special divisors. The easiest curves (after the hyperelliptic curves) are the trigonal curves. They occur in the set-up of ribbons and of 3-ropes. Here we study their Brill-Noether theory (for ropes: special generalized line bundles).

AMS Subject Classification: 14H51, 14HN05, 13H10
Key Words: unreduced curve, trigonal curve, Brill-Noether theory, ribbons, ropes

1. Introduction

In this paper we consider the stratification by gonality of locally Cohen-Macaulay projective curves, whose support is $\mathbf{P}^1$ (rational ribbons in the sense of [2] and [4] and ropes in the sense of [3]). The easiest curves are the hyperelliptic ones. For their well-known properties, see Proposition 1, and Proposition 2 and Remark 4. The next class of interesting curves is given by the trigonal curves and in this paper we will study their Brill-Noether theory of special divisors. For trigonal smooth curves the corresponding theory is due to Maroni (see [6]).

Received: May 13, 2003

In Section 2 we consider the case of rational ropes in the sense of [2] and [4], i.e. double structures on $\mathbf{P}^1$. In this case the critical object of study are the generalized bundles ([4], Section 1). First, we give a Clifford type inequality (see Lemma 1) and the well-known Brill-Noether theory of hyperelliptic rational ribbons (see Proposition 1, Proposition 2 and Remark 4). Then we describe the canonical model of a trigonal rational ribbon, i.e. of a rational ribbon with Clifford index one (see Remark 6). Our main result is the following classification theorem for spanned generalized line bundles (for the notation used in its statement, see Remark 6).

Theorem 1. *Let X be a rope of genus $g \geq 5$ with Clifford index one and F a spanned generalized line bundle on X. Let Δ be the singularity divisor of F. Set $\delta := \deg(\Delta)$ and $2b := \deg(F) - \delta$. Let P be the point of X_{red} such that the blowing-up of X at P gives a hyperelliptic curve of genus $g-1$. Assume the existence of an inclusion $j : F \to (\mathcal{I}_P)^2 \otimes \omega_X$. If $\delta > 0$ assume $0 \leq b \leq g-1-\delta$. Then either $F \cong \mathcal{O}_X$ or $\delta = 1$ and $F \cong F_b$.*

In Section 3 we consider the case of ropes in the sense of [3]. Since we are in curves with a trigonal pencil, we need to consider only the case of 3-ropes, X, with $X_{red} = \mathbf{P}^1$. The properties (and the existence) of trigonal curves heavily depend on the splitting type of their conormal module, which is a rank two vector bundle on $X_{red} \cong \mathbf{P}^1$.

2. Ribbons

A ribbon is a locally Cohen-Macaulay curve, which is a multiplicity two structure over a smooth curve. Set $D := \mathbf{P}^1$ and let X be a rational ribbon of genus g, i.e. a ribbon with $X_{red} = D$ and with $\mathcal{O}_D(-g-1)$ as conormal module (see [2]). Thus, we have an exact sequence

$$0 \to \mathcal{O}_D(-g-1) \to \mathcal{O}_X \to \mathcal{O}_D \to 0\,. \tag{1}$$

For any effective divisor Δ of D Bayer and Eisenbud defined the blowing-up $\phi_\Delta : X(\Delta) \to X$ of Δ ([2], Section 2); $X(\Delta)$ is a rational ribbon of genus $g - \deg(\Delta)$. As in [4], Section 1, we will call generalized line bundle on X a coherent sheaf on X, which is torsion free and such that, outside finitely many points of D, it is locally free and of rank one. For any generalized line bundle F on X set $\deg(F) = \chi(F) - \chi(\mathcal{O}_X) = \chi(F) + g - 1$. The integer $\deg(F)$ is called the degree of F. By [4], Section 1, every generalized line bundle F on X is associated to a pair (Δ, L) with Δ effective divisor on X and $L \in \mathrm{Pic}(X(\Delta))$;

we have $F \cong \phi_{\Delta *}(L)$, $\deg(F) = \deg(L) + \deg(\Delta)$ and $h^0(X, F) = h^0(X, L)$. The divisor Δ is called the singularity scheme or the singularity divisor of F and the integer $\deg(\Delta)$ is called the singularity degree of F. The sheaf F is locally free exactly outside the support of Δ.

