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WEAK OPEN SETS: A PRESENTATION USING OPERATORS

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Abstract: We study the family $\rho(X, \tau)$ of the ρ-open sets for any operator ρ on a topological space (X, τ). The concept of preclousure operator is introduced to provide X with a topology τ^ρ which is properly placed between τ and the family of the ρ-open sets. By means of operators we define a class of weak continuous functions and give some results concerning the properties they preserve. Some additional results are given.

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1. Introduction

In 1979 Kasahara [12] showed that some known properties of compact and generalized compact spaces can be unified with the help of an operator from the topology τ into the power set of $\cup\tau$. Operators have been used recently by

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other authors to investigate spaces satisfying generalized forms of compacity, separation properties and continuity (Dontchev [7], Fukutake [9], Ogata [18], and Rosas [19], [20]). The aim of this work is to continue the study of topological properties by mean of operators. In Section 2, we give some results for the class $\rho(X,\tau)$ of the ρ-open sets on a topological space (X,τ) for any operator ρ. In Section 3, we introduce the concept of preclosure operator to provide X with a topology τ^ρ which is placed between τ and $\rho(X,\tau)$. The concept of complemented operator is given to prove a characterization of the generalized closure of a subset of X. Considering a pair of operators ρ and δ on (X,τ), we show in Section 4 that operators allow to investigate simultaneously various classes of generalized closed sets. In Section 5 and Section 6 we study weak forms of continuity in terms of operators, and give some results in relation to a generalized compacity that this class of functions preservs. We end this work showing in Section 7 that the Uniform Bounded Theorem can be extended to a family of functions satisfying a weak form of continuity given by an operator.

2. The Family of the ρ-Open Sets

A class of weak open sets, the ρ-open sets, was introduced by Rosas [19] in terms of a map ρ between the power set $\mathcal{P}(X)$ of a topological space (X,τ), satisfying the condition $V \subset \rho V$ for all $V \in \tau$. If the map ρ is monotone on τ, the family $\rho(X,\tau)$ of the ρ-open sets exhibits interesting properties as we show in this section.

Definition 2.1. Let X be a set and let $\mathcal{F}$ be a family of subsets of X. A map $\rho : \mathcal{P}(X) \to \mathcal{P}(X)$ is said to be:

(a) *Expansion on* $\mathcal{F}$ if $A \subset \rho A$ for all $A \in \mathcal{F}$.

(b) *Monotone on* $\mathcal{F}$ if for all $A, B \in \mathcal{F}$, $A \subset B$ implies $\rho A \subset \rho B$.

(c) *Idempotent on* $\mathcal{F}$ if $\rho(\rho A) = \rho A$ for all $A \in \mathcal{F}$.

(d) *Additive on* $\mathcal{F}$ if $\rho(A \cup B) = \rho A \cup \rho B$ for all $A, B \in \mathcal{F}$.

If $\bigcup\limits_{i\in I} \rho A_i \subset \rho(\bigcup\limits_{i\in I} A_i)$ for any collection $\{A_i\}_{i\in I} \subset \mathcal{F}$, then ρ is said to be *subadditive on* $\mathcal{F}$.

Proposition 2.1. *Let $\mathcal{F}$ be a family of subsets of X and $\rho : \mathcal{P}(X) \to \mathcal{P}(X)$ a map. If $\mathcal{F}$ is a σ-algebra, then ρ is monotone on $\mathcal{F}$ if and only if ρ is subadditive on $\mathcal{F}$.*

Proof. Let ρ be monotone on $\mathcal{F}$ and let $\{A_i\}_{i\in I} \subset \mathcal{F}$. Then for each $i \in I$, $\rho A_i \subset \rho(\bigcup_{i\in I} A_i)$, and thus $\bigcup_{i\in I} \rho A_i \subset \rho(\bigcup_{i\in I} A_i)$. Reciprocally, if ρ is subadditive on $\mathcal{F}$ and $A, B \in \mathcal{F}$ with $A \subset B$, then $\rho A \subset \rho A \cup \rho B \subset \rho(A \cup B) = \rho B$. Thus ρ is monotone on $\mathcal{F}$. □

Remark 2.1. It is easy to see that if ρ is additive on $\mathcal{F}$ then ρ is monotone on $\mathcal{F}$.

In what follows (X, τ) denotes a topological space in which no separation axioms are asumed unless specified. The interior and closure of a subset A of X are denoted by $\operatorname{Int} A$ and $\operatorname{Cl} A$, respectively.

Let Ω be the set of all expansions $\rho : \mathcal{P}(X) \to \mathcal{P}(X)$ in the family $\mathcal{F} = \tau$. From now on, we will refer to ρ as an expansion on (X, τ). We define on Ω a partial order given by $\rho < \rho^{'}$ if and only if $\rho V \subset \rho^{'} V$ for all $V \in \tau$. Note that $\rho = \operatorname{Int}$ is the minimal element of Ω.

The following definition is a generalization of the concept of open set given in terms of an expansion ρ.

Definition 2.2. (Rosas [19]) Let ρ be an expansion on (X, τ). A subset A of X is said to be ρ-open if $A \subset \rho \operatorname{Int} A$. The complement of a ρ-open set is called ρ-closed.

Given an expansion ρ en (X, τ), we denote by $\rho(X, \tau)$ the family of the ρ-open subsets of X. From the definition it is obvious that $\tau \subset \rho(X, \tau)$. If $\tau^{'}$ is a topology on X weaker than τ, then ρ is an expansion on $(X, \tau^{'})$ and $\rho(X, \tau^{'}) \subset \rho(X, \tau)$. Note that for any ρ, $\rho' \in \Omega$, $\rho < \rho^{'}$ implies $\rho(X, \tau) \subset \rho^{'}(X, \tau)$. For the remainder of this section we will consider expansions which are monotone on τ.

Definition 2.3. Let (X, τ) be a topological space. An expansion ρ on (X, τ) is said to be *an operator on* (X, τ) if ρ is monotone for the family $\mathcal{F} = \tau$, i.e. $\rho U \subset \rho V$ whenever $U \subset V$ and $U, V \in \tau$.

Remark 2.2. (a) In any topological space (X, τ), the interior Int and the clousure Cl are operators on (X, τ). The operator defined by $\rho A = A$, for all $A \subset X$, is denoted by Id and called the identity operator. Note that the family $\operatorname{Id}(X, \tau)$ coincide with the family $\operatorname{Int}(X, \tau)$.

(b) From the definition of operator, it is clear that $\rho X = X$ for any operator ρ. For the purpose of simplicity it is assumed that $\rho\emptyset = \emptyset$ for any operator ρ. Since the family τ of the open subsets of X is σ-algebra, any operator is subadditive on τ.

Remark 2.3. If ρ and β are operators on (X, τ) and β is monotone on $\mathcal{P}(X)$, then the composite map $\beta\rho : \mathcal{P}(X) \to \mathcal{P}(X)$ defined by $\beta\rho A = \beta(\rho A)$ is an operator on (X, τ).

Remark 2.4. For some operators, $\rho(X, \tau)$ is a well known family. For instance, $\operatorname{Int}(X,\tau)$ is the family τ of the open sets, $\operatorname{Int}\operatorname{Cl}(X,\tau)$ is the family τ^α of the α-sets defined by Njastad [16], and $\operatorname{Cl}(X,\tau)$ is the family $SO(X,\tau)$ of the semi-open sets defined by Levine [14].

The following result characterizes the ρ-open sets for any operator ρ on (X,τ).

Proposition 2.2. *Let ρ be an operator on (X,τ). A subset A of X is ρ-open if and only if there exists an open set $O \in \tau$ such that $O \subset A \subset \rho O$.*

Proof. If A is a ρ-open set, then $A \subset \rho \operatorname{Int} A$. By taking $O = \operatorname{Int} A$, we have that $O \subset A \subset \rho O$. Conversely, let $O \in \tau$ such that $O \subset A \subset \rho O$. Then $O \subset \operatorname{Int} A$, and since ρ is monotone on τ it follows that $A \subset \rho O \subset \rho \operatorname{Int} A$. □

Proposition 2.3. *Let ρ be an operator on (X,τ). Then:*

(a) If A is non empty ρ-open set, then $\operatorname{Int} A$ is non empty.

(b) For each $x \in X$, the unitary set $\{x\}$ is ρ-open if and only if $\{x\}$ is open.

Proof. (a) If A is non empty ρ-open set, then $\emptyset \neq A \subset \rho \operatorname{Int} A$. Since $\rho\emptyset = \emptyset$ it follows that $\operatorname{Int} A \neq \emptyset$.

