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NOTES ON COMMUTATORS OF HARDY OPERATORS

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Abstract: Shunchao and Jian showed that the commutator of multiplication operator by b and Hardy operator is bounded on L^p if b is in one sided dyadic CMO^p. We prove the converse of this theorem.

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1. Introduction

Since Coifman, Rochberg and Weiss [1] introduced the commutator of multiplication by b and singular integral operator, many studies have been done about this commutator (see the references in [4]). Shunchao and Jian [4] considered the the commutator of multiplication operator by b and Hardy operator and introduced a new function space "one sided dyadic CMO". They proved this commutator is L^p bounded if b is a one sided dyadic CMO function.

In this paper we introduce a new function space "one sided dyadic Herz-Hardy space" and by using the duality argument we show the converse of their theorem.

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2. Definitions

The following notation is used: Let $R^+ = (0,\infty)$. The Hardy operators are defined by

$$Hf(x) = \frac{1}{x}\int_0^x f(t)dt \quad (x>0),$$

and (the adjoint operator)

$$H^*f(x) = \int_x^\infty \frac{f(t)}{t}dt \quad (x>0).$$

Let b be a locally integrable function on R^+. We define the commutator operator of multiplication by b and Hardy operator as follows:

$$H_b f(x) = b(x)Hf(x) - H(bf)(x) = \frac{1}{x}\int_0^x (b(x)-b(t))f(t)dt \quad (x>0),$$

and we also define:

$$H_b^* f(x) = b(x)H^*f(x) - H^*(bf)(x) = \int_x^\infty \frac{(b(x)-b(t))f(t)}{t}dt \quad (x>0).$$

Following [4], we define the one sided dyadic CMO^p space.

Definition 1. Let $1 \le p < \infty$. We say b is in $CMO^p(R^+)$, if

$$\sup_{j\in\mathbb{Z}}\Big(\frac{1}{2^j}\int_0^{2^j} |b(t)-b_{(0,2^j]}|^p dt\Big)^{1/p} = \|b\|_{CMO^p} < \infty,$$

where

$$b_{(0,2^j]} = \frac{1}{2^j}\int_0^{2^j} b(t)dt.$$

Remark . The John-Nirenberg space BMO is containd in CMO^p. $BMO(R^1) \subset CMO^q(R^+) \subset CMO^p(R^+)$, where $1 \le p < q < \infty$.

Next we define the one sided dyadic Herz-Hardy space HA^p.

Definition 2. Let $1 < p < \infty$. We say a is a one sided dyadic p-atom if there exists $j \in \mathbb{Z}$, which satisfies the following:

$$\text{supp}(a) \subset (0,2^j],$$

$$\Big(\frac{1}{2^j}\int_0^{2^j} |a(t)|^p dt\Big)^{1/p} \le 2^{-j},$$

$$\int a(t)dt = 0.$$

Definition 3. Let $1 < p < \infty$. We say f is in $HA^p(R^+)$ if f can be written as

$$f(x) = \sum_{j=-\infty}^{\infty} c_j a_j(x),$$

where a_j is a one sided dyadic p-atom supported in $(0, 2^j]$ and $\sum_j |c_j| < \infty$, and we define

$$\|f\|_{HA^p} = \inf \sum_j |c_j|,$$

where the infimum is taken over all representation of f.

Remark . $HA^p(R^+) \subset H^1(R^1)$, where $H^1(R^1)$ is the ordinary Hardy space.

The following proposition is easily proved by the standard argument (see for example [2] and [3], p. 289).

Proposition . *Let $1 < p < \infty$. The dual of $HA^p(R^+)$ is $CMO^{p'}(R^+)$, where $1/p + 1/p' = 1$.*

$$(HA^p(R^+))^* = CMO^{p'}(R^+).$$

3. Theorems

Shunchao and Jian [4] showed the following results.

Theorem A . *Let $1 < p < \infty$. If $b \in CMO^p(R^+) \cap CMO^{p'}(R^+)$, then H_b and H_b^* are bounded on $L^p(R^+)$.*

$$\Big(\int_0^\infty |H_b f(x)|^p dx\Big)^{1/p} \le C_p\big(\|b\|_{CMO^p} + \|b\|_{CMO^{p'}}\big)\Big(\int_0^\infty |f(x)|^p dx\Big)^{1/p},$$
$$\Big(\int_0^\infty |H_b^* f(x)|^p dx\Big)^{1/p} \le C_p\big(\|b\|_{CMO^p} + \|b\|_{CMO^{p'}}\big)\Big(\int_0^\infty |f(x)|^p dx\Big)^{1/p},$$

where C_p is a positive constant depending only on p.

We obtain the converse of this theorem.