Remark 1. Let F be a generalized line bundle on the ribbon X. Call G the image of the evaluation map $\alpha : H^0(X, F) \otimes \mathcal{O}_X \to F$. Since F is torsion free, G is torsion free (or the zero sheaf). Assume that F is spanned by its global sections outside finitely many points. Then G coincides with F outside finitely many points and hence it is a generalized line bundle. We have $h^0(X, G) = h^0(X, F)$.

Remark 2. Fix $L \in \mathrm{Pic}(X)$ such that $\deg(L) \leq 2g$. Hence, $\deg(L|D) = \deg(L)/2 \leq g$. If $\mathcal{I}_{D,X}$ is the ideal sheaf of D in X, then $\mathcal{I}_{D,X} \otimes L$ is a line bundle on D with degree $\deg(L|D) - g - 1$. By tensoring (1) with L we obtain that the restriction map $H^0(X, L) \to H^0(D, L|D)$ is injective. Now fix $L, M \in \mathrm{Pic}(X)$ such that $\deg(L) \leq 2g$ and $\deg(M) \leq 2g$. Since D is reduced and connected, the pairing $H^0(D, L|D) \otimes H^0(D, M|D) \to H^0(D, L \otimes M|D)$ is non-degenerate in both variables. Since the restiction maps $H^0(X, L) \to H^0(D, L|D)$ and $H^0(X, M) \to H^0(D, M|D)$ are injective, the pairing $\alpha : H^0(X, L) \otimes H^0(X, M) \to H^0(X, L \otimes M)$ is non-degenerate in both variables. Hence, by a classical lemma of Hopf we have

$$\dim(\mathrm{Im}(\alpha)) \geq h^0(X, L) + h^0(X, M) - 1$$

and in particular $h^0(X, L \otimes M) \geq h^0(X, L) + h^0(X, M) - 1$.

We have $\deg(\omega_X) = 2g - 2$. For every $L \in \mathrm{Pic}(X)$ we have $\deg(\omega_X \otimes L^*) = 2g - 2 - \deg(L)$. Since X is locally Cohen-Macaulay, we have $h^1(X, L) = h^0(X, \omega \otimes L^*)$ for every $L \in \mathrm{Pic}(X)$. Hence, by Riemann-Roch and Remark 2 we obtain the following result.

Lemma 1. (Clifford Inequality) *Let X be a rational ribbon of genus g. For every $L \in Pic(X)$ such that* $0 \leq \deg(L) \leq 2g$ *we have* $h^0(X, L) - 1 \leq \deg(L)/2$.

A ribbon (rational, as always in this section!) is called split if the inclusion of D in X has a retraction. This is equivalent to require that $\mathcal{O}_X$ is an $\mathcal{O}_D$-module and that for this $\mathcal{O}_D$-module structure the exact sequence (1) splits. This is equivalent to the existence of a degree two spanned line bundle on X. Hence a ribbon is called hyperelliptic if and only if it is a split ribbon. Now, we write down the well-known Brill-Noether theory of hyperelliptic ribbons, both for line bundles and for generalized line bundles.

Remark 3. Let $\phi_\Delta : X(\Delta) \to X$ be the blowing-up of Δ. For any $L \in \mathrm{Pic}(X)$ the sheaf $\phi_\Delta^*(L)$ is a line bundle on $X(\Delta)$ with $\deg(\phi_\Delta^*(L)) = \deg(L)$ and $h^0(X(\Delta), \phi_\Delta^a st(L)) \ge h^0(X, L)$. Hence if X is a split ribbon, then $X(\Delta)$ is a split ribbon.

Proposition 1. *Let X be a hyperelliptic ribbon of genus $g \ge 4$ and E its degree two hyperelliptic line bundle. Let L be a spanned line bundle on X such that $0 < deg(L) \le 2g-2$ and $\omega_X^* \otimes L^*$ is generically spanned by its global sections. Then $L \cong E^{\otimes b}$ with $b := deg(L)/2$.*

Proof. Since L is spanned, it induces a morphisn

$$h_L : X \to \mathbf{P}(H^0(X, L)^*).$$

Composing with an inclusion of L in ω_X we see that h_L factors through the degree two morphism induced by E. □

Proposition 2. *Let X be a hyperelliptic ribbon of genus $g \ge 4$. Let F be a generalized line bundle on X which is spanned outside finitely many points and such that there is an inclusion of sheaves $i : F \to \omega_X$. Let G be the image of the evaluation map $H^0(X, F) \otimes \mathcal{O}_X \to F$. Let Δ be the singularity divisor of G. Assume* $\deg(\Delta) \le 2g-2$. *Let $\phi_\Delta : X(\Delta) \to X$ be the blowing-up of Δ and $L \in \mathrm{Pic}(X)$ such that $G \cong \phi_{\Delta *}(L)$. Let E be the hyperelliptic pencil on $X(\Delta)$. Then $L \cong E^{\otimes b}$ with $b := (\deg(G) - \deg(\Delta))/2$.*

