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NONLINEAR QUASIVARIATIONAL INEQUALITIES AND NONLINEAR RESOLVENT EQUATIONS

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Abstract: In this paper, we introduce and study nonlinear quasivariational inequalities and nonlinear resolvent equations involving both relaxed Lipschitz and relaxed monotone and strongly monotone mappings. The resolvent operator technique is used to establish the equivalence among the nonlinear quasivariational inequalities, the fixed-point problems and the nonlinear resolvent equations. This equivalence is used to suggest some iterative algorithms for computing the aproximate solutions. We establish a few existence theorems of solutions for the nonlinear quasivariational inequalities and the nonlinear resolvent equations and prove some convergence results of iterative sequences generated by the algorithms. The results obtained in this paper represent significant improvements and refinements of the recently known results in this area.

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1. Introduction

Recently, variational inequality theory has been extended and applied in various directions, see [1], [2], [4]-[8], [11]-[22] and the references therein. It is worth mentioning that one of the most important problems in variational inequality theory is the development of an efficient and implementable iterative algorithm for solving variational inequalities.

In this paper, we introduce and study nonlinear quasivariational inequalities and nonlinear resolvent equations with both relaxed Lipschitz and relaxed monotone and strongly monotone mappings, which are more general and include the previously known classes of variational inequalities, quasivariational inequalities, Wiener-Hopf equations and resolvent equations as special cases. By applying the resolvent operator technique, it is shown the equivalence among the nonlinear quasivariational inequalities, the fixed-point problems and the nonlinear resolvent equations. A few new iterative algorithms for finding approximate solutions which converge strongly to the exact solutions of the nonlinear quasivariational inequalities and the nonlinear resolvent equations are proposed. Our results extend, improve and unify a host of results due to Ahmad-Ansari [1], Bai-Tang-Liu [2], Ding [4], Huang [5], [6], Huang-Bai-Cho-Kang [7], Jou-Yao [8], Noor [11]-[14], Noor-Al-Said [15], Noor-Noor [16], Noor-Noor-Rassias [17], Siddiqi-Ansari [18], Verma [19], [20], Yao [21], Zhang [22] and others.

2. Preliminaries

Let H be a real Hilbert space on which the inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let 2^H denotes the family of all nonempty subsets of H. Given mappings $g, a, b, c, d, h : H \to H$, and nonlinear mappings $N, M : H \times H \to H$. Suppose that $W : H \times H \to 2^H$ is a multivalued mpping such that for every fixed $t \in H$, $W(\cdot, t) : H \to 2^H$ is a maximal monotone mapping and $g(H) \cap \mathrm{dom}(W(\cdot, t)) \neq \emptyset$. For any fixed $f \in H$, we consider the following problem.

Find $u \in H$ such that $gu \in \mathrm{dom}(W(\cdot, hu))$ and

$$f \in gu - N(au, bu) + M(cu, du) + W(gu, hu), \tag{2.1}$$

which is known as the *nonlinear quasivariational inequality.*

Definition 2.1. see ([3]) If $W : H \to 2^H$ is a maximal monotone mapping, the resolvent operator J_W associated with W is defined by

$$J_W(x) = (I + \rho W)^{-1}(x), \quad x \in H,$$

where $\rho > 0$ is a constant.

Related to the nonlinear quasivariational inequality (2.1), we consider the following, called the *nonlinear resolvent equation.*

Find $P, u \in H$ such that

$$gu + \rho^{-1} R_{W(\cdot,hu)}(P) = N(au, bu) - M(cu, du) + f, \tag{2.2}$$

where $\rho > 0$ is a constant, $R_{W(\cdot,hu)} = I - J_{W(\cdot,hu)}$ and $J_{W(\cdot,hu)} = (I + \rho W(\cdot, hu))^{-1}$.

It is clear that the nonlinear quasivariational ineqality (2.2) and the nonlinear resolvent equation (2.8) include many classes of variational inequalities, quasivariational inequalities and resolvent equations, respectively, in [1], [2], [4]-[8], [11]-[22] as special cases.

