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CONTINUOUS HOMOGENEOUS POLYNOMIALS AND ZERO-DIMENSIONAL ANALYTIC SUBSETS OF INFINITE-DIMENSIONAL PROJECTIVE SPACES

E. Ballico
Department of Mathematics
University of Trento
380 50 Povo (Trento) - Via Sommarive, 14, ITALY
e-mail: ballico@science.unitn.it

Abstract: Let V be a locally convex and Hausdorff complex topological vector space and Z a zero-dimensional closed analytic subscheme of $\mathbf{P}(V)$. Here we prove that the restriction map

$$\rho_{Z,t} : H^0(\mathbf{P}(V), \mathcal{O}_{\mathbf{P}(V)}(t)) \to H^0(Z, \mathcal{O}_Z(t))$$

has dense image if and only if for every finite-dimensional linear subspace W of V the restriction map

$$\rho_{Z\cap W,t;W} : H^0(\mathbf{P}(W), \mathcal{O}_{\mathbf{P}(W)}(t)) \to H^0(Z \cap W, \mathcal{O}_{Z\cap W}(t))$$

is surjective.

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1. Introduction

Let V be a locally convex and Hausdorff complex topological vector space, $\mathbf{P}(V)$ the projective space of all one-dimensional linear subspaces of V and $Z \subset \mathbf{P}(V)$ a closed analytic subset of $\mathbf{P}(V)$. For any integer $d \geq 0$ let $S^d(V)$ be the set of all continuous degree d homogeneous polynomials on V (see [1]). For any

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integer t let $\mathcal{O}_{\mathbf{P}(V)}(t)$ be the degree t holomorphic line bundle on $\mathbf{P}(V)$. We have $H^0(\mathbf{P}(V), \mathcal{O}_{\mathbf{P}(V)}(t)) = 0$ if $t < 0$ and $H^0(\mathbf{P}(V), \mathcal{O}_{\mathbf{P}(V)}(t)) \cong S^t(V)$ if $t \geq 0$. Call $\rho_{Z,t} : H^0(\mathbf{P}(V), \mathcal{O}_{\mathbf{P}(V)}(t)) \to H^0(Z, \mathcal{O}_Z(t))$ the restriction map. Now, we assume that all connected components of Z are zero-dimensional, i.e. we assume that the set $S := Z_{red}$ is a discrete subset of $\mathbf{P}(V)$. We allow the case in which Z is not reduced; it just means that any connected component of Z is a zero-dimensional analytic space (in the sense of finite-dimensional analytic spaces) or, equivalently, a zero-dimensional complex scheme (in the sense of algebraic geometry). If $Z = S$ is reduced and S is infinite and countable, then for any integer t we have $H^0(Z, \mathcal{O}_Z(t)) \cong H^0(Z, \mathcal{O}_Z) \cong \mathbf{C}^{\mathbf{N}}$ (with the product topology). For more on infinite-dimensional holomorphy, see [3], [2] or [1]. In this note we shall give the following criterion for the density of $\mathrm{Im}(\rho_{Z,t})$.

Theorem 1. *Let V be a locally convex and Hausdorff complex topological vector space and Z a zero-dimensional closed analytic subscheme of $\mathbf{P}(V)$. The restriction map*

$$\rho_{Z,t} : H^0(\mathbf{P}(V), \mathcal{O}_{\mathbf{P}(V)}(t)) \to H^0(Z, \mathcal{O}_Z(t))$$

has dense image if and only if for every finite-dimensional linear subspace W of V the restriction map

$$\rho_{Z\cap\mathbf{P}(W),t;W} : H^0(\mathbf{P}(W), \mathcal{O}_{\mathbf{P}(W)}(t)) \to H^0(Z \cap \mathbf{P}(W), \mathcal{O}_{Z\cap\mathbf{P}(W)}(t))$$

is surjective.

In the statement of Theorem 1 $Z \cap \mathbf{P}(W)$ denotes the scheme-theoretic intersection. Notice that $H^0(Z \cap W, \mathcal{O}_{Z\cap\mathbf{P}(W)}(t))$ is a finite-dimensional vector space. The proof of Theorem 1 will show that to obtain the denseness of $\mathrm{Im}(\rho_{Z,t})$ it is sufficient to assume the surjectivity of the map $\rho_{Z\cap\mathbf{P}(W),t;W}$ for all finite-dimensional linear subspaces W of V such that $\mathbf{P}(W)$ is spanned by finitely many connected components of Z.

Very seldom the map $\rho_{Z,t}$ is surjective, unless S is finite (see Example 1 and Example 2).