(b) From the definition it is obvious that any open set is ρ-open. if $\{x\}$ is ρ-open. Reciprocally, if $\{x\}$ is open it follows from (a) that $\operatorname{Int}\{x\} \neq \emptyset$ and thus $\{x\}$ is open. □

Theorem 2.4. *Let ρ be an operator on (X,τ). Then the union of any collection of ρ-open sets is a ρ-open set.*

Proof. Let $\{A_i\}_{i\in I}$ be a collection of ρ-open sets. Then $A_i \subset \rho(\operatorname{Int} A_i)$ for each $i \in I$. Since ρ is subadditive on τ, it follows that $\bigcup_{i\in I} A_i \subset \bigcup_{i\in I} \rho(\operatorname{Int} A_i) \subset \rho(\bigcup_{i\in I} \operatorname{Int} A_i) = \rho(\operatorname{Int}(\bigcup_{i\in I} A_i))$. Therefore $\bigcup_{i\in I} A_i$ is ρ-open. □

It is known that the finite intersection of semi-open sets is not in general a semi-open set (Levine [14]). Thus the family $\rho(X,\tau)$ is not in general a topology on X.

Proposition 2.5. *Let ρ be an operator on (X,τ) and let $A \in \rho(X,\tau)$. If $A \subset B \subset \rho \operatorname{Int} A$, then $B \in \rho(X,\tau)$.*

Proof. Since ρ is monotone on τ, $A \subset B$ implies $\rho \operatorname{Int} A \subset \rho \operatorname{Int} B$. Thus $B \subset \rho \operatorname{Int} B$, and therefore $B \in \rho(X,\tau)$. □

Remark 2.5. Let ρ be an operator monotone on (X,τ), and let $\mathcal{B}$ be a collection of sets in X such that: (1) $\tau \subset \mathcal{B}$, and (2) if $B \in \mathcal{B}$ and $B \subset D \subset \rho \operatorname{Int} B$, then $D \in \mathcal{B}$. From the above proposition it follows that $\rho(X,\tau) \subset \mathcal{B}$. Thus, for any operator ρ, the family $\rho(X,\tau)$ is the smallest class of subsets of X satisfying (1) and (2).

The following definition gives a generalization of the concepts of interior and closure of a subset of a topological space.

Definition 2.4. (Rosas [9]) Let ρ be an operator and $A \subset X$. The ρ-*interior of* A, denoted by $\operatorname{Int}_\rho A$, is the union of all ρ-open sets contained in A. Similarly, the ρ-*closure of* A, denoted by $\operatorname{Cl}_\rho A$, is the intersection of all ρ-closed sets containing A.

Remark 2.6. From Theorem 2.4, it follows that $\operatorname{Int}_\rho A$ is a ρ-open set, and therefore $\operatorname{Cl}_\rho A$ is a ρ-closed set, for any $A \subset X$. Thus A is ρ-open (ρ-closed) if and only if $A = \operatorname{Int}_\rho A$ (respectively $A = \operatorname{Cl}_\rho A$).

Remark 2.7. (a) Let ρ and ρ' be two operators on (X, τ). If $\rho < \rho'$ then $\operatorname{Int}_\rho A \subset \operatorname{Int}_{\rho'} A$ and $\operatorname{Cl}_{\rho'} A \subset \operatorname{Cl}_\rho A$, for any subset A of X.

(b) Since $\operatorname{Int}_{\operatorname{Int}} A = \operatorname{Int} A$ and $\operatorname{Cl}_{\operatorname{Int}} A = \operatorname{Cl} A$ for all $A \subset X$, then for any operator ρ the inclusion $\operatorname{Int} A \subset \operatorname{Int}_\rho A \subset A \subset \operatorname{Cl}_\rho A \subset \operatorname{Cl} A$ holds for all $A \subset X$.

Theorem 2.6. *Let ρ be an operator on (X, τ) and let $A \subset X$. Then $z \in \operatorname{Cl}_\rho A$ if and only if $D \cap A \neq \emptyset$ for each ρ-open set D which contains z.*

Proof. Let $z \in \operatorname{Cl}_\rho A$ and suppose that $D \cap A = \emptyset$ for some ρ-open set D which contains z. Then D^c is ρ-closed and $A \subset D^c$, thus $\operatorname{Cl}_\rho A \subset D^c$. But this implies that $z \in D^c$, a contradiction. Therefore $D \cap A \neq \emptyset$.

Reciprocally, let $A \subset X$ and $z \in X$ such that for each ρ-open set D which contains z, $D \cap A \neq \emptyset$. If $z \notin \operatorname{Cl}_\rho A$, there is a ρ-closed set S such that $A \subset S$ and $z \notin S$. Then S^c is a ρ-open set with $z \in S^c$, and thus $S^c \cap A \neq \emptyset$, which is a contradiction. □

Corollary 2.7. *Let ρ be an operator on (X, τ) and let $A \subset X$. Then $\operatorname{Cl}_\rho A \backslash A$ does not contains nonempty ρ-open sets.*

Proof. The proof follows directly from the above theorem. □

Theorem 2.8. *For any operator ρ on (X, τ), the map $\operatorname{Cl}_\rho : \mathcal{P}(X) \to \mathcal{P}(X)$ is an operator on (X, τ) satisfying $(a), (b)$ and (c) of Definition 2.1 for the family $\mathcal{F} = \mathcal{P}(X)$.*

Proof. From the definition of ρ-closure of a set it is clear that $A \subset \operatorname{Cl}_\rho A$ for any $A \in \mathcal{P}(X)$ and $\operatorname{Cl}_\rho A \subset \operatorname{Cl}_\rho B$ whenever $A \subset B$, $A, B \in \mathcal{P}(X)$, thus Cl_ρ is a monotone expansion on $\mathcal{P}(X)$. We show that Cl_ρ is idempotent on $\mathcal{P}(X)$. Given $A \in \mathcal{P}(X)$ it is obvious that $\operatorname{Cl}_\rho A \subset \operatorname{Cl}_\rho(\operatorname{Cl}_\rho A)$. Let $x \in \operatorname{Cl}_\rho(\operatorname{Cl}_\rho A)$, and let D be any ρ-open set containing x, then by Theorem 2.6 there is $z \in D \cap \operatorname{Cl}_\rho A$. Since $z \in \operatorname{Cl}_\rho A$ and D is a ρ-open set containing z, we have that $D \cap A \neq \emptyset$, thus $x \in \operatorname{Cl}_\rho A$. Therefore $\operatorname{Cl}_\rho(\operatorname{Cl}_\rho A) \subset \operatorname{Cl}_\rho A$. □

In general, Cl_ρ is not an additive operator on $\mathcal{P}(X)$ as the next example shows.

Example 2.1. Let $X = \{a, b, c\}$ and $\tau = \{\{a\}, \{b\}, \{a, b\}, X, \emptyset\}$. Considere on (X, τ) the operator ρ defined by $\rho\emptyset = \emptyset$, $\rho A = X$ for all $A \in \tau$, $A \neq \emptyset$, and $\rho A = A$ for all $A \notin \tau$. Note that $\rho(X, \tau) = \tau \cup \{a, c\} \cup \{b, c\}$. Take $A = \{a\}$ and $B = \{b\}$, then $\mathrm{Cl}_\rho\{a\} = \{a\}$ and $\mathrm{Cl}_\rho\{b\} = \{b\}$, but $\mathrm{Cl}_\rho(\{a\} \cup \{b\}) = X$, thus $\mathrm{Cl}_\rho(A \cup B) \neq \mathrm{Cl}_\rho A \cup \mathrm{Cl}_\rho B$.

Remark 2.8. If Cl_ρ is an additive operator on $\mathcal{P}(X)$ then $\rho(X, \tau)$ is a topology on X. In fact, additivity of the operator Cl_ρ on $\mathcal{P}(X)$ implies that Cl_ρ is an operator of closure of Kuratowski, and thus the family $\mathcal{T} = \{(\mathrm{Cl}_\rho)^c : A \subset X\}$ is a topology on X (Dugundji [8]). But $\mathcal{T} = \rho(X, \tau)$ for any operator ρ, thus $\rho(X, \tau)$ is a topology on X.

We finish this section with a note about the ρ-open sets in the relative topology. Let Y be a subspace of (X, τ) with the relative topology τ_Y. Then $\rho(Y, \tau_Y)$ is not necessarily a subfamily of $\rho(X, \tau)$ (on (R, τ), the set of real numbers with the usual topology, consider the operator $\rho = \mathrm{Cl}$ and let $Y = [1, 2] \cup \{2\}$, then $\{2\} \in \mathrm{Cl}(Y, \tau_Y)$ but $\{2\} \notin \mathrm{Cl}(R, \tau)$). If Y is an open subset of X then, from the definition of ρ-open set and the fact that any open set on Y is open on X, it follows that $\rho(Y, \tau_Y) \subset \rho(X, \tau)$ for any operator ρ on (X, τ). On the other hand, the intersection of a ρ-open subset of X with a subspace $Y \subset X$ is not necessarily ρ-open on Y, even if Y is open on X, as the next example shows.