Proof. By Remark 1 G is a generalized line bundle. The composition of the inclusion of G in F with the map i gives an inclusion of sheaves $j : G \to \omega_X$. The case $\Delta = \emptyset$ is just Proposition 1. Assume $\Delta \ne \emptyset$. The pair $(G, H^0(X, G))$ defines on $X \backslash \mathrm{Supp}(\Delta)$ a morphism $u : X \backslash \mathrm{Supp}(\Delta) \to \mathbf{P}(H^0(X(\Delta), L)^*)$. Composing with j we see that the scheme-theoretic fibers of u are union of fibers of the hyperelliptic map. Since F is spanned, L is spanned. Hence L induces a morphism $h_L : X(\Delta) \to \mathbf{P}(H^0(X(\Delta), L)^*)$. We have $H^0(X(\Delta), L) \cong H^0(X, G) \cong H^0(X, F)$. The ribbon $X(\Delta)$ is a hyperelliptic ribbon of genus $g - \deg(\Delta) \ge 2$ (Remark 3). Hence h_L factors through a degree two map and hence it is composed with the hyperelliptic pencil. □

Remark 4. Let X be a hyperelliptic ribbon of genus $g \ge 4$. Let Δ be an effective Cartier divisor on D such that $\delta := \deg(\Delta) \le 2g-2$. Let $\phi_\Delta : X(\Delta) \to X$ be the blowing-up of Δ and E_Δ the hyperelliptic degree two spanned line bundle on $X(\Delta)$. Then for all integers b such that $0 \le b \le p_a(X(\Delta)) = g - \delta$ we have $h^0(X(\Delta), E_\Delta^{\otimes b}) = b+1$. The line bundle $E_\Delta^{\otimes b}$ is spanned. The rank one torsion free sheaf $\phi_{\Delta *}(E_\Delta^{\otimes b})$ is a generalized line bundle with Δ as singularity divisor, $\deg(\phi_{\Delta *}(E_\Delta^{\otimes b})) = 2b+\delta$ and $h^0(X, \phi_{\Delta *}(E_\Delta^{\otimes b})) = b+1$. Since $\phi_{\Delta *}(E_\Delta^{\otimes b})$ contains $E^{\otimes b}$ (see the proof of Proposition 2) and $h^0(X, E^{\otimes b}) = b+1$, we see that if $\delta > 0$ and $b+\delta \le g$, then $E^{\otimes b}$ is the subsheaf of $\phi_{\Delta *}(E_\Delta^{\otimes b})$ generated by $H^0(X, \phi_\Delta^*(E_\Delta^{\otimes b}))$.

Let X be any ribbon supported by D and with $\mathcal{O}_D(-g-1)$ as conormal module (see [2]). There is an exact sequence (see [2], p. 724)

$$0 \to \mathcal{O}_D(-g-1) \to \Omega_X|D \to \Omega_D \to 0\,, \tag{2}$$

and hence, we have $\Omega_X|D \cong \mathcal{O}_D(-e-2) \oplus \mathcal{O}_D(-g-1+e)$ for some integer e such that $2e \le (g-1)/2$. The integer $\mathrm{Cliff}(X) := e$ is called the Clifford index of X. At least if $g \ge 7$ we will call trigonal a rational ribbon with Clifford index one.

Remark 5. Let Y be a rational ribbon with genus g and Clifford index e. By [5], Theorem 2, Y is a flat limit of a flat family of smooth curves of genus g with constant Clifford index e. Indeed, as pointed out in [5], p. 307, the proof of [5], Theorem 2, shows that Y is the flat limit of a flat family of smooth curves of genus g with gonality $e+2$.

