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NODAL CURVES AND CURVES WITH FEW SINGULARITIES IN $\mathbf{P}^n$

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Abstract: Here we construct nodal integral non-degenerate curves $C \subset \mathbf{P}^n$, $n \geq 4$, with prescribed degree, arithmetic genus and number of nodes and with the same minimal free resolution as certain smooth curves.

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1. Introduction

Here we construct nodal integral non-degenerate curves $C \subset \mathbf{P}^n$, $n \geq 4$, with prescribed degree, arithmetic genus and number of nodes and with the same minimal free resolution as certain smooth curves.

Theorem 1. *Fix integers $n \geq 4$, $r \geq 4$ and $g_0 \geq 0$ and $d_0 \geq g_0 + n$. For $n = 4, 5$ take $x = n$ or $x = n + 2$. For $n \geq 6$ set $x = 2n - 3$ and $w := x + n - 1$. Define inductively the integers G_i, $0 \leq i \leq r$ by the formulas $G_0 := g_0$, $G_i := G_{i-1} + d_0 + (i-1)x + w$ for every $i > 0$. Set $d := d_0 + rx$ and $p_a := G_r$. Fix an integer δ such that $0 \leq \delta \leq r(x + 2n - 5)/2)$. Then there exists an integral non-degenerate curve $C \subset \mathbf{P}^n$ such that $\deg(C) = d$, $p_a(C) = p_a$ and C has exactly δ nodes as only singularities. Furthermore, if $d \geq (4 + n)x$ for every integer δ' such that $\delta < \delta' \leq q$ there exists a flat family $\{C_t\}_{t \in T}$ of integral non-degenerate degree d curves in $\mathbf{P}^n$ such that:*

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(i) *there is* $o \in T$ *such that* $C_o = C$;

(ii) *for every* $t \in T\backslash\{o\}$ *the integral curve* C_t *have exactly* δ' *nodes as only singularities;*

(iii) *all* C_t, $t \in T$ *have the same Betti numbers, i.e. the minimal free resolutions of all* C_t *have the same numerical invariants.*

Here we ask the following question concerning the exixtence of curves with small numbers of singular points and contained in a certain linear system of a smooth surface.

Question 1. Let F be a smooth and connected projective surface and $D \subset F$ a smooth and connected curve such that the linear system$|D|$ "moves sufficiently" (e.g. it is very ample):

(a) For which integers b such that $0 \leq b < p_a(D)$ there is an integral curve $C \in |D|$ such that C has a unique singular point and the normalization of C has genus b?

(b) For which integers $s \geq 2$, $q_i \geq b_i \geq 0$, $1 \leq i \leq s$, such that $\sum_{i=1}^{s} q_i < p_a(D)$ there is a curve $C \in |D|$ with a unique singular point and exactly s irreducible components, say $C_1, \dots C_s$, such that for every i we have $p_a(C_i) = q_i$ and the normalization of C_i has genus b_i?

Of course, this is not always possible. For instance, an Abelian variety contains no integral curve whose normalization is rational and hence if the Albanese mapping of F is an embedding we cannot take $b = 0$ or $b_i = 0$ for at least one index i in Question 1.

We work over an algebraically closed field $\mathbf{K}$ with $\mathrm{char}(\mathbf{K}) = 0$.

2. Proof of Theorem 1

Remark 1. Let $S \subset \mathbf{P}^n$ one of the following smooth non-degenerate rational surfaces:

(i) a Del Pezzo surface in $\mathbf{P}^n$, $n = 4$ or $n = 5$; here $\deg(S) = n$;

(ii) a Bordiga surface in $\mathbf{P}^4$ (see [7], 1.2), i.e. the blowing-up of $\mathbf{P}^2$ at ten points $P_1, \dots, P_{10}$ in general position embedded by the linear system of all quartic curves passing through these ten points;

(iii) a Bordiga surface in $\mathbf{P}^5$ (see [7], 2.2), i.e. the blowing-up of $\mathbf{P}^2$ at nine points $P_1, \dots, P_9$ in general position embedded by the linear system of all quartic curves passing through these nine points;

(iv) $n \geq 5$ and S is the degree $2n-3$ surface considered in part (i) of the Main Theorem of [3], i.e. the blowing-up of $\mathbf{P}^2$ at nine points $Q_1, \dots, Q_9$ which are the base points of a general pencil of plane cubic; S is embedded by the linear system $D_i - m\omega_S$ with D_i as described in [3], Theorem 8; here $n = 3m + i$ with $m \in \mathbf{N}$ and $i \in \{0, 1, 2\}$.