Example 2.2. Let $X = \{a, b, c\}$ and $\tau = \{\{a\}, \{b, c\}, X, \emptyset\}$. Let (X, τ) and ρ as in Example 2.1. Note that $\{a, c\} \in \rho(X, \tau)$. Let $Y = \{b, c\}$, an open subspace of X. Then $Y \cap \{a, c\} = \{c\} \notin \rho(X, \tau_Y)$.

3. Preclosure Operator, Topology Associated to the Operator, and Complemented Operator

Although the finite intersection of semi-open sets is not in general a semi-open set, the intersection of an open set and a semi open set is semi-open. This last fact can be easily proved using a property of the closure operator: $U \cap \mathrm{Cl}\, V \subset \mathrm{Cl}(U \cap V)$ for any U, $V \in \tau$. An operator ρ satisfying a likewise property will be called here a preclosure operator. We give in this section some results related with this class of operators.

Definition 3.1. Let ρ be an operator on (X, τ). If $U \cap \rho V \subset \rho(U \cap V)$ for any pair $U, V \in \tau$, then ρ is said to be a preclosure operator.

Remark 3.1. If ρ is a preclosure operator, then $\operatorname{Int}\rho$ is also a preclosure operator. In fact, given $U, V \in \tau$ and $x \in U \cap \operatorname{Int}\rho V$, let $O \in \tau$ such that $x \in O \subset \rho V$. Then $x \in O \cap U \in \tau$ and $U \cap O \subset U \cap \rho V \subset \rho(U \cap V)$, thus $x \in \operatorname{Int}\rho(U \cap V)$.

Proposition 3.1. *Let ρ be a preclosure operator on (X, τ) and let $V \in \tau$. Then:*

(i) *$V \cap B \in \rho(X, \tau)$ for all $B \in \rho(X, \tau)$.*

(ii) *$x \in \rho V$ implies $U \cap V \neq \emptyset$, for each $U \in \tau$ which contains x.*

(iii) *$U \cap \rho V = \emptyset = V \cap \rho U$ whenever $U \cap V = \emptyset$, $U \in \tau$.*

(iv) *$\rho V \subset \operatorname{Cl} V$.*

Proof. (i) Given $V \in \tau$ and $B \in \rho(X, \tau)$, let $U \in \tau$ such that $U \subset B \subset \rho U$. Then $V \cap U \subset V \cap B \subset V \cap \rho U \subset \rho(U \cap V)$, which shows that $V \cap B \in \rho(X, \tau)$.

(ii) Let $x \in \rho V$ and suppose there is an open set U which contains x such that $U \cap V = \emptyset$. Then $x \in U \cap \rho V \subset \rho(U \cap V) = \rho\emptyset = \emptyset$, a contradiction.

(iii) Follows from the fact that $\rho\emptyset = \emptyset$.

(iv) Follows directly from (ii). □

Remark 3.2. Note that $\beta = \operatorname{Cl}_\rho$ is a preclosure operator whenever ρ is. In fact, let $U, V \in \tau$ and let $x \in U \cap \operatorname{Cl}_\rho V$. Given $B \in \rho(X, \tau)$, with $x \in B$, we have that $x \in B \cap U \in \rho(X, \tau)$ (proposition 3.1(ii)). Then $B \cap (U \cap V) \neq \varnothing$ and therefore $x \in \operatorname{Cl}_\rho(U \cap V)$.

Recall that if Y is a subspace of X and $A \subset Y$, then the closure of A on Y, denoted by $(\operatorname{Cl} A)^Y$, satisfies $(\operatorname{Cl} A)^Y = \operatorname{Cl} A \cap Y$. Thus, in the context of ρ-closure for the operator $\rho = \operatorname{Int}$, we have that $(\operatorname{Cl}_{\operatorname{Int}} A)^Y = \operatorname{Cl}_{\operatorname{Int}} A \cap Y$. This result generalizes if ρ is a preclosure operator and Y is an open subspace of X as we prove below. First we give a lemma.

Lemma 3.2. *Let ρ be a preclosure operator on (X, τ), and let Y be an open subspace of X. If B is ρ-open on X then $Y \cap B$ is ρ-open on Y.*

Proof. Let O be an open subset of X such that $O \subset B \subset \rho O$. Then $Y \cap O$ is open on Y and satisfies $Y \cap O \subset Y \cap B \subset Y \cap \rho O \subset \rho(Y \cap O)$. □

Theorem 3.3. *Let ρ be a preclosure operator on (X, τ), and let Y be an open subspace of X. Then for any subset A of Y,*

$$Y \cap \operatorname*{Cl}_{\rho} A = (\operatorname*{Cl}_{\rho} A)^Y ,$$

where $(\mathrm{Cl}_\rho A)^Y$ denotes the ρ-closure of A on the subspace Y.

Proof. Let $A \subset Y$ and $z \in Y \cap \mathrm{Cl}_\rho(A)$. Since Y is open on X, given any ρ-open set B on Y with $z \in B$, it follows that B is ρ-open on X, and thus $A \cap B \neq \emptyset$. By Theorem 2.6, $z \in (\mathrm{Cl}_\rho A)^Y$. Therefore $Y \cap \mathrm{Cl}_\rho A \subset (\mathrm{Cl}_\rho A)^Y$.

On the other hand, let $z \in (\mathrm{Cl}_\rho A)^Y \subset Y$, and let B be any ρ-open subset of X with $z \in B$. Since ρ is a preclosure operator, then $Y \cap B$ is ρ-open on Y (Lemma 3.2), thus $(Y \cap B) \cap A \neq \emptyset$. In particular, $B \cap A \neq \emptyset$ which shows that $z \in \mathrm{Cl}_\rho(A)$. Since $z \in Y$, then $(\mathrm{Cl}_\rho A)^Y \subset Y \cap \mathrm{Cl}_\rho A$. □

Next example shows that the condition on ρ to be preclosure cannot be removed from the above theorem.

Example 3.1. Let X, τ, ρ and $Y \subset X$ as in Example 2.2, and let $A = \{b\}$. Then $\mathrm{Cl}_\rho\{b\} = \{b\}$ but $(\mathrm{Cl}_\rho\{b\})^Y = \{b, c\}$. Thus $(\mathrm{Cl}_\rho A)^Y \neq Y \cap \mathrm{Cl}_\rho A$.

Given any operator ρ on (X, τ), let $\tau^\rho = \{A \subset X : A \cap B \in \rho(X, \tau)$ for all $B \in \rho(X, \tau)\}$. It is clear that $\tau^\rho \subset \rho(X, \tau)$. Since the family $\rho(X, \tau)$ contains the empty set and is closed under arbitrary unions, it is a standard result that τ^ρ is a topology. We call τ^ρ the *topology associated to the operator* ρ. Note that for any operator ρ, the family $\rho(X, \tau)$ is a topology if and only if $\rho(X, \tau) = \tau^\rho$. If ρ is a preclosure operator, τ^ρ is topology on X finer than τ as we show in the next result.

Theorem 3.4. *If ρ is a preclosure operator on (X, τ), then the topology τ^ρ associated to the operator ρ is finer than τ.*

Proof. If ρ is a preclosure operator and $V \in \tau$ then, from Proposition 3.1 (i), $V \cap B \in \rho(X, \tau)$ for all $B \in \rho(X, \tau)$. Thus $V \in \tau^\rho$ and therefore $\tau \subset \tau^\rho$. □

From the fact that $\mathrm{Cl}_\rho < \mathrm{Cl}$, for any operator ρ on (X, τ), it is clear that τ^{Cl_ρ} is a topology on X satisfying $\tau^{\mathrm{Cl}_\rho} \subset \mathrm{Cl}(X, \tau)$ for any operator ρ on (X, τ). We prove here that this topology is contained in the family $\mathrm{Int}_\rho \mathrm{Cl}(X, \tau)$.