Lemma 2.1. (see [9]) *Let $W : H \to 2^H$ be a maximal monotone mapping. Then the resolvent operator J_W is single-valued and nonexpansive.*

Definition 2.2. A mapping $g : H \to H$ is said to be *strongly monotone* and *Lipschitz continuous* if there exist constants $r > 0$ and $s > 0$ such that

$$\langle gu - gv, u - v\rangle \geq r\|u - v\|^2 \quad \text{and} \quad \|gu - gv\| \leq s\|u - v\|, \quad u, v \in H,$$

respectively.

Definition 2.3. Let $N : H \times H \to H$ and $a : H \to H$ be mappings.

(1) N is said to be *Lipschitz continuous with respect to the first argument* if there exists a constant $r > 0$ such that

$$\|N(x, u) - N(y, u)\| \leq r\|x - y\|, \quad x, y, u \in H;$$

(2) a is said to be *relaxed Lipschitz with respect to the first argument* of N if there exists a constant $r > 0$ such that

$$\langle N(ax, s) - N(ay, s), u - v\rangle \leq -r\|u - v\|^2, \quad u, v, s, x, y \in H;$$

(3) a is said to be *strongly monotone with respect to the first argument* of N if there exists a constant $r > 0$ such that

$$\langle N(ax, s) - N(ay, s), u - v\rangle \geq r\|u - v\|^2, \quad u, v, s, x, y \in H;$$

(4) a is said to be *relaxed monotone with respect to the first argument* of $N : H \times H \to H$ if there exists a constant $r > 0$ such that

$$\langle N(ax, s) - N(ay, s), u - v\rangle \geq -r\|u - v\|^2, \quad u, v, s, x, y \in H.$$

Similarly, we can define that N is Lipschitz continuous with respect to the second argument, a is relaxed Lipschitz (resp., strongly monotone or relaxed monotone) with respect to the second argument.

3. Fixed Point Methods

In this section, we use the resolvent operator technique to establish the equivalence between the nonlinear quasivariational ineqality (2.1) and the fixed-point problems. This equivalence is used to suggest and analyze a new iterative algorithm for solving the nonlinear quasivariational inequality.

Lemma 3.1. *Let t and ρ be positive parameters. Then the following statements are equivalent:*

(a) *the nonlinear qusivariational inequality* (2.1) *has a solution $u \in H$ with $gu \in \mathrm{dom}(W(\cdot, hu))$;*

(b) *there exists $u \in H$ satisfying*

$$gu = J_{W(\cdot,hu)}((1-\rho)gu + \rho N(au, bu) - \rho M(cu, du) + \rho f);$$

(c) *the mapping $F : H \to H$ defined by*

$$\begin{aligned} Fp = (1-t)p + t(p - gp + J_{W(\cdot,hp)}((1-\rho)gp \\ + \rho N(ap, bp) - \rho M(cp, dp) + \rho f)), \quad p \in H \end{aligned}$$

has a fixed point $u \in H$.

Proof. From the definition of resolvent operator $J_{W(\cdot,hu)}$ and Lemma 2.1, we infer that

$$\begin{aligned} f &\in gu - N(au, bu) + M(cu, du) + W(gu, hu) \\ &\Leftrightarrow (1-\rho)gu + \rho N(au, bu) - \rho M(cu, du) + \rho f \in (I + \rho W(\cdot, hu))gu \\ &\Leftrightarrow gu = J_{W(\cdot,hu)}((1-\rho)gu + \rho N(au, bu) - \rho M(cu, du) + \rho f), \end{aligned}$$

from which it follows that (a) and (b) are equivalent. On the other hand, (c) holds if and only if there exists $u \in H$ such that

$$u = (1-t)u + t(u - gu + J_{W(\cdot,hu)}((1-\rho)gu + \rho N(au, bu) - \rho M(cu, du) + \rho f))),$$

which is equivalent to

$$gu = J_{W(\cdot,hu)}((1-\rho)gu + \rho N(au, bu) - \rho M(cu, du) + \rho f).$$

Hence (b) and (c) are equivalent. This completes the proof. □

Lemma 3.1 is very important from the numerical and approximation point of view. Based on Lemma 3.1, we now suggest the following new general and unified algorithm for the nonlinear quasivariational inequality (2.1).