Main Theorem . *Let $1 < p < \infty$. If H_b and H_b^* are bounded operators on $L^p(R^+)$, then b is in $CMO^p(R^+) \cap CMO^{p'}(R^+)$. Furthermore*

$$\|b\|_{CMO^p} + \|b\|_{CMO^{p'}} \le C_p\big(\|H_b\|_{L^p \to L^p} + \|H_b^*\|_{L^p \to L^p}\big).$$

4. Proof of Main Theorem

To prove Main Theorem we shall prove the following theorem.

Theorem . *Let $1 < p < \infty$. If H_b^* is a bounded operator on $L^p(R^+)$, then b is in $CMO^p(R^+)$. Furthermore*

$$\|b\|_{CMO^p} \leq C_p \|H_b^*\|_{L^p \to L^p}.$$

By using this theorem, we can prove Main Theorem. If H_b is bounded on L^p, then H_b^* is bounded on $L^{p'}$. Therefore by the theorem we obtain $b \in CMO^{p'}$.

Now we shall prove the theorem.

Proof of the theorem. By the duality between $HA^{p'}$ and CMO^p (see Proposition in Section 2), it suffices to show the following:

For any one sided dyadic p'-atom a,

$$\left| \int a(x)b(x)dx \right| \leq C_p \|H_b^*\|_{L^p \to L^p}. \tag{1}$$

To prove (1) we need the next lemma.

Lemma . *Let $1 < p < \infty$. For any one sided dyadic p'-atom a, there exist f and g such that*

$$a(x) = f(x)H^*g(x) - g(x)Hf(x),$$
$$\|f\|_{L^{p'}} \cdot \|g\|_{L^p} \leq (\log \tfrac{3}{2})^{-1}.$$

Proof. Let a be a one sided dyadic p'-atom supported in $(0, 2^j]$. We set

$$f(x) = (\log \tfrac{3}{2})^{-1} a(x) \quad \text{and} \quad g(x) = \chi_{[2^{j+1}, 2^{j+1}+2^j]}(x),$$

where χ is the characteristic function of an interval.

Because $Hf(x) = 0$ if $x > 2^j$, we have $g(x)Hf(x) = 0$.

If $x \leq 2^j$,

$$H^*g(x) = \int_{2^{j+1}}^{2^{j+1}+2^j} \frac{1}{t} dt = \log \tfrac{3}{2}.$$

So, we have $f(x)H^*g(x) = a(x)$, and obtain $a = fH^*g - gHf$.

Furthermore, we have

$$\|f\|_{L^{p'}} = \left(\log \frac{3}{2}\right)^{-1} \|a\|_{L^{p'}} \leq \left(\log \frac{3}{2}\right)^{-1} 2^{j/p'} 2^{-j},$$
$$\|g\|_{L^p} = 2^{j/p}.$$

Therefore $\|f\|_{L^{p'}}\|g\|_{L^p} \le \left(\log \frac{3}{2}\right)^{-1}$. □

By using this lemma, we shall prove (1).

$$\begin{aligned}\Big|\int a(x)b(x)dx\Big| &= \Big|\int \big(f(x)H^*g(x) - g(x)Hf(x)\big)b(x)dx\Big| \\ &= \Big|\int f(x)H_b^*g(x)dx\Big| \le \|f\|_{L^{p'}}\|H_b^*g\|_{L^p} \\ &\le \|H_b^*\|_{L^p\to L^p}\|f\|_{L^{p'}}\|g\|_{L^p} \le (\log\frac{3}{2})^{-1}\|H_b^*\|_{L^p\to L^p}.\end{aligned}$$

So, we obtain the desired result. □

5. Remark

Shunchao and Jian [4] considered the fractional order Hardy operators

$$H^\alpha f(x) = \frac{1}{x^{1-\alpha}}\int_0^x f(t)dt \quad \text{and} \quad H^{\alpha*}f(x) = \int_x^\infty \frac{f(t)}{t^{1-\alpha}}dt,$$

and the commutators

$$\begin{aligned}H_b^\alpha f(x) &= b(x)H^\alpha f(x) - H^\alpha(bf)(x),\\ H_b^{\alpha*} f(x) &= b(x)H^{\alpha*} f(x) - H^{\alpha*}(bf)(x).\end{aligned}$$

They proved the boundedness of these commutators. By the same argument as above, we can show the following theorem.

Theorem . *Let $1 < p, q < \infty$ and $1/p - 1/q = \alpha$, where $0 < \alpha < 1$. If H_b^α and $H_b^{\alpha*}$ are bounded operators from $L^p(R^+)$ to $L^q(R^+)$, then b is in $CMO^q(R^+) \cap CMO^{p'}(R^+)$. Furthermore,*

$$\|b\|_{CMO^q} + \|b\|_{CMO^{p'}} \le C_{p,q}\big(\|H_b^\alpha\|_{L^p\to L^q} + \|H_b^{\alpha*}\|_{L^p\to L^q}\big).$$

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