Remark 6. Let X be a non-hyperelliptic rational ribbon of genus $g \ge 5$. By [2], Theorem 5.3, the dualizing sheaf ω_X is a very ample line bundle and hence it induces an embedding $f : X \to \mathbf{P}^{g-1}$. By [2], Theorem 5.3, the curve $f(X)$ is arithmetically Cohen-Macaulay, i.e. for every integer $t \ge 1$ the restriction map $\rho_{X,t} : H^0(\mathbf{P}^{g-1}, \mathcal{O}_{\mathbf{P}^{g-1}}(t)) \to H^0(f(X), \mathcal{O}_{f(X)}(t))$ is surjective. Hence $\dim(\ker(\rho_{X,2})) = (g+1)g/2 - 3g + 3$ depends only on g. Now assume that X has Clifford index one. By the case $a = 1$ of [2], Proposition 2.6, any two ribbons of genus g and Clifford index one differ by an element of $\mathrm{Aut}(D)$; in a certain sense they are projectively equivalent and certainly they have the same Brill-Noether theory of generalized line bundles and generalized linear series in the sense of [4], Section 1. By Remark 5 and the very ampleness of ω_X, the curve $f(X)$ is in the closure in $\mathrm{Hilb}(\mathbf{P}^{g-1})$ of a flat family of canonical models, say $\{X_\lambda\}$ of trigonal smooth curves of genus g. Notice that each smooth X_λ is contained in a minimal degree surface $T_\lambda \subset \mathbf{P}^{g-1}$ with T_λ smooth and isomorphic to a certain Hirzebruch surface. The surjectivity of $\rho_{X,2}$ shows that the family T_λ has a flat limit $T_0 \subset \mathbf{P}^{g-1}$ with T_0 surface of degree $g-2$. Since we take the limit in $\mathrm{Hilb}(\mathbf{P}^{g-1})$, we cannot exclude that T_0 has embedded components. We shall use a different approach to identify a minimal degree surface of $\mathbf{P}^{g-1}$ containing the canonical model $f(X)$ of X. Let $T \subset \mathbf{P}^{g-1}$ be a cone over a rational normal curve of $\mathbf{P}^{g-2}$. Hence $\deg(T) = g-2$. Any two such cones are projectively equivalent. Set $G := \{h \in \mathrm{Aut}(\mathbf{P}^{g-1}) : h(T) \subseteq T\}$. Let P

be the vertex of T. Let $C \subset T$ be a rational normal curve of $\mathbf{P}^{g-1}$ with $P \in C$. Any two such curves differ by an element of G. The curve C is an effective Weil divisor of T. Let W be the unique Weil divisor of T corresponding to $2C$. Hence $W \subset \mathbf{P}^{g-1}$ is a locally Cohen-Macaulay curve such that $\deg(W) = 2g-2$. Take two linear subspaces A, B of $\mathbf{P}^g$ with $\dim(A) = 1$, $\dim(B) = g-2$ and $A \cap B = \emptyset$, i.e. with $A \cup B$ spanning $\mathbf{P}^g$. Let $J \subset \mathbf{P}^g$ be the join of A and a rational normal curve B' of B, i.e. the union of all lines spanned by a point of A and a point of B'. Thus, J is a 3-dimensional irreducible variety. Since A is a line, J is just the cone with vertex A and base B. Hence for every hyperplane H of $\mathbf{P}^g$ with A not contained in H the pair $(J \cap H, H)$ is projectively equivalent to the pair $(T, \mathbf{P}^{g-1})$. By [2], p. 746, there is an anticanonical divisor Z of J such that for every canonical model of a genus g ribbon X' with Clifford index one there is a hyperplane M of $\mathbf{P}^g$ such that $Z \cap M$ is the canonical model of X'. Notice that $Z \cap M$ is contained in $J \cap M$. Since $\mathrm{Cliff}(X) = 1$ and all triples $(C, T, \mathbf{P}^{g-1})$ are projectively equivalent, we obtain that, up to a projective transformation, $f(X)$ is the Weil divisor $W = 2C$ of the cone T. It is easy to compute the minimal free resolution of $f(X) = 2C$ as in [2], Section 8. A reader may ask: where is the trigonal pencil? We claim that it corresponds to the lines of the cone T. Let $u : T' \to T$ be the blowing-up of the vertex P of the cone T and let $C' \subset T'$ be the strict transform of C in T'. The surface T' is a smooth surface isomorphic to the Hirzebruch surface F_{g-2}. We have $\mathrm{Pic}(T') \cong \mathbf{Z}^{\oplus 2}$ and we take as a basis of $\mathrm{Pic}(T')$ a fiber R of the ruling of T' and the section, S, with minimal self-intersection of the ruling of T'. Thus, $S^2 = -g+2$, $S \cdot R = 1$ and $R^2 = 0$. The composition of u with the inclusion of T in $\mathbf{P}^{g-1}$ is the morphism $T' \to \mathbf{P}^{g-1}$ associated to the linear system $|2S + (2g-2)R|$. We have $C' \in |S + (g-1)R|$. The Cartier divisor $2C'$ of T' is an element of $|2S + (2g-2)R|$ and hence $p_a(2C') = g-1$. The locally Cohen-Macaulay curve $2C'$ is a ribbon and the ruling $|R|$ of T' shows that $2C'$ is a hyperelliptic ribbon, i.e. it is a split ribbon. The ribbon $2C'$ is obtained from the ribbon $2C' \cong X$ by blowing-up P in the sense of [2], Section 2. By [4], Section 1, the hyperelliptic line bundle of degree two on $2C'$ corresponds to a rank one torsion free sheaf F on X with $h^0(X, F) = 2$ and $\deg(F) = 3$ (a generalized line bundle with the terminology of [4]). Conversely, if $\mathrm{char}(\mathbf{K}) = 0$ a ribbon has Clifford index one if and only if it has a rank one torsion free sheaf F with $h^0(X, F) = 2$ and $\deg(F) = 3$ ([4], Theorem 1.2). Call E the line bundle on $2C'$ induced by the ruling $|R|$ of T'. Since $T' \cong F_{g-2}$, we have $\omega_{T'} \cong \mathcal{O}_{T'}(-2S - gR)$ and hence $h^1(T', \mathcal{O}_{T'}(-2S + zR)) = h^1(T', \mathcal{O}_{T'}((-z+g)R)))$ for every integer z (Serre duality). The Leray spectral sequence of the ruling of T' gives $h^1(T', \mathcal{O}_{T'}(wR)) = 0$ for every integer $w \geq -1$. Hence from the exact sequence