Algorithm 3.1. Let $g, a, b, c, d, h : H \to H$, $N, M : H \times H \to H$. For given $u_0 \in H$, compute $\{u_n\}_{n\geq 0}$ from the iterative scheme

$$u_{n+1} = (1-t)u_n + t(u_n - gu_n + J_{W(\cdot,hu_n)}((1-\rho)gu_n + \rho N(au_n, bu_n) - \rho M(cu_n, du_n) + \rho f)), \quad (3.1)$$

for all $n \geq 0$, where t and ρ are positive parameters with $t \leq 1$.

We now study those conditions under which the aproximate solutions obtained from Algorithm 3.1 converge strongly to the exact solutions of the nonlinear quasivariational inequality (2.1).

Theorem 3.1. *Let $g, a, b, c, d, h : H \to H$ be Lipschitz continuous with constants i, p, q, r, s, e, respectively, and g be strongly monotone with constant m. Let $N, M : H \times H \to H$ be Lipschitz continuous with respect to the first and second arguments with constants ξ, ζ, η and σ, respectively. Assume that a is relaxed Lipschitz with constant l with respect to the first argument of N, b is relaxed monotone with constant τ with respect to the second argument of N, c is relaxed monotone with constant w with respect to the first argument of M, and d is strongly monotone with constant λ with respect to the second argument of M. Let $W : H \times H \to 2^H$ be a multivalued mapping such that for each $x \in H$, $W(\cdot, x) : H \to 2^H$ is a maximal monotone mapping with $g(H) \cap \mathrm{dom}(W(\cdot, x)) \neq \emptyset$ and*

$$\|J_{W(\cdot,x)}(z) - J_{W(\cdot,y)}(z)\| \leq \mu\|x - y\|, \quad x, y, z \in H, \quad (3.2)$$

where $\mu > 0$ is a constant. Suppose that

$$\begin{aligned} k &= 2\sqrt{1-2m+i^2} + \mu e, \\ j &= \sqrt{1+2\tau+\zeta^2 q^2} + \sqrt{1-2\lambda+\sigma^2 s^2} - \sqrt{1-2m+i^2} \geq 0, \end{aligned} \quad (3.3)$$

$$\begin{aligned} L &= 1 + 2(l-\omega) + (\xi p + \eta r)^2 - j^2, \\ T &= 1 + l - \omega - (1-k)j, \quad S = k^2 - 2k, \end{aligned} \quad (3.4)$$

where $\sigma s \geq \lambda$. If there exists a constant $\rho \in (0, 1]$ satisfying

$$k + \rho j < 1, \quad (3.5)$$

and one of the following conditions:

$$L > 0, \quad |T| > \sqrt{-SL}, \quad |\rho - TL^{-1}| < L^{-1}\sqrt{T^2 + SL}; \tag{3.6}$$
$$L = 0, \quad T > 0, \quad \rho > -(2T)^{-1}S; \tag{3.7}$$
$$L < 0, \quad |\rho - TL^{-1}| > -L^{-1}\sqrt{T^2 + SL}, \tag{3.8}$$

then for each $f \in H$, the nonlinear quasivariational inequality (2.1) *has a solution $u \in H$ with $g(u) \in \mathrm{dom}(W(\cdot, hu))$ and the sequence $\{u_n\}_{n\geq 0}$ generated by Algorithm 3.1 converge strongly to u.*

Proof. Put $E_n = (1-\rho)gu_n + \rho N(au_n, bu_n) - \rho M(cu_n, du_n) + \rho f$. On account of (3.1), (3.2) and Lemma 2.1, we conclude that