Theorem 3.5. *Let ρ be an operator on (X, τ). Then $\tau^{\mathrm{Cl}_\rho} \subset \mathrm{Int}_\rho \mathrm{Cl}(X, \tau)$.*

Proof. Let $A \in \tau^{\mathrm{Cl}_\rho}$. In particular $A \in \mathrm{Cl}_\rho(X, \tau)$ and thus $A \subset \mathrm{Cl}_\rho \mathrm{Int}\, A$. Suppose there is $x \in A \cap (\mathrm{Int}_\rho \mathrm{Cl}\, \mathrm{Int}\, A)^c = \mathrm{Cl}_\rho \mathrm{Int}\, \mathrm{Cl}\, A^c$. Then the set $B = \mathrm{Int}\, \mathrm{Cl}\, A^c$ is open on X and satisfies $B \subset \{x\} \cup B \subset \mathrm{Cl}_\rho B$. Thus $\{x\} \cup B \in \mathrm{Cl}_\rho(X, \tau)$ and we have that $(\{x\} \cup B) \cap A \in \mathrm{Cl}_\rho(X, \tau)$. But $(\{x\} \cup B) \cap A = \{x\}$. By Proposition 2.3(b), the set $\{x\}$ is open, and since $\{x\} \subset \mathrm{Cl}_\rho \mathrm{Int}\, A$ it must be that $\{x\} \subset \mathrm{Int}\, \mathrm{Cl}_\rho \mathrm{Int}\, A \subset \mathrm{Int}_\rho \mathrm{Cl}\, \mathrm{Int}\, A$, which gives a contradiction. □

Theorem 3.6. *Let ρ be a preclosure operator on (X,τ) such that (i) ρ is monotone on $\mathcal{P}(X)$, (ii) ρ is idempotent on τ. Then* $\operatorname{Int}\rho(X,\tau) \subset \tau^{\rho}$.

Proof. Let $B \in \operatorname{Int}\rho(X,\tau)$ and let $A \in \rho(X,\tau)$. Then $A \cap B \subset (\operatorname{Int}\rho \operatorname{Int} B) \cap (\operatorname{Int}\rho A) \subset \rho(\operatorname{Int}\rho\operatorname{Int} B \cap \operatorname{Int} A) \subset \rho(\rho\operatorname{Int} B \cap \operatorname{Int} A) \subset \rho(\rho(\operatorname{Int} B \cap \operatorname{Int} A)) = \rho(\operatorname{Int}(A \cap B))$. Thus $A \cap B \in \rho(X,\tau)$, and therefore $B \in \tau^{\rho}$. □

If ρ is a preclosure operator, then Cl_{ρ} satisfies all the conditions on the hypothesis of the above theorem. Thus, applying Theorem 3.5 and Theorem 3.6, we have the following result.

Theorem 3.7. *Let ρ be a preclosure operator on (X,τ). Then* $\operatorname{Int}\operatorname{Cl}_{\rho}(X,\tau) \subset \tau^{\operatorname{Cl}_{\rho}} \subset \operatorname{Int}_{\rho}\operatorname{Cl}(X,\tau)$.

Corollary 3.8. (Proposition 2 in Njastad [16]) *For any topological space (X,τ), the family τ^{α} of the α-sets is a topology on X.*

Proof. For the operator $\rho = \operatorname{Int}$, the above theorem gives $\tau^{\operatorname{Cl}} = \operatorname{Int}\operatorname{Cl}(X,\tau)$. Since $\operatorname{Int}\operatorname{Cl}(X,\tau) = \tau^{\alpha}$, it follows that τ^{α} is a topology on X. □

Remark 3.3. Given a topological space (X,τ), it is known that the family τ^{α} of the α-sets is in general properly contained between τ and the family $SO(X,\tau)$ of the semiopen sets (Njastad [16]), i.e. in general $\tau \subsetneqq \tau^{\operatorname{Cl}} \subsetneqq \operatorname{Cl}(X,\tau)$. Thus shows that, for a preclosure operator ρ, the inclusion $\tau \subset \tau^{\rho} \subset \rho(X,\tau)$ is in general strict.

Theorem 3.9. *Let ρ be an operator on (X,τ) such that $\rho(X,\tau)$ is a topology on X. If $\operatorname{Int}\rho V$ is ρ-closed for all $V \in \tau$, then $\rho V \in \tau$ for all $V \in \tau$.*

Proof. Suppose $\rho V \notin \tau$ for some $V \in \tau$ and let $x \in \rho V \setminus \operatorname{Int}\rho V$. Then the set $B = \{x\} \cup \operatorname{Int}\rho V$ satisfies $V \subset B \subset \rho V$ and thus $B \in \rho(X,\tau)$. By hypothesis, $C = (\operatorname{Int}\rho V)^{c} \in \rho(X,\tau)$ and, since $\rho(X,\tau)$ is a topology on X, it follows that $\{x\} = C \cap B \in \rho(X,\tau)$. By Proposition 2.3(b), $\{x\}$ is open and then $\{x\} \subset \operatorname{Int}\rho V$, a contradiction. □

Note that the operator ρ on Example 2.2 satisfies $\rho V \in \tau$ for all $V \in \tau$ but $\rho(X,\tau)$ is not a topology on X since $\{a,c\}$, $\{b,c\} \in \rho(X,\tau)$ but $\{a,c\} \cap \{b,c\} = \{c\} \notin \rho(X,\tau)$. Next example shows that if $\rho(X,\tau)$ is a topology it is not necessarily true that $\rho V \in \tau$ for all $V \in \tau$. Thus the condition $\operatorname{Int}\rho V$ is ρ-closed for all $V \in \tau$ can not be removed on Theorem 3.9.

Example 3.2. Let $X = \{a,b,c\}$ and $\tau = \{\{a\}, X, \emptyset\}$. Consider in (X,τ) the operator $\rho\{a\} = \{a,b\}$ and $\rho B = B$ for all $B \neq \{a\}$. It is easy to see that $\rho(X,\tau)$ is a topology on X but $\rho\{a\} \notin \tau$.

We finish this section introducing the concept of complemented operator, which we use to give a characterization of the ρ-closure of a subset of X.

Definition 3.2. Let ρ be an operator on (X, τ). We say that ρ is *complemented* if the map $\gamma : \mathcal{P}(X) \to \mathcal{P}(X)$ defined by $\gamma A = (\rho A^c)^c$ for all $A \subset X$ is an operator on (X, τ). In this case we say that ρ is complemented by γ or that γ is a complement of ρ.

Example 3.3. For any operator ρ, the operator Cl_ρ is complemented by the operator Int_ρ.

Remark 3.4. (a) Note that $(\gamma A)^c = \rho A^c$, for all $A \subset X$, implies that $(\rho A)^c = \gamma A^c$, for all $A \subset X$ (replace A by A^c). Thus the relation "ρ is complemented by γ" is symmetric.

(b) If ρ and δ are operators complemented by γ and β, respectively, such that ρ and γ are monotone on $\mathcal{P}(X)$, then the composition operator $\rho\delta$ is complemented by the composition operator $\gamma\beta$.

Next we give a necessary and sufficient condition for an operator to be complemented.

Theorem 3.10. *Let ρ be an operator monotone on $\mathcal{P}(X)$. Then ρ is complemented if and only if $\rho D \subset D$ for all closed sets $D \subset X$.*

Proof. Let ρ be complemented by γ and let D be a closed subset of X. Then $D^c \subset \gamma D^c$ and thus $\rho D = (\gamma D^c)^c \subset D$. Conversely, suppose $\rho D \subset D$ for all closed sets $D \subset X$. Since $\rho X = X$, $\gamma\emptyset = (\rho\emptyset^c)^c = \emptyset$. If $V \in \tau$, then $\rho V^c \subset V^c$and thus $V \subset (\rho V^c)^c = \gamma V$, which shows that γ is an expansion on (X, τ). Let $A, B \in \mathcal{P}(X)$, with $A \subset B$. Since ρ is monotone on $\mathcal{P}(X)$, $\rho B^c \subset \rho A^c$ and then $\gamma A = (\rho A^c)^c \subset (\rho B^c)^c = \gamma B$ which gives the monotony of γ on $\mathcal{P}(X)$. Thus γ is an operator on (X, τ) and therefore ρ is complemented. $\square$

Theorem 3.11. *Let ρ be an operator on (X, τ). If ρ is complemented by γ, then:*

(i) D is ρ-closed if and only if $\gamma\,\mathrm{Cl}\,D \subset D$.

(ii) If D is ρ-closed and $A \subset D$, then $\gamma\,\mathrm{Cl}\,A \subset D$.

Proof. (i) D is ρ-closed $\Leftrightarrow$ D^c is ρ-open $\Leftrightarrow$ $D^c \subset \rho\,\mathrm{Int}\,D^c = (\gamma\,\mathrm{Cl}\,D)^c$ $\Leftrightarrow \gamma\,\mathrm{Cl}\,D \subset D$.

