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YANG-MILLS CONNECTIONS IN HOMOGENEOUS PRINCIPAL FIBRE BUNDLES

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Abstract: Let K be a compact connected Lie-group of automorphisms of a principal fibre bundle $P(M,G)$ which acts fibre-transitively on P. We obtain a necessary and sufficient condition for a K-invariant connection in $P(M,G)$ to be a Yang-Mills connection, and give such examples.

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1. Introduction

Yang-Mills connections in a G-principal fibre bundle P over a compact Riemannian manifold M are the extrema of the Yang-Mills functional. In case of $\dim(M) = 4$, (anti-) self-dual connections are always Yang-Mills connections and the theory of (anti-) self-dual connections has been greatly developed. Nevertheless, to solve the Yang-Mills equation, itself, is still a difficult problem because of its non-linearity.

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principal fibre bundle $P(M,G)$ which acts fibre-transitively on P, we obtain a necessary and sufficient condition for a K-invariant connection in $P(M,G)$ to be a Yang-Mills connection (cf. Theorem 4.1). Moreover as examples, we treat the case $K = SU(3)$ and $G = SU(2)$, and the base manifold $M = CP^2$ (cf. Proposition 5.2 and Theorem 5.4).

2. Yang-Mills Connections

For a compact Lie group G with the Lie algebra $\mathbf{g}$, let $P(M,G,\pi)$ be a principal G-bundle over a compact Riemannian manifold (M,h). A $\mathbf{g}$-valued 1-form ω on P is called a connection (form) if it satisfies

$$\omega(A^*) = A, \quad A \in \mathbf{g}, \tag{2.1}$$

where A^* is a vector field on P given by $A^*{}_u = d(u \cdot \exp(tA))/dt|_{t=0}$, $u \in P$, and the pull back $R_a{}^*\omega$ of ω by the action R_a, $a \in G$, of G satisfies

$$R_a{}^*\omega = \mathrm{Ad}\,(a^{-1})\omega, \quad a \in G. \tag{2.2}$$

We denote by $\mathcal{C}$ the set of all connections on P. Let $\{U_\alpha\}_{\alpha\in\mathcal{F}}$ be an open covering of M with a family of isomorphisms ψ_α: $\pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times G$ and the corresponding family of transition functions $\psi_{\alpha\beta}$: $U_\alpha \cap U_\beta \longrightarrow G$, $(\alpha, \beta \in \mathcal{F})$. For each $\alpha(\in \mathcal{F})$, let σ_α: $U_\alpha \longrightarrow P$ be the cross section on U_α defined by $\sigma_\alpha(x) := \psi_\alpha{}^{-1}(x, 1_G)$, $x \in U_\alpha$, where 1_G is the identity of G. For each $\alpha(\in \mathcal{F})$, a $\mathbf{g}$-valued 1-form ω_α on U_α is defined by $\omega_\alpha := \sigma_\alpha{}^*\omega$. Then on each non-empty $U_\alpha \cap U_\beta$, $(\alpha, \beta \in \mathcal{F})$,

$$\omega_\beta = (L_{\psi_{\beta\alpha}})_* d\psi_{\alpha\beta} + \mathrm{Ad}\,(\psi_{\beta\alpha})\omega_\alpha. \tag{2.3}$$

Then, on $\pi^{-1}(U_\alpha)$, $(\alpha \in \mathcal{F})$,

$$\omega = \mathrm{Ad}\,(s_\alpha{}^{-1})\pi^*\omega_\alpha + (L_{s_\alpha{}^{-1}})_* ds_\alpha \quad (\omega \in \mathcal{C}), \tag{2.4}$$

where s_α is the G-coordinate of the isomorphism $\psi_\alpha : \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times G$, $(\alpha \in \mathcal{F})$. The $\mathbf{g}$-valued 2-form Ω^ω on P, which is called the curvature form, is defined

$$\Omega^\omega = d\omega + (1/2)[\omega, \omega]. \tag{2.5}$$

On $\pi^{-1}(U_\alpha)$, $(\alpha \in \mathcal{F})$,

$$\Omega^\omega = \mathrm{Ad}\,(s_\alpha{}^{-1})\ \pi^*\Omega_\alpha, \tag{2.6}$$

where $\Omega_\alpha := d\omega_\alpha + (1/2)[\omega_\alpha, \omega_\alpha]$. Fix an $\mathrm{Ad}\,(G)$-invariant inner product $< \, , \, >$ on $\mathbf{g}$. Since the inner product $< \Omega^\omega, \Omega^\omega > (u)$ depends only on the point $x = \pi(u)$ because of (2.2) and (2.6), the following functional called the Yang-Mills functional, is well defined:

$$\mathcal{YM}(\omega) = (1/2)\int_M < \Omega^\omega, \Omega^\omega > dv_h,$$

$$(\mathcal{YM}(\omega_\alpha) = (1/2)\int_M < \Omega_\alpha, \Omega_\alpha > dv_h), \tag{2.7}$$

where dv_h is the volume element of (M, h). A connection ω in $\mathcal{C}$ is a *Yang-Mills connection* if it is a critical point of the Yang-Mills funtional $\mathcal{YM}$. The following theorem is well known (cf. [2,4]):