$$\begin{aligned}
&\|u_{n+1} - u_n\| \\
&\leq (1-t)\|u_n - u_{n-1}\| + t\|u_n - u_{n-1} - (gu_n - gu_{n-1})\| \\
&\quad + t\|J_{W(\cdot,hu_n)}(E_n) - J_{W(\cdot,hu_n)}(E_{n-1})\| \\
&\quad + t\|J_{W(\cdot,hu_n)}(E_{n-1}) - J_{W(\cdot,hu_{n-1})}(E_{n-1})\| \\
&\leq (1-t)\|u_n - u_{n-1}\| + t\|u_n - u_{n-1} - (gu_n - gu_{n-1})\| \\
&\quad + t\|E_n - E_{n-1}\| + te\|hu_n - hu_{n-1}\| \\
&\leq (1-t)\|u_n - u_{n-1}\| + t(2-\rho)\|u_n - u_{n-1} - (gu_n - gu_{n-1})\| \\
&\quad + t\|(1-\rho)(u_n - u_{n-1}) + \rho(N(au_n, bu_n) - N(au_{n-1}, bu_n) \\
&\quad - M(cu_n, du_n) + M(cu_{n-1}, du_n))\| \\
&\quad + t\rho\|N(au_{n-1}, bu_n) - N(au_{n-1}, bu_{n-1}) - (u_n - u_{n-1})\| \\
&\quad + t\rho\|u_n - u_{n-1} - (M(chu_{n-1}, dhu_n) - M(chu_{n-1}, dhu_{n-1}))\| \\
&\quad + te\|hu_n - hu_{n-1}\|.
\end{aligned} \tag{3.9}$$

By the Lipschitz continuity and strong monotonicity of g, we have

$$\|u_n - u_{n-1} - (gu_n - u_{n-1})\|^2 \leq (1 - 2m + i^2)\|u_n - u_{n-1}\|^2. \tag{3.10}$$

Since a is relaxed Lipschitz with respect to the first argument of N, c is relaxed monotone with respect to the first argument of M, and N and M are Lipschitz continuous with respect to the first arguments, we infer that

$$\begin{aligned}
&\|(1-\rho)(u_n-u_{n-1})+\rho(N(au_n,bu_n)-N(au_{n-1},bu_n)\\
&\quad -M(cu_n,du_n)+M(cu_{n-1},du_n))\|^2\\
&=(1-\rho)^2\|u_n-u_{n-1}\|^2+2(1-\rho)\rho\langle N(au_n,bu_n)\\
&\quad -N(au_{n-1},bu_n),u_n-u_{n-1}\rangle-2\rho(1-\rho)\langle M(cu_n,du_n)\\
&\quad -M(cu_{n-1},du_n),u_n-u_{n-1}\rangle+\rho^2\|N(au_n,bu_n)\\
&\quad -N(au_{n-1},bu_n)-M(cu_n,du_n)+M(cu_{n-1},du_n)\|^2\\
&\le[(1-\rho)^2-2(1-\rho)\rho(l-\omega)+\rho^2(\xi p+\eta r)^2+\|u_n-u_{n-1}\|^2.
\end{aligned}\tag{3.11}$$

Notice that b is relaxed monotone with respect to the second argument of N, d is strongly monotone with respect to the second argument of M, and N and M are Lipschitz continuous with respect to the second arguments. It follows that

$$\begin{aligned}
&\|N(au_{n-1},bu_n)-N(au_{n-1},bu_{n-1})-(u_n-u_{n-1})\|^2\\
&\qquad\qquad\le(1+2\tau+\zeta^2q^2)\|u_n-u_{n-1}\|^2,
\end{aligned}\tag{3.12}$$

$$\begin{aligned}
&\|u_n-u_{n-1}-(M(cu_{n-1},du_n)-M(cu_{n-1},du_{n-1}))\|^2\\
&\qquad\qquad\le(1-2\lambda+\sigma^2s^2)\|u_n-u_{n-1}\|^2.
\end{aligned}\tag{3.13}$$

In view of (3.9)-(3.13), we get that

$$\|u_{n+1}-u_n\|\le(1-(1-\theta)t)\|u_n-u_{n-1}\|,\tag{3.14}$$

where

$$\theta=k+\sqrt{(1-\rho)^2-2(1-\rho)\rho(l-w)+\rho^2(\xi p+\eta r)^2}+\rho j.\tag{3.15}$$

According to (3.3)-(3.5), we obtain that

$$\begin{aligned}
\theta<1\quad\Leftrightarrow\quad&\sqrt{(1-\rho)^2-2(1-\rho)\rho(l-\omega)+\rho^2(\xi p+\eta r)^2}\\
&\qquad<1-k-\rho j\quad\Leftrightarrow\quad L\rho^2-2\rho T<S.
\end{aligned}\tag{3.16}$$

Note that (3.16) and one of (3.6)-(3.8) ensures that $\theta<1$. From (3.14), we know that $\{u_n\}_{n\ge0}$ is a Cauchy sequence in H. Let $u_n\to u\in H$ as $n\to\infty$.