(v) $n \geq 6$, say $n = 3k + h$ with $h \in \{0, 1, 2\}$, and S is described in [2], 1.a, i.e. S is the blowing-up of $\mathbf{P}^2$ at $6 - h$ general points $P_1, \dots, P_{6-h}$ embedded by the linear systems of all degree $k + 2$ plane curves passing through P_1 with multiplicity at least k and containing $P_2, \dots, P_{6-h}$.

The paper [1] contains an easy proof that S is arithmetically Cohen-Macaulay (see [1], the lemmas).

Lemma 1. *Let $S \subset \mathbf{P}^n$ any of the smooth rational surfaces listed in Remark 1. Set $x := \deg(S)$ and $w := x - n + 1$. Then for every integer y such that $0 \leq y \leq w$. Then there is a hyperplane $H \subset \mathbf{P}^n$ such that the scheme-theoretic intersection $D := H \cap S$ is an integral curve with exactly y nodes as only singularities. More, precisely, we do not claim that the result is true for all surfaces S as in Remark 1, but only that for each case (i), (ii), (iii), (iv) and (v) listed in Remark 1 and for every $y \leq w$ there is at least one S as in that case for which the statement is true and that the surfaces S obtained in this way are enough to prove Theorem 1.*

Proof. Let $M \subset \mathbf{P}^n$ be any hyperplane such that the scheme-theoretic intersection is integral. We have $p_a(M \cap X) = w$. In case (i) (resp. (ii), resp. (iii), resp (iv), resp. (v)) of Remarks 1 we have $w = 1$ (resp. $w = 3$, $w = 3$, $w = 3n - 4$, $w = k$).

(a) Here we consider cases (ii) and (iii) of Remark 1. Let $C \subset \mathbf{P}^2$ be an integral quartic plane curve with exactly $y \leq 3$ nodes as only singularities. Take as S for case (ii) (resp. (iii)) the blowing-up of $\mathbf{P}^2$ at ten (resp. nine) general points of C and as $S \cap H$ the strict transform of C.

(b) Here we consider case (iv) of Remark 1. Set $n = 3m$ with $m \in \mathbf{N}$ and $i \in \{0, 1, 2\}$. Let M be the blowing-up of $\mathbf{P}^2$ at nine points $P_1, \dots, P_9$ which are the base points of a "sufficiently generic" (see below) pencil of plane cubics. With the notation of [3], Section 1, we have $\omega_M = (-3, 1, \dots, 1)$ and $H := D_i -$

ω_M, where $D_0 = (0,0,0,0,0,0,0,-1,-1,-1)$, $D_1 = (1,1,-1,0,0,0,0,0,0,0)$ and $D_2 = (1,0,0,0,0,0,0,0,0,0)$ (i.e. D_2 is the total transform of any line) is a 3-system; H is very ample. Here $n \geq 2$; if $i = 0$, then $|H|$ is the linear system of all plane curves of degree $3m$ with multiplicity at least m at $P_1, \ldots, P_6$ and multiplicity at least $m+1$ at P_7, P_8 and P_9; if $i = 1$, then $|H|$ is the linear system of all plain curves of degree $3m+3$ with multiplicity at least $m-1$ at P_1, multiplicity at least $m+1$ at P_2 and multiplicity at least m at $P_3, \ldots, P_9$; if $i = 2$, then $|H|$ is the linear system of all plane curves of degree $3m+1$ with multiplicity at least m at $P_1, \ldots, P_9$.