Theorem 2.1. *A connection ω in $\mathcal{C}$ is a Yang- Mills connection if and only if*

$$\sum_{j=1}^{m}\{(\nabla_{e_j}\Omega_\alpha)(e_j, e_i) + [\omega_\alpha(e_j), \Omega_\alpha(e_j, e_i)]\} = 0 \quad (i = 1, 2, \cdots, m), \tag{2.8}$$

where ∇ is the Levi-Civita connection of (M, h) and $\{e_i |\ i = 1, 2, \cdots, m\}$ is a local orthonormal frame field on (M, h).

3. Invariant Connections in $P = K \times_{(\lambda,H)} G$

The situation of this paper is the following. Let K be a compact connected Lie group acting on a principal fibre bundle $P(M, G, \pi)$ as a group of automorphisms which acts fibre-transitively on P, i.e., (i) each $k(\in K)$ is a diffeomorphism such that $k(ua) = k(u)a$ $(u \in P, a \in G)$, and (ii) for any two fibres of P, there is an element of K which maps one fibre into the other. Every element k of K induces a transformation of M in a natural manner because of (i), which is denoted by τ_k. For an arbitrary fixed point u_0 in P, with the projection $\pi(u_0) = x_0$, let H be the isotropy subgroup of K at x_0, i.e., $H := \{k \in K | \tau_k x_0 = x_0\}$. Then $K/H = M$ and $x_0 = \{H\}$. Moreover $P = K \times_{(\lambda,H)} G$. In fact, the identification Ψ of P with $K \times_{(\lambda,H)} G$ is given by

$$\Psi(u) := [(k, a)] \quad (u = k(u_0)a \in P, \quad k \in K, \ a \in G). \tag{3.1}$$

This correspondence (3.1) is well defined, then $u_0 = [(1_K, 1_G)]$ and $\pi(u) = \pi[(k, a)] = \pi_0(k)$, $(u = k(u_0)a)$. In this paper, π_0 is the natural projection of K onto K/H. Let λ be the holonomy representation of H into G, i.e., $h(u_0) = u_0\lambda(h)$ $(h \in H)$. Now let us recall a work of H.C. Wang (cf. [7]) in which he considered the K-invariant connections on P. A connection $\omega(\in \mathcal{C})$ is K-invariant if the full back $k^*\omega$, $(k \in K)$, coincides with ω. We denote by C_K the set of all K-invariant connections of the principal fibre bundle $P(M, G, \pi)$. For every $\omega \in \mathcal{C}_K$, a linear map $\wedge$ of $\mathbf{k}$ into $\mathbf{g}$ is defined by $\wedge(X) := \omega_{u_0}(\widetilde{X})$, $X \in \mathbf{k}$, where $\mathbf{k}$ is the Lie algebra of K and $\widetilde{X}$ is a vector field on P defind by

$$\widetilde{X}_u := d(\exp tX)(u)/dt|_{t=0}, \quad (u \in P). \tag{3.2}$$

Then, (cf. [7]),

$$\begin{cases} \wedge(X) = \lambda(X), \quad (X \in \mathbf{h}), \quad \text{and} \\ \wedge(\operatorname{Ad}(h)X) = \operatorname{Ad}(\lambda(h)) \wedge (X), \quad (h \in H,\ X \in \mathbf{k}). \end{cases} \tag{3.3}$$

Since K is compact, the Lie algebra $\mathbf{k}$ of K can be decomposed into a direct sum of the Lie algebra $\mathbf{h}$ of H and an $\operatorname{Ad}(H)$-invariant subspace $\mathbf{m}$ as vector spaces, that is, $\mathbf{k} = \mathbf{h} \oplus \mathbf{m}$ and $\operatorname{Ad}(H)\mathbf{m} \subset \mathbf{m}$. Then we have the following theorem.