(ii) Follows from (i) and the fact $A \subset D \Rightarrow D^c \subset A^c \Rightarrow (\gamma\,\mathrm{Cl}\,D)^c = \rho(\mathrm{Cl}\,D)^c = \rho\,\mathrm{Int}\,D^c \subset \rho\,\mathrm{Int}\,A^c = \rho(\mathrm{Cl}\,A)^c = (\gamma\,\mathrm{Cl}\,A)^c \Rightarrow \gamma\,\mathrm{Cl}\,A \subset \gamma\,\mathrm{Cl}\,D$. $\square$

Corollary 3.12. *Let ρ be a complemented operator which is monotone on $\mathcal{P}(X)$. If $D \subset X$ is ρ-closed and not nowhere dense subset of X, then $\mathrm{Int}\,D \neq \emptyset$.*

Proof. Let γ be the complement of ρ. By Theorem 3.11, $\gamma\,\mathrm{Cl}\,D \subset D$. Since

Int $<\gamma$, it follows that $\emptyset \neq \operatorname{Int}\operatorname{Cl} D \subset \gamma \operatorname{Cl} D \subset D$. Therefore $\operatorname{Int} D$ is not empty. □

Theorem 3.13. *Let ρ be an operator on (X,τ). If ρ is complemented by γ, then $\operatorname{Cl}_\rho A = A \cup \gamma \operatorname{Cl} A$, for all $A \subset X$.*

Proof. We first show that $A \cup \gamma \operatorname{Cl} A$ is ρ-closed for all $A \subset X$. Let $D = (A \cup \gamma \operatorname{Cl} A)^c = A^c \cap (\gamma \operatorname{Cl} A)^c = A^c \cap \rho \operatorname{Int} A^c$. Since $\operatorname{Int} A^c \subset D \subset \rho \operatorname{Int} A^c$, then D is ρ-open and therefore D^c is ρ-closed. Now, since $A \subset A \cup \gamma \operatorname{Cl} A$ it must be that $\operatorname{Cl}_\rho A \subset A \cup \gamma \operatorname{Cl} A$.

On the other hand, if D is ρ-closed and $A \subset D$, then $\gamma \operatorname{Cl} A \subset D$ (Theorem 3.11(*ii*)) and it follows that $A \cup \gamma \operatorname{Cl} \subset D$. Thus $A \cup \gamma \operatorname{Cl} A \subset \operatorname{Cl}_\rho A$. □

4. (δ, ρ)-Closed Sets

In 1970, Levine [15] introduced the concept of generalized closed set on a topological space comparing the clousure of a set with its open super sets. A subset A of (X,τ) is generalized closed (abreviated g-closed) if $\operatorname{Cl} A \subset U$ whenever $A \subset U$ and U is open. By mean of other generalized closure operators and other open generalized sets, variuos notions similar to the Levine g-closed set have been defined and studied. We state some of them.

Definition 4.1. Let (X,τ) be a topological space. Let Cl_s denotes the ρ-closure for the operator $\rho = \operatorname{Cl}$ and let Cl_α denotes the ρ-closure for the operator $\rho = \operatorname{Int}\operatorname{Cl}$. A subset A of X is said:

(*i*) semi-generalized closed (sg-closed) if $\operatorname{Cl}_s A \subset U$ whenever $A \subset U$ and U is semi open (Bhattacharya [3]).

(*ii*) generalized semi-closed (gs-closed) if $\operatorname{Cl}_s A \subset U$ whenever $A \subset U$ and U is open (Arya [1]).

(*iii*) generalizaded α-closed ($g\alpha$-closed) if $\operatorname{Cl}_\alpha A \subset U$ whenever $A \subset U$ and U is α-open (Herrington [11]).

(*iv*) α-generalizaded closed (αg-closed) if $\operatorname{Cl}_\alpha A \subset U$ whenever $A \subset U$ and U is open (Kasahara [12]).

In [5], Cao, Greenwood and Reilly introduced the concept of qr-closed sets, which allows them to investigate various classes of generalized closed sets in a unified way. A subset A of (X,τ) is said to be qr-closed if $\operatorname{Cl}_q A \subset U$ whenever $A \subset U$ and U is r-open, where $q, r \in \mathcal{M} = \{\tau, \alpha, s, p, \beta\}$, and τ, α, s, p and β refers to the τ-open, α-open, semi-open, pre-open and β-open sets, respectively (Gauld [10]). In this section we define the (δ, ρ)-closed sets for any pair of operators δ y ρ on (X, τ), and show the relationship between (δ, ρ)-closed, δ-closed and ρ-closed sets. An additional result is given.

Definition 4.2. Let δ and ρ be operators on (X, τ). A subset A of X is said to be (δ, ρ)-closed if $\operatorname{Cl}_\delta A \subset U$ whenever $A \subset U$ and U is ρ-open.

Remark 4.1. (a) Since $\operatorname{Cl}_\delta A \subset \operatorname{Cl} A$ for any set $A \subset X$ and for any operator δ, then a closed set is (δ, ρ)-closed for any pair of operators δ and ρ on (X, τ). Moreover, if A is (δ, ρ)-closed, then A is (δ, ρ')-closed and (δ', ρ)-closed for all operators δ' and ρ' such that $\rho' < \rho$ and $\delta < \delta'$.

(b) For some operators the property (δ, ρ)-closed is equivalent to a generalized closed set. For instance, $(\operatorname{Int}, \operatorname{Int})$-closed is equivalent to g-closed, $(\operatorname{Int}\operatorname{Cl}, \operatorname{Int})$-closed is equivalent to αg-closed, $(\operatorname{Int}\operatorname{Cl}, \operatorname{Int}\operatorname{Cl})$-closed is equivalent to $g\alpha$-closed, $(\operatorname{Cl}, \operatorname{Cl})$-closed is equivalent to sg-closed and $(\operatorname{Cl}, \operatorname{Int})$-closed is equivalent to gs-closed (Cao [4]).

Theorem 4.1. *Let δ and ρ be operators on (X, τ). Then every (δ, ρ)-closed subset of X is δ-closed if and only if each singleton of X is either δ-open or ρ-closed.*

Proof. Suppose that every (δ, ρ)-closed set of X is δ-closed. If $x \in X$ and $\{x\}$ is not ρ-closed, then $X \backslash \{x\}$ is not ρ-open. Thus, the only ρ-open set containing $X \backslash \{x\}$ is X, which implies that $X \backslash \{x\}$ is (δ, ρ)-closed. By hypothesis, $X \backslash \{x\}$ is δ-closed. Therefore $\{x\}$ is δ-open.

Conversely, suppose that each singleton of X is either δ-open or ρ-closed. Let A be (δ, ρ)-closed and $x \in \operatorname{Cl}_\delta A$. If $\{x\}$ is δ-open, then $A \cap \{x\} \neq \emptyset$ and thus $x \in A$ (Theorem 2.6). If $\{x\}$ is ρ-closed and $x \notin A$, then $A \subset X \backslash \{x\}$. Since A is (δ, ρ)-closed, then $x \in \operatorname{Cl}_\delta A \subset X \backslash \{x\}$ which is a contradiction. Therefore $A = \operatorname{Cl}_\delta A$ and thus A is δ-closed. □

In Battacharya [3], the semi-$T_{1/2}$ space is defined as the space in which every singleton is semi-open or semi-closed. Next result follows from the above theorem.

Corollary 4.2. *A topological space is semi-$T_{1/2}$ if and only if every sg-closed set is semi-closed.*

Lemma 4.3. *Let ζ be an operator which is complemented. If* $\operatorname{Int} \zeta \operatorname{Cl}\{x\} \neq \emptyset$, *then* $\{x\} \subset \operatorname{Int} \zeta \operatorname{Cl}\{x\}$.