(c) Here we consider case (v) of Remark 1. Fix $P \subset \mathbf{P}^2$ and let $u : T \to \mathbf{P}^2$ be the blowing-up of P. The smooth surface S is isomorphic to the Hirzebruch surface F_1 and we take as a basis of $\mathrm{Pic}(T)$ the exceptional divisor $h := u^{-1}(P)$ and the strict transform f of a line through P. Hence $h^2 = -1$, $h \cdot f = 1$ and $f^2 = 0$. It is easy to check that the linear system $|2h + (k+2)f|$ is very ample and that for every integer y such that $1 \leq y \leq k$ there is an integral $C \in |2h + (k+2)f|$ with exactly y nodes as only singularities (use for instance [9], Remark 3.2, or construct a singular nodal hyperelliptic curve of genus k with exactly y nodes). Take as S the blowing-up of T at $5-h$ general points of C and as $C \cap H$ the strict transform of C.

(d) We leave the very easy case (i) to the reader (here either $y = 0$ or $y = 1$). □

Remark 2. Here we give the background needed to prove the part of Theorem 1 concerning the minimal free resolution of the curves C_t. Let $S \subset \mathbf{P}^n$ be a smooth surface and $C, D \subset S$ effective Cartier divisors such that $\mathcal{O}_S(C) \cong \mathcal{O}_S(-D))$. Hence for every vector bundle E on S we have $h^1(S, \mathcal{I}_C \otimes E) = h^1(S, \mathcal{I}_D \otimes E)$. As general references on syzygies of projective schemes one can use the original paper by M. Green (see [5]) or the introduction of [6]. Set $\mathcal{O} := \mathcal{O}_{\mathbf{P}^n}$. For every integer k such that $0 \leq k \leq n$ set $\Omega^k := \wedge^k(\Omega_{\mathbf{P}^n})$. Let $X \subset \mathbf{P}^n$ be any closed subscheme. The minimal free resolution of $\mathcal{I}_X$ is essentially determined bu the ranks of all restriction maps. $H^0(\mathbf{P}^n, \Omega^k(t)) \to H^0(X, \Omega^k(t)|X)$ for all integers k, t such that $0 \leq k \leq n$. In our set-up it is even sufficient to know the ranks of these linear maps for a small number of pairs (k,t) or, equivalently, to know the value of the integer $h^1(\mathbf{P}^n, \mathcal{I}_X \otimes \Omega^k(t))$ for a small number of pairs (k,t) (see Remark 3)). The bundle $\Omega(2)$ is spanned by its global sections. Hence the bundle $\Omega^k(2k)$ is spanned by its global sections. Notice that $H^0(\mathbf{P}^n, \mathcal{O}(2))$ spans the 2-jets of the line bundle $\mathcal{O}(2)$. Thus if $\mathbf{K} = \mathbf{C}$ for all integers k, t such that $0 \leq k \leq n$ and $t \geq 2$ the bundles $\Omega^k(2k+t)$ and $\Omega^k(2k+t)$ are Nakano positive. Hence reducing to the case $\mathbf{K} = \mathbf{C}$ and applying Nakano

Vanishing Theorem ([8], Corollary 6.14) we obtain $h^1(\mathbf{P}^n, \Omega^k(2k+t-n-1)) = 0$ for every $t \geq 2$ and $h^1(S, (\Omega^k(2k+t)|S) \otimes K_S) = 0$ for every $t \geq 2$. If $T \subset S$ is a reduced curve and z a positive integer such that $z \cdot \deg(S) < \deg(T)$, then every degree t hypersurface containing S contains T. Hence the homogeneous ideal of T is generated by any set of generators of the homogeneous ideal of X and by forms of degree at least $\deg(T)/\deg(S)$.