Theorem 3.1. (cf. [7]) *On the G-principal fibre bundle $P = K \times_{(\lambda,H)} G$ over M, the correspondence above $\omega \longrightarrow \wedge$ gives a bijection between $\mathcal{C}_K$ and the set of all linear maps $\wedge$ satisfying*

$$\wedge_{\mathbf{m}}(\operatorname{Ad}(h)X) = \operatorname{Ad}(\lambda(h))(\wedge_{\mathbf{m}}(X)) \quad (h \in H,\ X \in \mathbf{m}), \tag{3.4}$$

and the curvature form Ω of the K-invariant connection defined by $\wedge_{\mathbf{m}}$ satisfies the following $2\,\Omega_{u_0}(\widetilde{X}, \widetilde{Y}) = [\wedge_{\mathbf{m}}(X), \wedge_{\mathbf{m}}(Y)] - \wedge_{\mathbf{m}}([X, Y]_{\mathbf{m}}) - \lambda([X, Y]_{\mathbf{h}})$,

$$(X, Y \in \mathbf{m}), \tag{3.5}$$

where $\wedge_{\mathbf{m}}$ is the restriction of $\wedge$ to $\mathbf{m}$, and $[X, Y]_{\mathbf{m}}$ (resp. $[X, Y]_{\mathbf{h}}$) denotes the $\mathbf{m}$-component (resp. $\mathbf{h}$-component) of $[X, Y] \in \mathbf{k}$.

4. Main Results

We preserve the notations as in Section 2 and Section 3. For the H-principal fibre bundle $K(K/H, H, \pi_0)$, the following lemma (cf. [3, Lemma 4.1, p. 123]) is well known.

Lemma 4.1. *There is a neighbourhood V of 0 in the vector space $\mathbf{m}$ which is mapped diffeomorphically under $\exp|_{\mathbf{m}}$ and such that π_0 maps $N := \exp(V)$ diffeomorphically onto a neighbourhood U of the point $\{H\}$ in K/H.*

Let σ_0 be a cross section of the neighbourhood U of $\{H\}$ in Lemma 4.1 into ${\pi_0}^{-1}(U)$ $(\subset K)$ which is defined by $\sigma_0(\pi_0(\exp X)) = \exp(X)$ $(X \in V)$. For each $u = [(k,a)] \in P, \pi(u) = \pi_0(k)$. For convenience in this paper, we denote by U_α the neighbourhood U of $\{H\}$ $(\in M)$ in Lemma 4.1. Using the mapping σ_0, we can define a cross section σ_α of the neighborhood U_α into $\pi^{-1}(U_\alpha)(\subset P)$, which is defined by $\sigma_\alpha(\pi_0(\exp X)) := \exp X(u_0)$, $(X \in V)$. Evidently, $\sigma_\alpha(x_0) = u_0$.

For the calculus, we define a vector field X^*, $X \in \mathbf{m} = T_{\{H\}}M$, on the neighborhood U_α of $\{H\}$ in K/H by

$$X^*{}_{xH} := (\tau_x)_* X \;\in\; T_{xH}(M), \quad x \in \exp(V) = N. \tag{4.1}$$

Let $< , >$ be an inner product which is $\mathrm{Ad}\,(H)$-invariant on $\mathbf{m}$, This inner product $< , >$ determines a K-invariant Riemannian metric $h_{< , >}$ on K/H. Let $\{X_i\}_{i=1}^m$ be an orthonormal basis on $(\mathbf{m}, < , >)$. Then $\{{X_i}^*\}_{i=1}^m$ is an orthonormal frame field on the neighborhood U_α of $\{H\}$ in $(K/H, h_{< , >})$. Let $\{\theta^{j*}\}_{j=1}^m$ be a system of 1-forms on U_α which is dual to $\{{X_i}^*\}_{i=1}^m$. Then, the Levi-Civita connection ∇ of $(K/H, h_{< , >})$ is given by

$$(\nabla_{X^*} Y^*)_{x_0} = (1/2)[X,Y]_{\mathbf{m}} + U(X,Y) \quad (X, Y \in \mathbf{m}), \tag{4.2}$$

where $U(X,Y)$ is determined by

$$2 < U(X,Y), Z > = < [Z,X]_{\mathbf{m}}, Y > + < X, [Z,Y]_{\mathbf{m}} > \qquad (X, Y, Z \in \mathbf{m}). \tag{4.3}$$