$$0 \to \mathcal{O}_{T'}(-2C' - tR) \to \mathcal{O}_{T'}(tR) \to E^{\otimes t} \to 0\,, \tag{3}$$

we obtain $h^0(C', E^{\otimes t}) = t + 1$ for $1 \le t \le g - 1$. We have $\deg(E^{\otimes t}) = 2t$. Since $2C'$ is obtained from $2C \cong X$ by blowing-up the point P, the line bundle $E^{\otimes t}$ corresponds to a rank one torsion free sheaf $F_t := \phi_{P*}(E^{\otimes t})$ on X with $h^0(X, F_t) = h^0(2C, E^{\otimes t}$ and $\deg(F_t) = \deg(E^{\otimes t}) + 1 = 2t + 1$. F_t is locally free outside P. Since $E^{\otimes t}$ is spanned by its global sections, F_t, $t > 0$, is spanned outside P. Now, we shall prove that F_t is spanned at P if $t > 0$. Assume that this is not the case. Hence there is a shear $G \subset F_t$ such that $F_t/G \cong \mathcal{O}_P$ (and hence $\deg(G) = 2b$) with $h^0(X, G) = t + 1$. Since the sections of F_t comes from sections of $E^{\otimes t}$, they induces (at least outside P) a two-to-one map. We easily obtain that X is hyperelliptic, contradiction.

Proof of Theorem 1. Identify X with its canonical model $f(X) \subset \mathbf{P}^{g-1}$ and $f(X)$ with the Weil divisor $2C$ of the cone T. Hence $\mathcal{O}_{f(X)}(1) \cong \omega_X$. Let $\phi_P : X(P) \cong 2C' \to X \cong 2C$ be the blowing-up of P. Call E the hyperelliptic line bundle on the split genus $g - 1$ rational ribbon $X(P)$.

(a) Here we assume that the support of the singularity divisor δ of F contains P. Set $\Delta' := \Delta - P$. See Δ' as a Cartier divisor of the support of $X(P)$. There is a generalized line bundle F' on X' with $\deg(F') = \deg(F) - 1$ and $\phi_{P*}(F') = F$. Since F is spanned by its global sections and the tensor product is a right exact functor, F' is spanned by its global sections. The inclusion j induces an inclusion of F' into $\omega_{X(P)}$. Hence $F' \cong E^{\otimes b}$ (Proposition 1), i.e. $F \cong F_b$.