Lemma 2. *Let $S \subset \mathbf{P}^n$ be any of the smooth surfaces listed in Remark 1. Then its Betti numbers $b_{p,q}(S)$ satify $b_{p,q}(S) = 0$ unles $q = p+2$ or $q = p+3$; here the subscripts in the Betti numbers are normalized in such a way that the condition "$b_{0,q}(S) = 0$ unless $q = 2$ or $q = 3$" means that the homogeneous ideal of S is generated by forms of degree 2 and degree 3.*

Proof. Since S is arithmetically Cohen-Macaulay, $b_{p,q}(S) = b_{p,q}(S \cap H)$, where $H \subset \mathbf{P}^n$ is a general hyperplane and $b_{p,q}(S \cap H)$ are computed seeing $S \cap H$ as a subscheme of the projective space. The smooth curve $S \cap H$ satisfies $h^1(S \cap H, \mathcal{O}_H(1)) = 0$ and it is linearly normal in H; for the latter assertion we use that S is rational and hence $h^1(S, \mathcal{O}_S) = 0$. Furthermore, $h^1(S \cap H, \mathcal{O}_{S \cap H}(2)) = 0$ because $S \cap H$ is projectively normal (here we use that S is arithmetically Cohen-Macaulay). Hence $b_{p,q}(S) = 0$ if $q \geq p+4$ by Catelnuovo-Mumford Lemma. We have $b_{p,q}(S) = 0$ if $q \leq p+1$ because S is not contained in a hyperplane. □

Remark 3. Let $A \subset S$ be an effective divisor of degree d and t any integer such that $tz < d$. Every degree t hypersurface of $\mathbf{P}^n$ contains S. Hence, if $d > 3x$ the homogeneous ideal of A is generated by the forms containing S and by forms of degree at least 4. By Remark 2 we have $h^1(S, \Omega^k(k+z)|S) = 0$ for every $z \geq 4$. By Lemma 2 the restriction map $\rho_{S,k,z} : H^0(\mathbf{P}^n, \Omega^k(k+z)) \to H^0(S, \Omega^k(k+z)|S)$ for every $z \geq 4$. Hence for all effective divisors A, B of S with $\mathcal{O}_S(-A) \cong \mathcal{O}_S(-B)$ we have $\mathrm{Corank}(\rho_{A,k,z}) = \mathrm{Corank}(\rho_{A,k,z})$. Hence to prove the last part of Theorem 1 it is sufficient to find C and C_t, $t \in T$, such that $|C| = |C_t|$ for every $t \in T$.

Proof of Theorem 1. Let $S \subset \mathbf{P}^n$ one of the surfaces in cases (i), (ii), (iii) or (iv) for with there is a smooth connected and non-degenerate curve $D \subset \mathbf{P}^n$ (see [7] for $n = 4, 5$ and [3] for $n \geq 5$). By the proofs in [7] and [3] we have $N \in |A + rH|$, where H is a hyperplane section of S and A is a smooth and connected curve with degree d_0 and genus g_0. By Lemma 1 there is a nodal integral curve $B_i \in |H|$ with exactly w nodes (i.e. a nodal rational curve). We may also find $B_1, \dots, B-r$ such that the reduced curve $A \cup B_1 \cup \dots \cup B_r$. We have $H^2 + 2 \cdot K_S = 2p_a(H) - 2 = 2w - 2$ (adjunction formula), i.e. $H \cdot K_S = 2w - 2 - x = x + 2n - 2$. Fix a finite set $E \subseteq \mathrm{Sing}(C)$ and let $f : F \to S$ the blowing-up of E. Let $T \subset F$ denote the strict transform of N. Thus T is nodal and $\mathrm{card}(\mathrm{Sing}(T)) = \mathrm{card}(\mathrm{Sing}(N)) - \mathrm{card}(A)$. If for every irreducible component B of N have weight $(E \cap B) < -B \cdot K_S$, where the weight of a point P of $E \cap B$ is one if B is smooth at P and two if B is nodal at P, then every irreducible component B' of T satisfies $T \cdot K_F < 0$. Hence we may apply [4], Lemma 3, and obtain that a general $M \in |T|$ is smooth. Set $C' := f(M)$. C' is nodal and with exactly $\mathrm{card}(A)$ singular points. If for every irreducible component B of N we have $N \cap (\mathrm{Sing}(N) \backslash A) \neq \emptyset$, then M is connected and hence C' is irreducible. The upper bound on δ is exactly the one which gives the condition weight $(E \cap B) < -B \cdot K_S$ for all B and allows us to satisfies also the connectedness condition. For the last assertion use Remark 2, Remark 3 and Lemma 2. □