Let $E = (1-\rho)gu + \rho N(au, bu) - \rho M(cu, du) + \rho f$. It is easy to verify that

$$
\begin{aligned}
&\|J_{W(\cdot,hu_n)}(E_n) - J_{W(\cdot,hu)}(E)\| \\
&\le \|J_{W(\cdot,hu_n)}(E_n) - J_{W(\cdot,hu_n)}(E)\| + \|J_{W(\cdot,hu_n)}(E) - J_{W(\cdot,hu)}(E)\| \\
&\le (1-\rho)\|gu_n - gu\| + \rho\|N(au_n, bu_n) - N(au_{n-1}, bu_n) \\
&\quad + N(au_{n-1}, bu_n) - N(au_{n-1}, bu_{n-1})\| \\
&\quad + \rho\|M(cu_n, du_n) - M(cu_{n-1}, du_n) + M(cu_{n-1}, du_n) \\
&\quad - M(cu_{n-1}, du_{n-1})\| + \mu\|hu_n - hu\| \\
&\le (1-\rho)\|gu_n - gu\| + \rho\xi\|au_n - u_{n-1}\| + \rho\zeta\|bu_n - bu_{n-1}\| \\
&\quad + \rho\eta\|cu_n - cu_{n-1}\| + \rho\sigma\|du_n - du_{n-1}\| + \mu\|hu_n - hu\| \\
&\to 0,
\end{aligned}
\tag{3.17}
$$

as $n \to \infty$. It follows from (3.1) and (3.17) that

$$u = (1-t)u + t(u - gu + J_{W(\cdot,hu)}((1-\rho)gu + \rho N(au, bu) - \rho M(cu, du) + \rho f)).$$

The above equation and Lemma 3.1 yield that $u \in H$ with $g(u) \in \mathrm{dom}(W(\cdot, hu))$ is a solution of the nonlinear quasivariational inequality (2.1). This completes the proof. ☐

Theorem 3.2. *Let $k, L, T, S, f, g, a, b, c, d, h, N, M, W$ be as in Theorem 3.1. Let $j = \zeta q + \sigma s - \sqrt{1 - 2m + i^2} \ge 0$. If there exists a constant $\rho \in (0, 1]$ satisfying (3.5) and one of (3.6)-(3.8), then the nonlinear quasivariational inequality (2.1) has a solution $u \in H$ with $g(u) \in \mathrm{dom}(W(\cdot, hu))$ and the sequence $\{u_n\}_{n\ge 0}$ generated by Algorithm 3.1 converges strongly to u.*

Proof. Because b, d, N and M are Lipschitz continuous, it follows that

$$
\begin{aligned}
&\|N(au_{n-1}, bu_n) - N(au_{n-1}, bu_{n-1})\| \le \zeta q, \\
&\|M(cz_{n-1}, dv_n) - M(cz_{n-1}, dv_{n-1})\| \le \sigma s.
\end{aligned}
\tag{3.18}
$$

Using (3.1), (3.2), (3.10), (3.11), (3.18) and Lemma 2.1, we have

$$
\begin{aligned}
&\|u_{n+1} - u_n\| \\
&\le (1-t)\|u_n - u_{n-1}\| + t(2-\rho)\|u_n - u_{n-1} - (gu_n - gu_{n-1})\| \\
&\quad + t\|(1-\rho)(u_n - u_{n-1}) + \rho(N(au_n, bu_n) - N(au_{n-1}, bu_n) \\
&\quad - M(cu_n, du_n) + M(cu_{n-1}, du_n))\| + t\rho(\|N(au_{n-1}, bu_n) \\
&\quad - N(au_{n-1}, bu_{n-1})\| \\
&\quad + \|M(cu_{n-1}, du_n) - M(cu_{n-1}, du_{n-1})\|) + t\mu\|hu_n - hu_{n-1}\| \\
&\le (1 - (1-\theta)t)\|u_n - u_{n-1}\|,
\end{aligned}
$$

where

$$\theta = k + \sqrt{(1-\rho)^2 - 2(1-\rho)\rho(l-w) + \rho^2(\xi p + \eta r)^2} + \rho j.$$