(b) Here we assume that P is not in the support of Δ. Hence $\phi_P^*(F)$ is a generalized line bundle on $X(P)$ with degree $\deg(F) = 2b + \delta$. We see Δ as a Cartier divisor of the support C' of $X(P) \cong 2C'$. With this abuse of notation Δ is the singularity divisor of $\phi_P^*(F)$. Since the tensor product is a right exact functor and F is spanned, $\phi_P^*(F)$ is spanned. Since $b+\delta \le g-1 = p_a(X(P))$ and the inclusion j induces an inclusion of F' into $\omega_{X(P)}$, Proposition 1 gives $\delta = 0$ and $\phi_P^*(F) \cong E^{\otimes b}$. Thusl, F is a subsheaf of F_b with $F_b/F \cong \mathcal{O}_P$. If $b = 0$, then $F \cong \mathcal{O}_X$. Assume $b > 0$. Fix a Cartier divisor Z associated to E and containing $P \in C'$. The multiplication by the equation of Z induces an inclusion of E and of F_{b-1} into F_b. With this identification we have $F_{b-1} \subsetneq F \subsetneq F_b$ because $F_b/F \cong \mathcal{O}_P$. Notice that $h^0(X, F_b) - h^0(X, F_{b-1}) = 1$ and that F_b is spanned. Hence $h^0(X, F) = h^0(X, F_{b-1})$. Since F is spanned and $F_{b-1} \ne F$, we have $h^0(X, F_{b-1} < h^0(X, F)$, contradiction. □

3. Ropes

In this section we consider the case of ropes in the sense of [3]. Hence a 2-rope is just a ribbon. Let X be an x-rope with $x \geq 3$. To have any chance to have a degree three line bundle on X we need to assume $x = 3$. To have any chance to have a degree three spanned line bundle we are forced to assume $X_{red} \cong \mathbf{P}^1$. Hence in this section we only consider rational 3-ropes.

Set $D := \mathbf{P}^1$ and $F := \mathcal{O}_D(a) \oplus \mathcal{O}_D(b)$. Let X be the 3-rope on D with F as conormal module (see [3]), i.e. let X be the locally Cohen-Macaulay scheme such that $X_{red} = D$ and the ideal sheaf $\mathcal{I}$ of D on X satisfied $\mathcal{I}^2 = 0$ and, as an $\mathcal{O}_D$-module, it is isomorphic to F. Thus, we have an exact sequence

$$0 \to F \to \mathcal{O}_X \to \mathcal{O}_D \to 0\,. \tag{4}$$

We have $\chi(\mathcal{O}_X) = \chi(F) + \chi(\mathcal{O}_D) = a + b + 3$. Set $g := -a - b - 2$. The 3-rope X will be called a split rope if the natural surjection $\mathcal{O}_X \to \mathcal{O}_D$ has a retraction, i.e. if $\mathcal{O}_X$ is an $\mathcal{O}_D$-module and, seen $\mathcal{O}_X$ as an $\mathcal{O}_D$-module, the exacrt sequence is a split exact sequence of $\mathcal{O}_D$-modules. We have an exact sequence on D

$$0 \to F \to \Omega_X|D \to \Omega_D \to 0 \tag{5}$$

(the restricted cotangent sequence). Hence we may associate to any rational 3-rope X with F as conormal module and extension class $e_X \in \mathrm{Ext}^1(D;\Omega_D, F) \cong H^1(D, F(2))$. Since D is is smooth, the exact sequence (5) locally splits, the structure of 3-rope is locally split and one can copy [2], pp. 724–725, and obtain the following result.

Proposition 3. *For any rank two vector bundle F on $D \cong \mathbf{P}^1$ and every $e \in H^1(D, F(2))$ there is a unique (up to isomorphism) rational 3-rope X with F as conormal bundle and e as associated extension class.*

Since $\mathcal{I}^2 = 0$, from (4) we obtain the exact sequence

$$0 \to H^1(D, F) \to \mathrm{Pic}(X) \to \mathrm{Pic}(D) \to 0\,. \tag{6}$$

For every $L \in \mathrm{Pic}(X)$ the sheaf $\mathcal{I} \otimes L$ is a rank two vector bundle on D isomorphic to $F(c)$, where $c := \deg(L|D)$. Thus, $\chi(L) = \chi(\mathcal{I} \otimes L) + \chi(L|D) = 3c + 1 - g$.

For every $L \in \mathrm{Pic}(X)$ set $\deg(L) = \chi(L) - \chi(\mathcal{O}_X)$. Thus, $\deg(L) = 3(\deg(L|D))$.