2. Proof of Theorem 1 and the Examples

Lemma 1. *Let V be a locally convex and Hausdorff complex topological vector space and W a finite-dimensional subspace. Then for all integers t the restriction map $\rho_{\mathbf{P}(W),t} : H^0(\mathbf{P}(V), \mathcal{O}_{\mathbf{P}(V)}(t)) \to H^0(\mathbf{P}(W), \mathcal{O}_{\mathbf{P}(W)}(t))$ is surjective.*

Proof. The result is obvious if $t < 0$ because $H^0(\mathbf{P}(W), \mathcal{O}_{\mathbf{P}(W)}(t)) = \{0\}$ if $t < 0$ and if $t = 0$ because every holomorphic function on $\mathbf{P}(W)$ is constant. The result is true for $t = 1$ because W is topologically a direct factor of V (see [4], Theorem 5, p. 38). If $t \geq 2$ the surjectivity of $\rho_{\mathbf{P}(W),t}$ follows from the surjectivity of $\rho_{\mathbf{P}(W),1}$ and the fact that any degree t homogeneous polynomial on W is a linear combination of products of degree one forms. □

Proof of Theorem 1. Since the result is trivially true if $t \leq 0$, we may assume $t > 0$. The "only if part" is obvious. Let $\{Z_i\}_{i \in I}$ be the set of all connected components of Z. Hence $H^0(Z, \mathcal{O}_Z(t)) \cong \prod_{i \in I} H^0(Z_i, \mathcal{O}_{Z_i}(t))$. Thus, by the very definition of product topology to check the "if part" it is sufficient to show that if the condition is satisfied, then for any finite subset J of I the subset $\prod_{i \in J} H^0(Z_i, \mathcal{O}_{Z_i}(t))$ is contained in $\mathrm{Im}(\rho_{Z,t})$, with the convention that we see $\prod_{i \in J} H^0(Z_i, \mathcal{O}_{Z_i}(t))$ as a quotient of $\prod_{i \in I} H^0(Z_i, \mathcal{O}_{Z_i}(t))$ in which the factors Z_i with $i \in (I \backslash J)$ go to zero. Set $Z' := \cup_{i \in J} Z_i$. Let W be the linear subspace of V such that $\mathbf{P}(W)$ is the linear span of Z'. Hence W is finite-dimensional and $Z' = Z \cap \mathbf{P}(W)$. By Lemma 1 we have $\mathrm{Im}(\rho_{Z',t}) = \mathrm{Im}(\rho_{Z',t;W})$, concluding the proof. □

Example 1. Let V be a separable Hilbert space and call $\{e_i\}_{i \in \mathbf{N}}$ an orthonormal basis of V. Set $Z = \cup\{e_i\}_{i \in \mathbf{N}}$ with the reduced analytic structure. The dual space V' is an Hilbert space with the dual basis $\{e_i^*\}_{i \in \mathbf{N}}$ as an orthonormal basis. We have $H^0(Z, \mathcal{O}_Z(1)) \cong \prod_{i \in \mathbf{N}} \mathbf{C} e_i^*$. Notice $\mathrm{Im}(\rho_{Z,1})$ is a proper subset of $\prod_{i \in \mathbf{N}} \mathbf{C} e_i^*$ isomorphic to l^2.

Example 2. Take $V = \mathbf{C}^{\mathbf{N}}$. Thus, V is a Fréchet nuclear space. Every continuous homogeneous polynomial on V depends only on finitely many variables, i.e. for all integers $t \geq 0$ the vector space $S^t(V)$ is the order t algebraic symmetric product of $\mathbf{C}^{\oplus \mathbf{N}}$. Hence $\rho_{Z,t}$ is not surjective if Z is not finite.

Remark 1. Assume that $S = Z_{red}$ is finite. Hence Z is contained in a finite-dimensional linear subspace $\mathbf{P}(W)$ of $\mathbf{P}(V)$. Thus, $H^0(Z, \mathcal{O}_Z(t))$ is finite-dimensional. Hence the map $\rho_{Z,t}$ is surjective if and only if its image is dense. By Lemma 1 $\rho_{Z,t}$ is surjective if and only if the restriction map $H^0(\mathbf{P}(W), \mathcal{O}_{\mathbf{P}(W)}(t)) \to H^0(Z, \mathcal{O}_Z(t))$ is surjective.

Remark 2. Lemma 1 is true with the same proof for real topological vector spaces. Hence Theorem 1 is true for a real topological vector space V and for a real analytic closed subscheme of $\mathbf{P}(V)$.

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