For each $u \in \pi^{-1}(U_\alpha)$, there exist a unique pair $(X, a) \in (V \times G) \subset (\mathbf{m} \times G)$ such that $u = [(\exp X, a)]$. A diffeomorphism ψ_α of $\pi^{-1}(U_\alpha)$ onto $U_\alpha \times G$ is defined by

$$\psi_\alpha(u) = (\pi(u), a) = (\pi_0(\exp X), a), \tag{4.4}$$

for $u = \exp X(u_0)\cdot a = [(\exp X, a)]$, $(X \in V$ and $a \in G)$. Then $\sigma_\alpha(\pi_0(\exp X)) = {\psi_\alpha}^{-1}(\pi_0(\exp X), 1_G) = (\exp X)(u_0)$, $(X \in V)$. So, $s_\alpha(\sigma_\alpha(\tau_{\exp X}\{H\})) = 1_G$, $(X \in V)$. By virtue of (2.6), we have on U_α

$$\Omega^\omega(\sigma_{\alpha *}({X_i}^*), \sigma_{\alpha *}({X_j}^*)) = \Omega_\alpha({X_i}^*, {X_j}^*). \tag{4.5}$$

Since $\sigma_\alpha(\pi_0(\exp tX_i)) = \exp(tX_i)(u_0)$ for sufficiently small t, $\widetilde{X_i}(u_0) = \sigma_{\alpha *}({X_i}^*{}_{\{H\}})$ for each i. Hence, by(2.6) we get on U_α

$$\begin{aligned} &({\sigma_\alpha}^*\omega)({X_i}^*{}_{\{H\}}) = \wedge_{\mathbf{m}}(X_i), \\ &\Omega^\omega(\widetilde{X_i},\ \widetilde{X_j})(u_0) = \Omega_\alpha({X_i}^*, {X_j}^*)(x_0). \end{aligned} \tag{4.6}$$

From now on, we use the following notations:

$\{X_i\}_i$: an orthonormal basis of $(\mathbf{m}, <\ ,\ >)$, $\{Y_a\}_a$: a basis of $\mathbf{h}$

$\{E_\alpha\}_\alpha$: an orthonormal basis of the Lie algebra $\mathbf{g}$ of the structure group G with respect to an $\mathrm{Ad}\,(G)$-invariant inner product $<\ ,\ >$,

$$\begin{aligned}
&[X_i, X_j]_{\mathbf{m}} =: \sum_k C_{ij}{}^k X_k, && [X_i, X_j]_{\mathbf{h}} =: \sum_b C_{ij}{}^b Y_b,\\
&d\lambda(Y_a) =: \lambda(Y_a) \ =: \sum_\beta \lambda_a{}^\beta E_\beta, && [E_\alpha, E_\beta] =: \sum_\gamma G_{\alpha\beta}{}^\gamma E_\gamma,\\
&U(X_i, X_j) =: \sum_k U_{ij}{}^k X_k, && \wedge_{\mathbf{m}}\,(X_j) =: \sum_\beta \wedge_j{}^\beta E_\beta,\\
&\Omega_\alpha := \sum_{i,j,\beta} \Omega_{ij}{}^\beta (\theta^{i*} \wedge \theta^{j*}) \otimes E_\beta, && \Omega_\alpha(X_i{}^*, X_j{}^*) =: \Omega_{ij} \ \text{ on } \ U_\alpha,
\end{aligned}$$

$$(\nabla_{X_k{}^*\{H\}} \Omega_\alpha)(X_j{}^*{}_{\{H\}}, X_i{}^*{}_{\{H\}}) =: \nabla_k \Omega_{ji} \quad \text{on} \quad U_\alpha.$$

By (4.3), we get

$$U_{ij}{}^k = (1/2)(C_{ki}{}^j + C_{kj}{}^i). \tag{4.7}$$

We obtain by help of (4.2) and (4.7)

$$\begin{cases} (\nabla_{X_i{}^*} X_j{}^*) = (1/2) \sum_k (C_{ki}{}^j + C_{kj}{}^i + C_{ij}{}^k) X_k{}^*, \\ (\nabla_{X_i{}^*} \theta^{j*}) = (-1/2) \sum_k (C_{ji}{}^k + C_{jk}{}^i + C_{ik}{}^j) \theta^{k*}. \end{cases} \tag{4.8}$$

Using (3.5) and (4.6), we obtain

$$\Omega_{ij}{}^\alpha = (1/2)\Big(\sum_{\beta,\gamma} \wedge_i{}^\beta \wedge_j{}^\gamma G_{\beta\gamma}{}^\alpha - \sum_k C_{ij}{}^k \wedge_k{}^\alpha - \sum_a C_{ij}{}^a \lambda_a{}^\alpha\Big). \tag{4.9}$$