Remark 7. Here we assume $a \geq -1$ and $b \geq -1$. By Proposition 3 X is uniquely determined by the integers a, b, i.e. it is the split rational 3-rope with F as conormal module. By (6) for every integer c there is a unique

$L \in \mathrm{Pic}(X)$ such that $\deg(L|D) = c$, i.e. such that $\deg(L) = 3c$. If $c \geq 0$ we have $h^1(C, L) = 0$ and $h^0(X, L) = 3c + 3 + a + b = 3c + 1 - g$. It is easy to check that L is spanned if and only if $c \geq \max\{0, -a, -b\}$ and that L is very ample if and only if $c > \max\{0, -a, -b\}$. Hence the Brill-Noether theory of X is completely described. If $a \geq 0$ and $b \geq 0$ we may apply [1], Theorem 0.1, to study the property N_p for the minimal free resolution associated to any very ample line bundle on X. To cover the missing cases, in which either $a = -1$ or $b = -1$ we just state here the easy case concerning the postulation for arbitrary integers a, $b \geq -1$.

Proposition 4. *Let X be the rational 3-rope with $\mathcal{O}_D(a) \oplus \mathcal{O}_D(b)$ as conormal module, $a \geq -1$, $b \geq -1$. Fix an integer $c > 0$ and set $z := 1$ if $a \geq 0$ and $b \geq 0$ and $z := 2$ if either $a = -1$ or $b = -1$. Let $L \in Pic(X)$ be the only line bundle on X with degree $3c$. Then $L^{\otimes z}$ is very ample, $h^0(X, L^{\otimes z}) = 3cz + 1 - g$ and the embedding $\gamma : X \to \mathbf{P}^{3zc-g}$ is k-normal for every integer $k \geq 1$, i.e. for every integer $k \geq 1$ the restriction map $H^0(\mathbf{P}^{3zc-g}, \mathcal{O}_{\mathbf{P}^{3zc-g}}(k)) \to H^0(\gamma(X), \mathcal{O}_{\gamma(X)}(k))$ is surjective. However, the curve $\gamma(X) \subset \mathbf{P}^{3zc-g}$ is not artithmetically Cohen-Macaulay, unless $a = b = -1$ because $h^0(\gamma(X), \mathcal{O}_{\gamma(X)}) = 3+a+b$ and hence the restriction map $H^0(\mathbf{P}^{3zc-g}, \mathcal{O}_{\mathbf{P}^{3zc-g}}) \to H^0(\gamma(X), \mathcal{O}_{\gamma(X)})$ is not surjective, unless $a = b = -1$.*

Remark 8. Here we assume $a \geq -1$ and $b \leq -2$. There is a unique rank one subbundle $\mathcal{J}$ of F such that $\mathcal{J} \cong \mathcal{O}_D(a)$. Since $\mathcal{I}\mathcal{J} = 0$, $\mathcal{J}$ is an $\mathcal{O}_X$-sheaf. As an $\mathcal{O}_X$ sheaf the sheaf $\mathcal{J}$ defines a closed subscheme Z of X such that $\mathcal{O}_Z = \mathcal{O}_X/\mathcal{J}$. The local freeness of $\mathcal{O}_D(a)$ (or its homogeneity) implies that Z has no embedded point. Thus, Z is a rational rope with $\mathcal{O}_D(b)$ as conormal module and $p_a(Z) = -b + 1$. We have the exact sequences