3. One Singular Point?

Lemma 3. *Let $S \subset \mathbf{P}^n$, $n \geq 4$, be any of the smooth surfaces listed in Remark 1. Set $x := \deg(S)$ and $w := x + n - 1$. Fix an integer y such that $0 \leq y \leq w$. There is a hyperplane $H \subset \mathbf{P}^n$ and $Q \in S \cap H$ such that the scheme $S \cap H$ is integral, the normalization of $S \cap H$ has genus $w - y$ and $S \cap H$ is smooth outside Q. More, precisely, we do not claim that the result is true for all surfaces S as in Remark 1, but only that for each case (i), (ii), (iii), (iv) and (v) listed in Remark 1 and for every $y \leq w$ there is at least one S as in that case for which the statement is true and that the surfaces S obtained in this way are enough to prove Theorem 1 and Theorem 2.*

Proof. Case (i) of Remark 1 is contained in Lemma 2.2 (and $S \cap H$ may even be taken nodal) because $w = 1$ in this case.

(a) Here we consider cases (ii) and (iii) of Remark 1. Fix $Q \in \mathbf{P}^2$ and let $Z \subset \mathbf{P}^2$ a suitable zero-dimensional scheme with length 6 in case (ii) and length 7 in case (iii). For instance, if $y = 3$ we take as Z either the second infinitesimal neighborhood $3Q$ of Q in $\mathbf{P}^2$ or a general lingth 7 scheme containin $3Q$; if $y = 2$ we take as Z a sufficiently general length 6 (resp. 7) containing the first infinitesimal neighborhood $2Q$ of Q in $\mathbf{P}^2$ and for $y = 1$ we take as Z a general length 6 (or 7) scheme containing Z' with $2Q \subset Z' \subset 3Q$, Z' becoming a suitable conductor. Let $C, C' \subset \mathbf{P}^2$ general degree 4 plane curves containing Z. Hence $C \cap C'$ is the union of ten (resp. nine) different smooth points of C and C' and we take as S the blowing-up of $\mathbf{P}^2$ at these points.

(b) Here we consider cases (iv) and (v) of Remark 1. Just use that at a general $Q \in S$ we may prescribe the local parametrization of S up to order m using n affine coordinates, say x_i/x_0, $1 \le i \le n$ if $Q = (1; 0, \dots, 0) \in \mathbf{P}^n$. $\square$

Lemma 4. *Let $S \subset \mathbf{P}^n$, $n = 4$ or $n = 5$, be any of the smooth surfaces considered in cases (i), (ii) and (iii) of Remark 1. Set $x := \deg(S)$, i.e. set $x = n$ in case (i) and $x = 2 + n$ in cases (ii), (iii). Set $w := x + n - 1$. Fix an integer $r \ge 2$ and b_i, $1 \le i \le r$, such that $0 \le b_i \le r$. Then there exists integral hyperplane sections C_i, $1 \le i \le r$, of S such that the normalization of C_i has genus b_i and $C_1 \cup \dots \cup C_r$ has a unique singular point.*

Proof. Lemma 3 gives integral hyperplane sections $C_1, \dots C_r$ with prescribed geometriv genus and each of them with at most one singular point. We may also find $Q \in S$ such that every C_i has Q as the only singular point (if $b_i < w$). We need to find $C_1, \dots, C_r$ as above such that $C_i \cap C_j = \{Q\}$ for all $i \ne j$. This is done in part (a) of the proof of Lemma 3 for cases (ii) and (iii). We leave the easier case (i) to the reader. $\square$

Theorem 2. *Let $S \subset \mathbf{P}^n$, $n \ge 4$, be any of the smooth surfaces considered in Lemma 3 and listed in Remark 1. Set $x := \deg(S)$ (i.e. $x = n$ in case (i), $x = 6$ and $n = 4$ in case (ii), $x = 7$ and $n = 5$ in case (iii), $x = 2n - 3$ and $n \ge 6$ in case (iv), $x = 4m + h - 1$, $n = 3m + h$ with $m \ge 2$, m integer and $h \in \{0, 1, 2\}$ in case (v)). Set $w := x + n - 1$. Fix integers $r > 0$, d_0, g_0 such that:*