Proof. Let $z \in \operatorname{Int} \sigma \operatorname{Cl}\{x\}$ and let V_z an open set containing z such that $V_z \subset \sigma \operatorname{Cl}\{x\}$. By Theorem 3.11$(i)$, $\zeta \operatorname{Cl}\{x\} \subset \operatorname{Cl}\{x\}$, thus $z \in \operatorname{Cl}\{x\}$ and we have that $x \in V_z$. If $x \notin \operatorname{Int} \zeta \operatorname{Cl}\{x\}$ then $V_z \cap (\zeta \operatorname{Cl}\{x\})^c \neq \emptyset$, which is a contradiction. Therefore $\{x\} \subset \operatorname{Int} \zeta \operatorname{Cl}\{x\}$. □

Theorem 4.4. *Let δ and ρ be operators on (X,τ). If ρ is complemented, then $A \subset X$ is $(\delta, \mathrm{Cl}\,\rho)$-closed if and only if A is $(\delta, \mathrm{Int}\,\mathrm{Cl}\,\rho)$-closed.*

Proof. By definition, every $(\delta, \mathrm{Cl}\,\rho)$-closed is $(\delta, \mathrm{Int}\,\mathrm{Cl}\,\rho)$-closed. To show the converse, let $A \subset X$ be $(\delta, \mathrm{Int}\,\mathrm{Cl}\,\rho)$-closed and $A \subset U$, where U is $\mathrm{Cl}\,\rho$-closed, i.e. $U \subset \mathrm{Cl}\,\rho\,\mathrm{Int}\,U$. If there exists $x \in \mathrm{Cl}_\delta A \backslash U$, then $A \subset U \subset X \backslash \{x\}$. Let ρ be complemented by ζ. Consider two cases:

(i) $\mathrm{Int}\,\zeta\,\mathrm{Cl}\{x\} = \emptyset$. Then $\mathrm{Cl}\,\mathrm{Int}\,\zeta\,\mathrm{Cl}\{x\} \subset \{x\}$ and, by Theorem 3.11(i), $\{x\}$ is $\mathrm{Int}\,\mathrm{Cl}\,\rho$-closed and therefore $X \backslash \{x\}$ is $\mathrm{Int}\,\mathrm{Cl}\,\rho$-open. Since A is $(\delta, \mathrm{Int}\,\mathrm{Cl}\,\rho)$-closed, it follows that $x \in \mathrm{Cl}_\delta A \subset X \backslash \{x\}$, which is a contradiction.

(ii) $\mathrm{Int}\,\zeta\,\mathrm{Cl}\{x\} \neq \emptyset$. Then by Lemma 4.3 $\{x\} \subset \mathrm{Int}\,\zeta\,\mathrm{Cl}\{x\}$, and we have that $x \in \mathrm{Cl}_\delta A \subset \mathrm{Cl}\,A \subset \mathrm{Cl}\,U \subset \mathrm{Cl}\,\rho\,\mathrm{Int}\,U \subset \mathrm{Cl}\,\rho\,\mathrm{Int}\,X \backslash \{x\} = (\mathrm{Int}\,\zeta\,\mathrm{Cl}\{x\})^c$, which gives a contradiction.

Thus $\mathrm{Cl}_\delta A \backslash U = \emptyset$, and so $\mathrm{Cl}_\delta A \subset U$ which shows that A is $(\delta, \mathrm{Cl}\,\rho)$-closed. □

Corollary 4.5. *A subset A of X is qs-cerrado if and only if A is $q\alpha$-cerrado, for any $q \in \mathcal{M}'=\{\tau, \alpha, s\}$.*

Proof. The result follows applying the above theorem to $\delta = q \in \mathcal{M}'$ and $\rho = \mathrm{Int}$. □

5. (α,β)-Expansion Continuity

Let (X,τ) and (Y,σ) be topological spaces and let α and β be expansions on (X,τ) and (Y,σ) respectively. In [23], Tong defined a weak form of continuity, the β-expansion continuity, as follows: $f : X \to Y$ is β-expansion continuous if $f^{-1}(V) \subset \mathrm{Int}\,f^{-1}(\beta V)$ for all $V \in \sigma$. Using this concept, Tong [23] gave a decomposition of continuity: $f : X \to Y$ is continuous if and only if f is β-expansion continuous and β'-expansion continuous, where β and β' are expansions on (Y,σ) such that $\beta V \cap \beta' V = V$ for all $V \in \sigma$. In this section we define a weak form of continuity, the (α,β)-expansion continuity, which is weaker than the (α,β)-continuity in the sense of Ogata [18] and the (α,β)-weak continuity in the sense of Rosas [19] when α is expansion on $\mathcal{P}(X)$. Some aditional results related with the topological properties the (α,β)-expansion continuous functions preserve are given here.

Definition 5.1. (Ogata [18]) A function $f : X \to Y$ is said to be (α,β)-continuous if for each $x \in X$ and for each $V \in \sigma$ containing $f(x)$, there exists $U \in \tau$ containing x such that $\alpha U \subset f^{-1}(\beta V)$.

Definition 5.2. (Ogata [18]) A function $f : X \to Y$ is said to be (α,β)-weak continuous if $\alpha f^{-1}(V) \subset \operatorname{Int} f^{-1}(\beta V)$ for all $V \in \sigma$.

Definition 5.3. A function $f : X \to Y$ is said to be (α,β) -expansion continuous if $f^{-1}(V) \subset \alpha \operatorname{Int} f^{-1}(\beta V)$ for each $V \in \sigma$.

Remark 5.1. (1) If $\alpha = \mathrm{Id}$ on (X,τ) and $\beta = \mathrm{Id}$ on (Y,σ), the three definitions are equivalent to continuity.

(2) If $\alpha = \mathrm{Id}$ and β is any expansion on (Y,σ), then (α,β)-weak continuity and (α,β)-expansion continuity are both equivalent to the β-expansion continuity in the sense of Tong [23]. Thus we have, for instance, $\beta = \mathrm{Cl}$ is weak continuity in the sense of Levine [13], $\beta = \operatorname{Int}\mathrm{Cl}$ is almost continuity in the sense of Singal [21], $\beta = \operatorname{Ker}\mathrm{Cl}$ and $\beta = \operatorname{Int}\operatorname{Ker}\mathrm{Cl}$ are very weak continuity and weak almost continuity, respectively, in the sense of Tong [22].

(3) If $\alpha = \mathrm{Cl}$ and $\beta = \mathrm{Id}$, then (α,β)-expansion continuity is equivalent to semi-continuity in the sense of Levine [14].

(4) If $\alpha = \operatorname{Int}\mathrm{Cl}$ y $\beta = \mathrm{Id}$, then (α,β)-expansion continuity is equivalente to α-continuity in the sense of Njastad [16].

(5) If $\alpha = \operatorname{Int}\mathrm{Cl}$ y $\beta = \operatorname{Int}\mathrm{Cl}$, then (α,β)-expansion continuity is equivalente to almost α-continuity in the sense of Noiri [17].

Theorem 5.1. *For any pair of expansions α on (X,τ) and β on (Y,σ) the following implication holds:*

(α,β)-continuity $\Rightarrow$ β-expansion continuity $\Rightarrow$ (α,β)-expansion continuity.

Proof. Let $f : X \to Y$ a (α,β)-continuous function and let $V \in \sigma$. If $f^{-1}(V) = \emptyset$, then it is obvious that $f^{-1}(V) \subset \operatorname{Int} f^{-1}(\beta V)$. Otherwise, for each $x \in f^{-1}(V)$ let U_x be an open set on X which contains x and such that $U_x \subset \alpha U_x \subset f^{-1}(\beta V)$. Then $f^{-1}(V) \subset \bigcup_{x\in X} U_x \subset f^{-1}(\beta V)$. Therefore f is β-expansion continuous.

On the other hand, since given any function f the inclusion

$$\operatorname{Int} f^{-1}(\beta V) \subset \alpha \operatorname{Int} f^{-1}(\beta V)$$

holds for any $V \in \sigma$, it follows that any β-expansion continuous function is (α,β)-expansion continuous. $\square$

Remark 5.2. It follows from the definition that if $f : X \to Y$ is (α,β)-expansion continuous, then:

(a) f is (α',β)-expansion continuous, for all expansion α' on (X,τ) such that $\alpha < \alpha'$.

(b) f is (α,β')-expansion continuous, for all expansion β' on (Y,σ) such that $\beta < \beta'$.

Theorem 5.2. *Let α be an expansion on $\mathcal{P}(X)$ and β be an expansion on σ. If $f : X \to Y$ is (α, β)-weak continuous, then f is β-expansion continuous.*

Proof. Follows directly from the fact that

$$f^{-1}(V) \subset \alpha f^{-1}(V) \subset \operatorname{Int} f^{-1}(\beta V).$$ □

If α is expansion on τ but is not expansion on all $\mathcal{P}(X)$, the result on the above proposition might fail as we show in the next example.

Example 5.1. Let $X = Y = \{a, b\}$, and let $\tau = \{\{a\}, X\ , \emptyset\}$ and $\sigma = \{\{a\}, \{b\}, X, \emptyset\}$ be topologies on X and Y respectively. Let $f : (X, \tau) \to (Y, \sigma)$ be the identity function on (X, τ). Consider the expansion α definided by $\alpha V = V$ for each $V \in \tau$ and $\alpha\{b\} = \emptyset$, and let $\beta = \mathrm{Id}$, the identity expansion on (Y, σ). Note that $\{b\} \subsetneqq \alpha\{b\}$, then α is not expansion on all $\mathcal{P}(X)$. It is easy to see that $\alpha f^{-1}(V) \subset \operatorname{Int} f^{-1}(\beta V)$ for each $V \in \sigma$, and thus f is (α, β)-weak continuous. But since $f^{-1}(\{b\}) \subsetneqq \operatorname{Int}\ f^{-1}(\beta\{b\})$, then f is not β-expansion continuous

Let α and α' be expansions on (X, τ). The intersection expansion is defined by $(\alpha \wedge \alpha')A = \alpha A \cap \alpha' A$, for each $A \subset X$ (Rosas [20]). The expansions α and α' are said to be mutually dual on τ if $\alpha \wedge \alpha'$ is the identity expansion on τ (Tong [23]). Equivalently, if $\alpha V \cap \alpha' V = V$ for all $V \in \tau$. Note that the identity expansion is mutually dual to any expansion α.