$$0 \to \mathcal{O}_D(a) \to \mathcal{O}_X \to \mathcal{O}_Z \to 0\,, \tag{7}$$

$$0 \to \mathcal{O}_D(b) \to \mathcal{O}_Z \to \mathcal{O}_D \to 0\,. \tag{8}$$

By (8) $\mathrm{Pic}(Z)$ is an extension of $\mathrm{Pic}(D) \cong \mathbf{Z}$ by $H^1(D, \mathcal{O}_D(b)) \cong \mathbf{K}^{\oplus(-b-1)}$. By (7) $\mathrm{Pic}(X)$ is an extension of $\mathrm{Pic}(Z)$ by $H^1(D, \mathcal{O}_D(a)) = 0$, i.e. the restriction map $\mathrm{Pic}(X) \to \mathrm{Pic}(Z)$ is an isomorphism. Since $-b + 1 \geq 2$, there is a unique hyperelliptic ribbon of genus $-b+1$ (see [2], p. 729): the split ribbon with $\mathcal{O}_D(b)$ as conormal module. If we assume the existence of a $L \in \mathrm{Pic}(X)$ with $\deg(L) = 3$ and spanned (or at least generically spanned) we need to assume $h^0(Z, L|Z) \geq 2$, i.e. we need to assume that Z is the hyperelliptic ribbon and that $L|Z$ is the gree two hyperelliptic line bundle, R on Z which induces the splitting of Z. Assume Z hyperelliptic and call R its hyperelliptic line bundle. We just proved that there is a unique $L \in \mathrm{Pic}(X)$ such that $L|Z \cong R$. Since $b \leq -2$, the restriction map $H^0(Z,R) \to H^0(D,R|D) \cong \mathbf{K}^{\oplus 2}$ is an isomorphism. Hence $h^0(X,L) = a+4$. Notice that this is true for any 3-rope of genus g extending the hyperelliptic ribbon of genus $-b+1$. However, using Proposition 3 it is easy to see that there is a unique such 3-rope. Hence if $a \geq -1$ and $b \leq -2$ there is a unique trigonal 3-rope with $\mathcal{O}_D(a) \oplus \mathcal{O}_D(b)$ as conormal module and the trigonal line bundle L is unique. The restriction map $H^0(X,L) \to H^0(Z,L|Z)$ is surjective and the morphism $h_L : X \to \mathbf{P}(H^0(X,L)^*)$ associated to the spanned line bundle L when restricted to Z induces the splitting of Z, i.e. h_L collapses Z to D and sends X onto a double line of $\mathbf{P}^{a+3}$.

Remark 9. Here we assume $a \leq -2$ and $b \leq -2$. By (4) the restriction map $H^0(X,L) \to H^0(D,L|D)$ is injective for every degree three line bundle on X.

Proposition 5. *If $a \leq -2$ and $b \leq -2$ there is a spanned $L \in Pic(X)$ with $\deg(L) = 3$ if and only if is the split rational 3-ribbon. Assume $a \leq -2$, $b \leq -2$ and that X is the split rational 3-ribbon with $\mathcal{O}_D(a) \oplus \mathcal{O}_D(b)$ as conormal module. Then there is a unique spanned $L \in Pic(X)$ with $\deg(L) \leq 3$ and $h^0(X,L) \geq 2$; such an L has degree three and $h^0(X,L) = 2$ and the corresponding morphism $h_L : X \to \mathbf{P}^1$ induces the splitting of X.*

Proof. Assume the existence of $L \in \mathrm{Pic}(X)$ with $\deg(L) \leq 3$ and $h^0(X,L) \geq 2$. Since X is a 3-rope with $h^0(D,F) = 0$, F its conormal module, we have $\deg(L|D) > 0$. Thus, $\deg(L|D) = 1$ and $\deg(L) = 3$. By Remark 9 the restriction map $\rho : H^0(X,L) \to H^0(D,L|D)$ is injective. Hence ρ is bijective and $h^0(X,L) = 2$. Since L is spanned, we obtain a morphism $h_L : X \to \mathbf{P}^1$, whose restriction to D has degree one, i.e. it is an isomorphism. Hence h_L induces a splitting of X. Conversely, assume that X is the split 3-rope with $\mathcal{O}_D(a) \oplus \mathcal{O}_D(b)$ as conormal module. By assumption there is $f : X \to D$ such that $f|D$ is the identity. It is easy to check that $f^*(\mathcal{O}_D(1))$ is a spanned degree three line bundle on X. Furthermore, any two degree three morphism $X \to \mathbf{P}^1$ differs by an element of $\mathrm{Aut}(D)$ because for every $h \in \mathrm{Aut}(D)$ we have $h^*(\mathcal{O}_D(1)) \cong \mathcal{O}_D(1)$ and $H^0(D, \mathcal{O}_D(1))$ is an irreducible $SL(2)$-representation. □

Acknowledgements

The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

References

[1] E. Ballico, Embeddings of unreduced curves in projective space: postulation and minimal free resolution, *Bull. Inst. Math. Acad. Sinica*, To appear.

[2] D. Bayer, D. Eisenbud, Ribbons and their canonical embeddings, *Trans. Amer. Math. Soc.*, **347** (1995), 719–756.

[3] K.A. Chandler, Geometry of dots and ropes, *Trans. Amer. Math. Soc.*, **347** (1995), 767–784.

[4] D. Eisenbud, M. Green, Clifford indices of ribbons, *Trans. Amer. Math. Soc.*, **347** (1995), 757–765.

[5] L.-Y. Fong, Rational ribbons and deformations of hyperelliptic curves, *J. Algebraic Geometry*, **2** (1993), 295–307.

[6] G. Martens, F.-O. Schreyer, Line bundles and syzygies of trigonal curves, *Abh. Math. Sem.*, Univ. Hamburg, **56** (1986), 169–189.