By virtue of (4.6) and (4.8), we get

$$\sum_j \nabla_j \Omega_{ji} \ = \sum_{j,k,\alpha} \{\Omega_{ik}{}^\alpha C_{kj}{}^j + (1/2) \Omega_{kj}{}^\alpha (C_{kj}{}^i + C_{ki}{}^j + C_{ji}{}^k)\} E_\alpha, \tag{4.10}$$

$$\begin{aligned}
\sum_j [\wedge_{\mathbf{m}}(X_j), \Omega_{ji}] = (1/2) \sum_{j,\alpha,\beta,\delta} \wedge_j{}^\beta \Big(&\sum_{\gamma,\mu} \wedge_j{}^\gamma \wedge_i{}^\mu G_{\gamma\mu}{}^\delta \\
&- \sum_k C_{ji}{}^k \wedge_k{}^\delta - \sum_a C_{ji}{}^a \lambda_a{}^\delta\Big) G_{\beta\delta}{}^\alpha E_\alpha.
\end{aligned} \tag{4.11}$$

Thus, by (4.10) and (4.11) we obtain the following theorem.

Theorem 4.1. *Let K be a compact connected Liegroup of automorphisms of $P(M,G)$ which acts fibre-transitively on P. Then a K-invariant connection in the principal fibre bundle $P(M,G)$ is a Yang-Mills connection if and only if*

$$\sum_{k,j}\{2\Omega_{ik}{}^{\alpha}C_{kj}{}^{j}+\Omega_{kj}{}^{\alpha}(C_{kj}{}^{i}+C_{ki}{}^{j}+C_{ji}{}^{k})\}+\sum_{j,\beta,\delta}\wedge_j{}^{\beta}G_{\beta\delta}{}^{\alpha} \times(\sum_{\gamma,\mu}\wedge_j{}^{\gamma}\wedge_i{}^{\mu}G_{\gamma\mu}{}^{\delta}-\sum_{k}C_{ji}{}^{k}\wedge_k{}^{\delta}-\sum_{a}C_{ji}{}^{a}\lambda_a{}^{\delta})=0. \quad (4.12)$$

Corollary 4.2. *Assume the base manifold $(M,h_{<\ ,\ >})$ in the principal fibre bundle $P(M,G)$ is symmetric. Then a K-invariant connection in the bundle $P(M,G)$ is a Yang-Mills connection if and only if*

$$\sum_{j,\beta,\delta}\wedge_j{}^{\beta}G_{\beta\delta}{}^{\alpha}(\sum_{\gamma,\mu}\wedge_j{}^{\gamma}\wedge_i{}^{\mu}G_{\gamma\mu}{}^{\delta}-\sum_{a}C_{ji}{}^{a}\lambda_a{}^{\delta})=0. \quad (4.13)$$

5. Examples

We consider the case when $K=SU(3), H=S(U(1)\times U(2))$ and $G=SU(2)$. Note that $U(1)\times SU(2)$ is a double covering of $U(2)$ and $U(2)$ is isomorphic with $S(U(1)\times SU(2))$ by group homomorphisms

$$U(1)\times SU(2)\longrightarrow U(2)\overset{\sim}{\longrightarrow}S(U(1)\times U(2))\,,$$

which are given by

$$(e^{i\theta},A)\text{ or }(e^{i(\theta+\pi)},-A)\longmapsto e^{i\theta}A=:B\longmapsto\begin{pmatrix}\det(B^{-1}) & 0\\ 0 & B\end{pmatrix}.$$

If l is an even integer, a group homomorphism λ of $S(U(1)\times U(2))$ into $SU(2)$ via $e^{i\theta}A\longmapsto \mathrm{diag}(e^{il\theta},e^{-il\theta})$, $(e^{i\theta}\in U(1),\ A\in SU(2))$, is well defined.

Let E_{ij} denote a square matrix of order 3 with the (i,j)-entry being 1, and all the other entries being 0. Then we put:

$$X_1:=(1/\sqrt{12})(E_{12}-E_{21}),\quad X_2:=(\sqrt{-1}/\sqrt{12})(E_{12}+E_{21}),$$
$$X_3:=(1/\sqrt{12})(E_{13}-E_{31}),\quad X_4:=(\sqrt{-1}/\sqrt{12})(E_{13}+E_{31}),$$
$$Y_5:=(1/\sqrt{12})(E_{23}-E_{32}),\quad Y_6:=(\sqrt{-1}/\sqrt{12})(E_{23}+E_{32}),$$

$Y_7 := (\sqrt{-1}/\sqrt{12})\,\mathrm{diag}\,(0,1,-1), \quad Y_8 := (\sqrt{-1}/6)\,\mathrm{diag}\,(-2,1,1).$