Given any expansion α on (X, τ), a natural question arises: among all the expansions which are mutually dual to α, is there a maximal expansion α', in the sense that if λ is an expansion on (X, τ) which is mutually dual to α, then $\lambda < \alpha'$? The positive answer is given in the next theorem.

Theorem 5.3. *Let α be an expansion on (X, τ). Then the expansion $\alpha' V = V \cup (\alpha V)^c$ is the maximal expansion which is mutually dual to α.*

Proof. Let B_α be the set of all expansions on (X, τ) which are mutually dual to α. Since $V \subset \alpha V$, for each $V \in \tau$, αV can be written as $\alpha V = V \cup (\alpha V \backslash V)$. Let $\alpha' V = V \cup (\alpha V)^c = (\alpha V \backslash V)^c$. It is obvious that α' is an expansion on (X, τ) and $\alpha V \cap \alpha' V = V$ for all $V \in \tau$, and thus $\alpha' \in B_\alpha$. Now, given any expansion λ on (X, τ), write $\lambda V = V \cup (\lambda V \backslash V)$. If $\lambda \in B_\alpha$, then $(\alpha V \backslash V) \cap (\lambda V \backslash V) = \emptyset$, so that $\lambda V \backslash V \subset (\alpha V \backslash V)^c = \alpha' V$, and it follows that $\lambda V = V \cup (\lambda V \backslash V) \subset V \cup \alpha' V = \alpha' V$, which shows that $\lambda < \alpha'$. Therefore α' is the maximal element on B_α. □

Theorem 5.4. *Let α and α' be expansions on (X, τ) and β be an expansion on (X, σ). A function $f : (X, \tau) \to (Y, \sigma)$ is $(\alpha \wedge \alpha', \beta)$-expansion continuous if and only if f is (α, β)-expansion continuous and (α', β)-expansion continuous.*

Proof. The proof follows directly from the definitions. □

Corollary 5.5. (Decomposition of the β-Expansion Continuity) *Let α and α' be mutually dual expansions on (X,τ) and let β be expansion on (Y,σ). Then $f: X \to Y$ is β-expansion continuous if and only if f is (α,β)-expansion continuous and (α',β)-expansion continuous.*

Proof. The result follows from the above theorem and the fact that (Id,β)-expansion continuity is equivalent to β-expansion continuity. □

Theorem 5.6. *Let α be a preclosure operator on (X,τ), and let β y β' be expansions on (Y,σ). If $f: X \to Y$ is (α,β)-expansion continuous and (Id,β')-expansion continuous, then f is $(\alpha,\beta\wedge\beta')$-expansion continuous.*

Proof. Let $f: X \to Y$ be (α,β)-expansión continuous and (Id,β')-expansion continuous, and let $V \in \sigma$. Then $f^{-1}(V) \subset \alpha \operatorname{Int} f^{-1}(\beta V)$ and $f^{-1}(V) \subset \operatorname{Int} f^{-1}(\beta' V)$. Thus:

$$\begin{aligned} f^{-1}(V) &\subset \alpha \operatorname{Int} f^{-1}(\beta V) \cap \operatorname{Int} f^{-1}(\beta' V) \\ &\subset \alpha(\operatorname{Int} f^{-1}(\beta V) \cap \operatorname{Int} f^{-1}(\beta' V) = \alpha \operatorname{Int} f^{-1}((\beta\wedge\beta')(V)), \end{aligned}$$

which shows that f is $(\alpha,\beta\wedge\beta')$-expansion continuous. □

Corollary 5.7. (Theorem 1 in Tong [23]) *Let β and β' be mutually dual expansions on (Y,σ). Then $f:(X,\tau)\to(Y,\sigma)$ is continuous if and only if f is β-expansion continuous and β'-expansion continuous.*

Proof. If f is continuous, then it is obvious that f is β-expansion continuous for any expansion β on (Y,σ). Reciprocally, let f be a function which is β-expansion continuous and β'-expansion continuous. Take $\alpha = \mathrm{Id}$ and apply the previous theorem to get the result. □

The following definition is a generalization of the concept of closed β-expansion continuity given by Tong [23].

Definition 5.4. A function $f: X \to Y$ is said to be *closed* (α,β)-*continuous* if $f^{-1}((\beta V)^c)$ is α-closed on X, for all V open on Y.

Example 5.2. For $\alpha = \mathrm{Id}$ and $\beta V = (\partial V)^c$, the (α,β)-continuity is equivalent to weak*-continuity in the sense of Levine [13].

Theorem 5.8. *Let α and β expansions on (X,τ) y (Y,σ) respectively. If $f: X\to Y$ is closed (α,β)-continuous then f is (α,β)-expansion continuous.*

Proof. If $f^{-1}((\beta V)^c)$ is α-closed it follows that

$$f^{-1}(\beta V) \subset \alpha \operatorname{Int} f^{-1}(\beta V).$$

But $f^{-1}(V) \subset f^{-1}(\beta V)$, and thus $f^{-1}(V) \subset \alpha \operatorname{Int} f^{-1}(\beta V)$. □

The reciprocal of the above theorem is not in general true as the next example shows.

Example 5.3. Let $X = Y = R$, the set of the real numbers with the usual topology, and let $f : X \to Y$ be the identity function. Then f is continuous and thus f is (α, β)-expansion continuous for any pair of expansions α and β. In particular f is (Id, β)-expansion continuous. But f is not closed (Id, β)-continuous for the expansion $\beta V = V \cup (\operatorname{Int} \operatorname{Cl} V)^c$. To see this, let $V = \bigcup\limits_{n \geq 1} (\frac{1}{n+1}, \frac{1}{n})$. Then $f^{-1}((\beta V)^c) = (\beta V)^c = \operatorname{Int} \operatorname{Cl} V \diagdown V = \{\frac{1}{n} : n \geq 2\}$, which is not a closed subset of X.

Remark 5.3. The above example shows that Corollary 2.4 on [23] is false.

Next result gives sufficient conditions on the operator β on (Y, σ) such that (α, β)-expansion continuity implies closed (α, β)-continuity.

Theorem 5.9. *Let α and β expansions on (X, τ) and (Y, σ) respectively, such that $\beta V \in \sigma$ for all $V \in \sigma$ and β is idempotent on σ. If $f : X \to Y$ es (α, β)-expansion continuous, then f is closed (α, β)-continuous.*

Proof. Let $V \in \sigma$. Since β is idempotent and $\beta V \in \sigma$, it follows that $f^{-1}(\beta V) \subset \alpha \operatorname{Int} f^{-1}(\beta(\beta V)) = \alpha \operatorname{Int} f^{-1}(\beta V)$, and thus $f^{-1}(\beta V)$ is α-open. It follows that $f^{-1}((\beta V)^c) = (f^{-1}(\beta V))^c$ is α-closed. □

Corollary 5.10. *Let α and β be expansions on (X, τ) and (Y, σ) respectively. Suppose that $\beta V \in \sigma$ for all $V \in \sigma$, and β is idempotent on σ. Then $f : X \to Y$ is (α, β)-expansion continuous if and only if $f^{-1}(\beta V)$ is α-open for all $V \in \sigma$.*

Proof. The proof follows directly from Theorem 5.8 and Theorem 5.9. □

6. Operator-Compact Sets

In Kasahara [1] a concept of weak compacity was introduced using operators as follows. Let β be an operator on (Y, σ). A subset A of Y is said to be compact(β) if for any open cover $\mathcal{G}$ of A there exists a finite subfamily $\{V_1, ..., V_n\}$ of $\mathcal{G}$ such that $A \subset \bigcup_{i=1}^n \beta V_i$. It is obious that compacity coincides with compacity(β) for the operator $\beta = \mathrm{Id}$. Furthermore compact(β) implies compact(β') for all operators β' on (Y, σ) such that $\beta < \beta'$. In this section we consider a strong form of compacity, the α-compacity, and we give sufficient conditions such that the image of any α-compact set by an (α, β)-expansion continuous function is a compact(β) set.

Definition 6.1. Let α be an operator on (X,τ). A subset A of X is said to be α-compact if any cover of A by α-open subsets of X has a finite subfamily that covers A.