- *(i) in case (i) for $n = 4$ we assume $d_0 \ge 12$ and $0 \le g_0 \le d_0^2/8 - d_0/2 + 1$; in case (i) for $n = 5$ we assume $d_0 \ge 20$ and $(d_0 + 30)\sqrt{2d_0 + 40} - 23d_0/2 - 189 < g_0 \le d_0^2/10 - d/2 + 1$;*
- *(ii) in case (ii) we have $n = 4$; we assume $0 \le g_0 \le d^2/12 - d/6 + 11/12$;*
- *(iii) in case (iii) we have $n = 5$; we assume $d_0 \ge 5$ and $0 \le g_0 \le d_0^2/14 - 3d_0/14 + 9/56$;*
- *(iv) in case (iv) we have $n \ge 6$; we assume $d_0 \ge n$ and $0 \le g_0 \le (d_0 - n)^2/2(2n - 3)$;*
- *(v) in case (v) we have $n = 3m + h$ with $m \ge 2$, m integer and $h \in \{0, 1, 2\}$; we assume the conditions of [2], Theorem 1.1 for the pair (d_0, g_0).*

Set $d := d_0 + rx$, $G_0 := g_0$ and define inductively the integers G_i, $1 \le i \le r$, by the formula $G_i := G_{i-1} + d_0 + (i - 1)x + w$ for every $i > 0$. Set $g := G_r$. Fix integers b_i, $1 \le i \le r$; in cases (i), (ii) and (iii) ssume $0 \le b_i \le w$; in cases *(iv) and* (v) *assume the existence of an integer* m *such that* $m(m+1)/2 \le n$ *and* $w - b_i \le m(m-1)/2$. *Then there exists connected and reduced curves* $C, C' \subset S$ *and* $Q \in C'$ *such that* $\deg(C) = \deg(C') = d$, $p_a(C) = p_a(C') = g$, $\mathcal{O}_S(C) \cong \mathcal{O}_S(C')$, C *is smooth,* C' *has exactly* $r+1$ *irreducible components, say* $A, C_1, \dots, C_r$, A *is smooth of degree* d_0 *and genus* g_0, *each* C_i, $1 \le i \le r$, *is a hyperplane section of* S *(hence* $\deg(C_i) = x$ *and* $p_a(X) = w$*) and its normalization has genus* b_i, $\mathrm{Sing}(C_i)$ *is one point and the Betti numbers of the minimal free resolution of* C *and* C' *are the same. In cases (i), (ii) and (iii) we may find* C' *such that* $\mathrm{Sing}(C_1 \cup \dots \cup C_r)$ *is one point.*

Proof. Start with A and apply Lemma 3 r times obtaing curves $C_1, \dots, C_r$ with the same singular point. In cases (i), (ii) and (iii) for the last assertion use Lemma 4. □

Take $A = \emptyset$; the proof of Theorem 2 gives the following result.

Proposition 1. *Fix an integer* $r \ge 4$ *and* $S \subset \mathbf{P}^n$, $n \ge 4$, *be any of the smooth surfaces listed in Remark 1. Set* $x := \deg(S)$ *and* $w := x + n - 1$. *Set* $G_0 := g_0$; *for* $1 \le i \le r$, *define inductively the integer* G_i *by the formula* $G_i := G_{i-1} + d_0 + (i-1)x - 1 + w$. *Fix integers* b_i, $1 \le i \le r$, *such that* $0 \le b_i \le w$ *for every* i. *There are reduced, connected and non-degenerate curves* $C, C' \subset \mathbf{P}^n$ *with degree* rw *and arithmetic genus* G_r *and with the same Betti numbers of their minimal free resolution such that* C *is smooth,* C' *has a unique singular point,* C' *has exactly* r *irreducible components, say* C_i, $1 \le i \le r$, *and for every* i $p_a(C_i) = w$ *and the normalization of* C_i *has genus* b_i.

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