The rest of the argument is the same as in the proof of Theorem 3.1 and is therefore omitted. This completes the proof. □

Remark 3.1. Theorem 3.1 and Theorem 3.2 are extension and generalizations of Theorem 4.1 in [1], Theorems 4.1-4.3 in [2], Theorem 3.1 in [19] and [20], and Theorem 3.6 in [21].

Theorem 3.3. *Let $k, S, f, g, a, b, c, d, h, N, M, W$ be as in Theorem 3.1. Assume that b is strongly monotone with constant τ with respect to the second argument of N, and d is relaxed Lipschitz with constant λ with respect to the second argument of M. Let $\zeta q + \sigma s \geq \tau + \lambda$,*

$$\begin{aligned} &j = \sqrt{1 - 2(\tau+\lambda) + (\zeta q + \sigma s)^2} - \sqrt{1 - 2m + i^2} \geq 0, \\ &L = (\xi p + \eta r)^2 - j^2, \quad T = l - \omega - (1-k)j. \end{aligned} \tag{3.19}$$

If there exists a constant $\rho \in (0,1]$ satisfying (3.5) and one of (3.6)-(3.8), then the nonlinear quasivariational inequality (2.1) has a solution $u \in H$ with $g(u) \in$ $\mathrm{dom}(W(\cdot, hu))$ and the sequence $\{u_n\}_{n\geq 0}$ generated by Algorithm 3.1 converges strongly to u.

Proof. Since b is strongly monotone with respect to the second argument of N, and d is relaxed Lipschitz with respect to the second argument of M, we see that

$$\begin{aligned}
&\|N(au_{n-1},bu_n)-N(au_{n-1},bu_{n-1})-M(cu_{n-1},du_n)\\
&\quad+M(cu_{n-1},du_{n-1})-(u_n-u_{n-1})\|^2\\
&=\|u_n-u_{n-1}\|^2-2\langle N(au_{n-1},bu_n)-N(au_{n-1},bu_{n-1}),u_n-u_{n-1}\rangle\\
&\quad+2\langle M(cu_{n-1},du_n)-M(cu_{n-1},du_{n-1}),u_n-u_{n-1}\rangle\\
&\quad+\|N(au_{n-1},bu_n)-N(au_{n-1},bu_{n-1})-M(cu_{n-1},du_n)\\
&\quad+M(cu_{n-1},du_{n-1})\|^2\\
&\le(1-2(\tau+\lambda)+(\zeta q+\sigma s)^2)\|u_n-u_{n-1}\|^2.
\end{aligned}$$

In view of (3.1), (3.2), (3.10), (3.11), (3.19) and Lemma 2.1, we know that

$$\begin{aligned}
&\|u_{n+1}-u_n\|\\
&\le(1-t)\|u_n-u_{n-1}\|+t(2-\rho)\|u_n-u_{n-1}-(gu_n-gu_{n-1})\|\\
&\quad+t\|u_n-u_{n-1}+\rho(N(au_n,bu_n)-N(au_{n-1},bu_n)-M(cu_n,du_n)\\
&\quad+M(cu_{n-1},du_n))\|+t\rho\|N(au_{n-1},bu_n)-N(au_{n-1},bu_{n-1})\\
&\quad-M(cu_{n-1},du_n)+M(cu_{n-1},du_{n-1})-(u_n-u_{n-1})\|\\
&\quad+te\|hu_n-hu_{n-1}\|\\
&\le(1-(1-\theta)t)\|u_n-u_{n-1}\|,
\end{aligned}$$

where

$$\theta=k+\sqrt{1-2\rho(l-\omega)+\rho^2(\xi p+\eta r)^2}+\rho j.$$

The rest of the argument now follows as in the proof of Theorem 3.1 and is therefore omitted. This completes the proof. □