Remark 6.1. For any operator α the following implication holds:

$$\alpha\text{-compact} \quad \Rightarrow \quad \text{compact} \quad \Rightarrow \quad \text{compact}(\alpha).$$

Next result shows that ρ-compacity is hereditary on ρ-closed subsets.

Theorem 6.1. *If X is α-compact and D is a α-closed subset of X, then D is α-compact.*

Proof. Let $\mathcal{A} = \{B_\lambda : \lambda \in \Gamma\}$ be a cover of D by α-open subsets of X. Then the family $\mathcal{A}' = \{B_\lambda : \lambda \in \Gamma\} \cup \{X \backslash D\}$ is a cover of X by ρ-open subsets of X. From the α-compacity of X, there exists a finite subfamily $\mathcal{B}$ of $\mathcal{A}'$ that covers X. If $X \backslash D \notin \mathcal{B}$, it follows that $\mathcal{B}$ is a finite subfamily of $\mathcal{A}$ that covers D. If not, by removing $X \backslash D$ from $\mathcal{B}$ we have a finite subfamily of $\mathcal{A}$ that covers D. □

Theorem 6.2. *Let α and β be operators on (X,τ) and (Y,σ) respectively. Suppose that $\beta V \in \sigma$ for all $V \in \sigma$, and that β is idempotente on σ. If $f : X \to Y$ is (α,β)-expansion continuous and $K \subset X$ is α-compact, then $f(K)$ is compact(β).*

Proof. Let $f(K) \subset \bigcup_{\lambda\in\Gamma} V_\lambda$, where $V_\lambda \in \sigma$ for all $\lambda \in \Gamma$. Then $K \subset \bigcup_{\lambda\in\Gamma} f^{-1}(V_\lambda) \subset \bigcup_{\lambda\in\Gamma} f^{-1}(\beta V_\lambda)$. By Theorem 5.9, f is closed (α,β)-continuous, thus $f^{-1}(\beta V_\lambda)$ is α-open for each $\lambda \in \Gamma$. Since K is α-compact, there exists a finite subfamily $\{V_i\}_{i=1}^n$ of $\{V_\lambda\}_{\lambda\in\Gamma}$ such that

$$K \subset \bigcup_{i=1}^{n} f^{-1}(\beta V_{\lambda_i}).$$

Thus $f(K) \subset \bigcup_{i=1}^{n} \beta V_{\lambda_i}$ and therefore $f(K)$ is compact(β). □

Corollary 6.3. (Theorem 2.7 of Caldas [4]) *The semi-continuous image of a semi-compact set is compact.*

Proof. Since semi-continuity is equivalent to closed $(\mathrm{Cl},\mathrm{Id})$-continuity and semi-compacity is equivalent to Cl-compacity, the result follows from the above theorem for $\alpha = \mathrm{Cl}$ and $\beta = \mathrm{Id}$. □

The result in Theorem 6.2 is still true if the condition $\beta V \in \sigma$ for all $V \in \sigma$ is replaced by a condition on f that we call (α, β)-open and which we define bellow.

Definition 6.2. Let α and β operators on (X, τ) and (Y, σ) respectively. The function $f : X \to Y$ is said to be (α, β)-open if $f(\alpha U) \subset \beta f(U)$ for all $U \in \tau$.

Remark 6.2. (a) Any function is $(\mathrm{Id}, \mathrm{Cl})$-open. The condition on f to be open is equivalent to f is $(\mathrm{Id}, \mathrm{Int})$-open. If f is continous then f is $(\mathrm{Cl}, \mathrm{Cl})$-open.

(b) If f is (α, β)-open, then f is (α', β)-open and (α, β')-open for any operator α' on (X, τ) such that $\alpha' < \alpha$ and for any operator β' on (Y, σ) monotone on $\mathcal{P}(Y)$ such that $\beta < \beta'$.

Theorem 6.4. *Let α and β be operators on (X, τ) and (Y, σ) respectively such that β is idempotent on σ, and let $f : X \to Y$ be an (α, β)-open function. If f is (α, β)-expansion continuous and $K \subset X$ is α-compact then $f(K)$ is compact(β).*

Proof. Let K be a α-compact subset of X, and let $f(K) \subset \bigcup\limits_{\lambda \in \Gamma} V_\lambda$, where $V_\lambda \in \sigma$ for all $\lambda \in \Gamma$. Since f is (α, β)-expansion continuous then $K \subset \bigcup\limits_{\lambda \in \Gamma} f^{-1}(V_\lambda) \subset \bigcup\limits_{\lambda \in \Gamma} \alpha \operatorname{Int} f^{-1}(\beta V_\lambda)$. From the α-compacity of K we can choose a finite subfamily $\{V_i,\ i = 1, 2, ...n\}$ of $\{V_\lambda\}_{\lambda \in \Gamma}$ such that $K \subset \bigcup\limits_{i=1}^{n} \alpha \operatorname{Int} f^{-1}(\beta V_i)$. Since f is (α, β)-open and β is idempotent on σ, then $f(K) \subset \bigcup\limits_{i=1}^{n} f(\alpha \operatorname{Int} f^{-1}(\beta V_i)) \subset \bigcup\limits_{i=1}^{n} \beta f(\operatorname{Int} f^{-1}(\beta V_i)) \subset \bigcup\limits_{i=1}^{n} \beta\beta V_i = \bigcup\limits_{i=1}^{n} \beta V_i$, which shows that $f(K)$ is compact(β). □

Corollary 6.5. (Theorem 7 in Chew [6]) *Let $f : X \to Y$ be a weak continuous function. If K is a compact subset of X, then $f(K)$ is closure-compact.*

Proof. Since weak continuity is equivalent to $(\mathrm{Id}, \mathrm{Cl})$-expansion continuity and since closure-compact is equivalent to compact(Cl), the result follows from the above theorem for $\alpha = \mathrm{Id}$ and $\beta = \mathrm{Cl}$. □

Remark 6.3. The condition $\beta V \in \sigma$ for all $V \in \sigma$ is independent of the condition f is (α, β)-open. To see this, take $(X, \tau) = (Y, \sigma) = R$, the set of real numbers with the usual topology, and U an open interval in R. Consider on R the operators $\alpha = \mathrm{Cl}$ and $\beta = \operatorname{Int}\mathrm{Cl}$ and let $f : R \to R$ be the identity map. Then $\beta V = \operatorname{Int}\mathrm{Cl}\, V \in \sigma$ for all $V \in \sigma$, but $f(\mathrm{Cl}\, U) \nsubseteq \operatorname{Int}\mathrm{Cl}\, f(U)$. Thus

f is not (α, β)-open. On the other hand, if $\alpha = \mathrm{Id}$ and $\beta = \mathrm{Cl}$, we have that $f : R \to R$ is (α, β)-open but $\beta U = \mathrm{Cl}\, U \notin \sigma$.

7. An Aplicaction of the Baire Category Theorem

In aplications of the Baire Category Theorem it is relevant the fact that a closed set which is not nowhere dense has nonempty interior. In Baer [2] it was showed that the same holds if closed is replaced by semi-closed, and using this fact it was proved that the Uniform Bounded Theorem holds for a family of semi-continuous real valued functions defined on a Baire space (Theorem 4 in Baer [2]). Corollary 3.12 says that if ρ is a complemented operator which is monotone on $\mathcal{P}(X)$, then any ρ-closed and not nowhere dense set has nonempty interior. This fact allows to extend the Uniform Bounded Theorem to a more general class of weak continuous functions. In particular, Theorem 4 on [2] follows from next theorem for $\rho = \mathrm{Cl}$.

Theorem 7.1. *Let (X, τ) be a Baire space, ρ be a complemented operator which is monotone on $\mathcal{P}(X)$, and let F be a family of (ρ, Id)-expansion continuous real-valued functions. Suppose that for each x on X, there exists a real number $M_x \geq 0$ such that $|f(x)| \leq M_x$ for all f on F. Then, there is a nonempty open subset G of X and a constant M such that $|f(x)| \leq M$ for all x on G and for all f on F.*

Proof. For each positive integer number m and each $f \in F$, let $D_{m,f} = \{x \in X : |f(x)| \leq m\}$ and let $D_{m=} \bigcap_{f \in F} D_{m,f}$. By Theorem 5.9, f is closed (ρ, Id)-continuous for all f on F, and thus $D_{m,f}$ is ρ-closed for all m and all f on F. Furthermore D_m is ρ-closed for all m. Now $X = \bigcup_{m=1}^{\infty} D_m$, and since X is a Baire space there is a set D_k which is not nowhere dense. Since D_k is ρ-closed it follows from Corollary 3.12 that D_k contains a nonempty open set. □

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