Then $\{Y_5, Y_6, Y_7, Y_8\}_R = \mathbf{h}$. Let B be the Killing form of $\mathbf{su}(n)$, i.e, $B(X,Y) = \mathrm{Trace}(\,\mathrm{ad}\,(X)\,\mathrm{ad}\,(Y)) = 2n\mathrm{Trace}(XY)$ $(X, Y \in \mathbf{su}(n))$. We define an inner product $< \, , \, >$ on $\mathbf{su}(n)$ by

$$< X, Y >= -B(X,Y) = -2n \ \mathrm{Trace}\,(X,Y) \quad (X, Y \in \mathbf{su}(n)). \tag{5.1}$$

We put $\{X_1, X_2, X_3, X_4\}_R =: \mathbf{m}$, and then $[\mathbf{h}, \mathbf{m}] \subset \mathbf{m}$. Moreover, $\{X_i\}_{i=1}^4$ is an orthonormal basis of $(\mathbf{m}, < \, , \, >)$.

Similarly, we put $E_1 := (1/\sqrt{8})(E_{12} - E_{21})$, $E_2 := (\sqrt{-1}/\sqrt{8})(E_{12} + E_{21})$, $E_3 := (\sqrt{-1}/\sqrt{8})\,\mathrm{diag}\,(1,-1)$ in $\mathbf{su}(2)$. Then, $\{E_1, E_2, E_3\}$ is an orthonormal basis of $\mathbf{su}(2)$ with respect to the $\mathrm{Ad}\,(SU(2))$-invariant inner product $< \, , \, >$ which is induced by the Killing form B on $\mathbf{su}(2)$.

By straightforward computations, we have:

$$\begin{aligned}
&{C_{12}}^7 = (-\sqrt{12})^{-1}, \quad {C_{12}}^8 = (-1/2), \quad {C_{13}}^5 = (-\sqrt{12})^{-1},\\
&{C_{14}}^6 = (-\sqrt{12})^{-1}, {C_{23}}^6 = (\sqrt{12})^{-1}, \quad {C_{24}}^5 = (-\sqrt{12})^{-1},\\
&{C_{34}}^7 = (\sqrt{12})^{-1}, \quad {C_{34}}^8 = (-1/2), \quad \text{and the others are zero ;}\\
&{G_{12}}^3 = {G_{23}}^1 = {G_{31}}^2 = (1/\sqrt{2}), \quad \text{and the others are zero ;}\\
&{\lambda_8}^3 = (\sqrt{2}\ell)/3 \quad \text{and the others are zero.}
\end{aligned} \tag{5.2}$$

Using Theorem 4.1 and Corollary 4.2, we have the following result.

Proposition 5.1. *Let P be the principal fibre bundle*

$$SU(3) \times_{(\lambda, S(U(1)\times U(2)))} SU(2) =: P_\lambda$$

over the Riemannian manifold $(CP^2, \ g_{< \, , \, >})$. Then, a $SU(3)$-invariant connection w in P_λ is a Yang-Mills connection if and only if

$$3 \sum_{j(j\neq k)} [{\wedge_k}^2({\wedge_j}^1{\wedge_j}^2) + {\wedge_k}^3({\wedge_j}^1{\wedge_j}^3) - {\wedge_k}^1\{({\wedge_j}^2)^2 + ({\wedge_j}^3)^2\}] = \begin{cases} \ell \wedge_2{}^2 \ \ \textit{if} \ \ k=1, & -\ell \wedge_1{}^2 \ \ \textit{if} \ \ k=2, \\ \ell \wedge_4{}^2 \ \ \textit{if} \ \ k=3, & -\ell \wedge_3{}^2 \ \ \textit{if} \ \ k=4, \end{cases}$$

$$3 \sum_{j(j\neq k)} [{\wedge_k}^1({\wedge_j}^1{\wedge_j}^2) + {\wedge_k}^3({\wedge_j}^2{\wedge_j}^3) - {\wedge_k}^2\{({\wedge_j}^1)^2 + ({\wedge_j}^3)^2\}] = \begin{cases} -\ell \wedge_2{}^1 \ \ \textit{if} \ \ k=1, & \ell \wedge_1{}^1 \ \ \textit{if} \ \ k=2, \\ -\ell \wedge_4{}^1 \ \ \textit{if} \ \ k=3, & \ell \wedge_3{}^1 \ \ \textit{if} \ \ k=4, \end{cases}$$

and

$$\sum_{j(j\neq k)} [\wedge_k{}^1(\wedge_j{}^1\wedge_j{}^3) + \wedge_k{}^2(\wedge_j{}^2\wedge_j{}^3) - \wedge_k{}^3\{(\wedge_j{}^1)^2 + (\wedge_j{}^2)^2\}] = 0\,,$$

$$\textit{for each}\ \ k\ (k = 1,\ 2,\ 3,\ 4)\,. \quad (5.3)$$

The Hodge star operator $*$ satisfies $*^2 = id$. In case of $\dim(M) = 4$, if $*\Omega^\omega = \Omega^\omega$ ($resp. * \Omega^\omega = -\Omega^\omega$), w is *self-dual* (resp. *anti-self-dual*), which is always a Yang-Mills connection.

By help of (4.9) and (5.2), we have:

$$\begin{aligned}
&\Omega_{12}{}^1 = c(\wedge_1{}^2\wedge_2{}^3 - \wedge_1{}^3\wedge_2{}^2), \quad \Omega_{12}{}^2 = c(\wedge_1{}^3\wedge_2{}^1 - \wedge_1{}^1\wedge_2{}^3),\\
&\Omega_{12}{}^3 = c\{\wedge_1{}^1\wedge_2{}^2 - \wedge_1{}^2\wedge_2{}^1 + (\ell/3)\},\\
&\Omega_{13}{}^1 = c(\wedge_1{}^2\wedge_3{}^3 - \wedge_1{}^3\wedge_3{}^2), \quad \Omega_{13}{}^2 = c(\wedge_1{}^3\wedge_3{}^1 - \wedge_1{}^1\wedge_3{}^3),\\
&\Omega_{13}{}^3 = c(\wedge_1{}^1\wedge_3{}^2 - \wedge_1{}^2\wedge_3{}^1),\\
&\Omega_{14}{}^1 = c(\wedge_1{}^2\wedge_4{}^3 - \wedge_1{}^3\wedge_4{}^2), \quad \Omega_{14}{}^2 = c(\wedge_1{}^3\wedge_4{}^1 - \wedge_1{}^1\wedge_4{}^3),\\
&\Omega_{14}{}^3 = c(\wedge_1{}^1\wedge_4{}^2 - \wedge_1{}^2\wedge_4{}^1),\\
&\Omega_{23}{}^1 = c(\wedge_2{}^2\wedge_3{}^3 - \wedge_2{}^3\wedge_3{}^2), \quad \Omega_{23}{}^2 = c(\wedge_2{}^3\wedge_3{}^1 - \wedge_2{}^1\wedge_3{}^3),\\
&\Omega_{23}{}^3 = c(\wedge_2{}^1\wedge_3{}^2 - \wedge_2{}^2\wedge_3{}^1),\\
&\Omega_{24}{}^1 = c(\wedge_2{}^2\wedge_4{}^3 - \wedge_2{}^3\wedge_4{}^2), \quad \Omega_{24}{}^2 = c(\wedge_2{}^3\wedge_4{}^1 - \wedge_2{}^1\wedge_4{}^3),\\
&\Omega_{24}{}^3 = c(\wedge_2{}^1\wedge_4{}^2 - \wedge_2{}^2\wedge_4{}^1),\\
&\Omega_{34}{}^1 = c(\wedge_3{}^2\wedge_4{}^3 - \wedge_3{}^3\wedge_4{}^2), \quad \Omega_{34}{}^2 = c(\wedge_3{}^3\wedge_4{}^1 - \wedge_3{}^1\wedge_4{}^3),\\
&\Omega_{34}{}^3 = c\{\wedge_3{}^1\wedge_4{}^2 - \wedge_3{}^2\wedge_4{}^1 + (\ell/3)\}, \qquad (5.4)
\end{aligned}$$

where $c := (2\sqrt{2})^{-1}$. We obtain from (5.4) the following proposition.

Proposition 5.2. *In the principal fibre bundle*

$$SU(3) \times_{(\lambda, S(U(1)\times U(2))} SU(2),$$

over CP^2, w *is self dual (resp. anti-self-dual) if and only if*

$$\begin{aligned}
&\wedge_1{}^2\wedge_2{}^3 - \wedge_1{}^3\wedge_2{}^2 = \wedge_3{}^2\wedge_4{}^3 - \wedge_3{}^3\wedge_4{}^2\ (\textit{resp.}\ \wedge_3{}^3\wedge_4{}^2 - \wedge_3{}^2\wedge_4{}^3),\\
&\wedge_1{}^3\wedge_2{}^1 - \wedge_1{}^1\wedge_2{}^3 = \wedge_3{}^3\wedge_4{}^1 - \wedge_3{}^1\wedge_4{}^3\ (\textit{resp.}\ \wedge_3{}^1\wedge_4{}^3 - \wedge_3{}^3\wedge_4{}^1),\\
&\wedge_1{}^1\wedge_2{}^2 - \wedge_1{}^2\wedge_2{}^1 = \wedge_3{}^1\wedge_4{}^2 - \wedge_3{}^2\wedge_4{}^1\\
&\qquad\qquad (\textit{resp.}\ \wedge_3{}^2\wedge_4{}^1 - \wedge_3{}^1\wedge_4{}^2 - (2\ell)/3),\\
&\wedge_1{}^\alpha\wedge_3{}^\beta - \wedge_1{}^\beta\wedge_3{}^\alpha = \wedge_4{}^\alpha\wedge_2{}^\beta - \wedge_4{}^\beta\wedge_2{}^\alpha\ (\textit{resp.}\ \wedge_4{}^\beta\wedge_2{}^\alpha - \wedge_4{}^\alpha\wedge_2{}^\beta),\\
&\wedge_1{}^\alpha\wedge_4{}^\beta - \wedge_1{}^\beta\wedge_4{}^\alpha = \wedge_2{}^\alpha\wedge_3{}^\beta - \wedge_2{}^\beta\wedge_3{}^\alpha\ (\textit{resp.}\ \wedge_2{}^\beta\wedge_3{}^\alpha - \wedge_2{}^\alpha\wedge_3{}^\beta), \qquad (5.5)
\end{aligned}$$

where $\alpha,\ \beta = 1,2,3,$ *and* $\alpha \neq \beta$.

By Proposition 5.2, it can be shown that if $(\wedge_i{}^{\alpha})$ satisfies $\wedge_i{}^1 = \wedge_i{}^2 = 0$ for each i $(i = 1,2,3,4)$, i.e., $< \wedge_{\mathbf{m}}(X_i), E_1 > = < \wedge_{\mathbf{m}}(X_i), E_2 > = 0$ for each i, the $SU(3)$-invariant connection w in P_λ corresponding to $(\wedge_i{}^{\alpha})$ is self dual. Since a self-dual (or anti-self-dual) connection ω is a Yang-Mills connection, combining Proposition 5.1 and Proposition 5.2 we get the following result.

Proposition 5.3. *For a* $SU(3)$*-invariant connection in the principal fibre bundle* P_λ *over* CP^2*, the sufficient and necessary conditions to be a Yang-Mills connection are* $\ell = 0$*, or*

$$< \wedge_{\mathbf{m}}(X_i), E_1 > = < \wedge_{\mathbf{m}}(X_i), E_2 > = 0 \quad \text{for each} \quad i \ (i = \ 1,\ 2,\ 3,\ 4). \qquad (5.6)$$

By virtue of Proposition 5.1 and Proposition 5.3, we obtain the following theorem.

Theorem 5.4. *Let* P_λ *be the principal fibre bundle* $K \times_{(\lambda,H)} G$, $(K := SU(3),\ H := S(U(1) \times U(2)),\ G := SU(2))$, *over* $(CP^2, g_{<\ ,\ >})$. *Then the sufficient and necessary conditions for a* K*-invariant connection* ω *in* P_λ *to be a Yang-Mills connection are*

$$\begin{aligned}
&\sum_{j(j\neq k)} [\wedge_k{}^2(\wedge_j{}^1\wedge_j{}^2) + \wedge_k{}^3(\wedge_j{}^1\wedge_j{}^3) - \wedge_k{}^1\{(\wedge_j{}^2)^2 + (\wedge_j{}^3)^2\}] \\
&\qquad = \sum_{j(j\neq k)} [\wedge_k{}^1(\wedge_j{}^1\wedge_j{}^2) + \wedge_k{}^3(\wedge_j{}^2\wedge_j{}^3) - \wedge_k{}^2\{(\wedge_j{}^1)^2 + (\wedge_j{}^3)^2\}] \\
&\qquad = \sum_{j(j\neq k)} [\wedge_k{}^1(\wedge_j{}^1\wedge_j{}^3) + \wedge_k{}^2(\wedge_j{}^2\wedge_j{}^3) - \wedge_k{}^3\{(\wedge_j{}^1)^2 + (\wedge_j{}^2)^2\}] \\
&\qquad = 0, \qquad (5.7)
\end{aligned}$$

for each k $(k = 1,\ 2,\ 3,\ 4)$.

By Proposition 5.3, we derive the following corollary.

Corollary 5.5. *Let* ℓ *be a non-zero even integer. Then for each* $SU(3)$*-invariant connection* ω *in* P_λ*, it is a Yang-Mills connection if and only if* ω *is self-dual.*

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