Theorem 3.4. *Let f,g,a,b,c,d,h,N,M,W be as in Theorem 3.1. Suppose that $k=\sqrt{1-2m+i^2}+\mu e$, $\xi p+\eta r\ge l-\omega$, $\sigma s\ge\lambda$,*

$$\begin{aligned}
&j=\sqrt{1-2(l-\omega)+(\xi p+\eta r)^2}+\sqrt{1+2\tau+(\zeta q)^2}\\
&\qquad+\sqrt{1-2\lambda+(\sigma s)^2},\\
&L=i^2+2m+1-j^2,\quad T=i^2+m-(1-k)j,\quad S=(1-k)^2-i^2.
\end{aligned}$$

If there exists a constant $\rho>0$ satisfying (3.5) and one of (3.6)-(3.8), then the nonlinear quasivariational inequality (2.1) has a solution $u\in H$ with $g(u)\in$ $\mathrm{dom}(W(\cdot,hu))$ and the sequence $\{u_n\}_{n\ge0}$ generated by Algorithm 3.1 converges strongly to u.

Proof. Since g is strongly monotone, it follows that

$$\begin{aligned} &\|(1-\rho)(gu_n - gu_{n-1}) - \rho(u_n - u_{n-1})\| \\ &\leq \sqrt{(1-\rho)^2 i^2 - 2\rho(1-\rho)m + \rho^2}\|u_n - u_{n-1}\|. \end{aligned} \tag{3.20}$$

Note that

$$\begin{aligned} &\|u_n - u_{n-1} + N(au_n, bu_n) - N(au_{n-1}, bu_n) \\ &\quad - M(cu_n, du_n) + M(cu_{n-1}, du_n)\|^2 \\ &\leq \sqrt{1 - 2(l-\omega) + (\xi p + \eta r)^2}\|u_n - u_{n-1}\|. \end{aligned} \tag{3.21}$$

By virtue of (3.1), (3.2), (3.10)-(3.13), (3.20), (3.21) and Lemma 2.1, we infer that

$$\begin{aligned} &\|u_{n+1} - u_n\| \\ &\leq (1-t)\|u_n - u_{n+1}\| + t\|u_n - u_{n-1} - (gu_n - gu_{n-1})\| \\ &\quad + t\|(1-\rho)(gu_n - gu_{n-1}) - \rho(u_n - u_{n-1})\| + t\rho\|u_n - u_{n-1} \\ &\quad + N(au_n, bu_n) \\ &\quad - N(au_{n-1}, bu_n) - M(cu_n, du_n) + M(cu_{n-1}, du_n)\| \\ &\quad + t\rho\|N(au_{n-1}, bu_n) - N(au_{n-1}, bu_{n-1}) - (u_n - u_{n-1})\| \\ &\quad + t\rho\|u_n - u_{n-1} - M(cu_{n-1}, du_n) + M(cu_{n-1}, du_{n-1})\| \\ &\quad + t\mu\|hu_n - hu_{n-1}\| \\ &\leq (1 - (1-\theta)t)\|u_n - u_{n-1}\|, \end{aligned}$$

where

$$\theta = k + \sqrt{(1-\rho)^2 i^2 - 2m\rho(1-\rho) + \rho^2} + \rho j.$$

The rest of the proof follows precisely as in the proof of Theorem 3.1. This completes the proof. □

Remark 3.2. We claim that $S \leq 0$ in Theorem 3.4. In fact, if $i \geq 1$, then $S \leq 0$; if $i \in (0,1)$, by (3.5) and (3.18) we know that

$$\begin{aligned} S \leq 0 \quad &\Leftrightarrow \quad (1-k)^2 \leq i^2 \Leftrightarrow 1 - k \leq i \\ &\Leftrightarrow \quad (1-i)^2 \leq k^2 \Leftrightarrow 2m \leq 2i + 2\mu e\sqrt{1 - 2m + i^2} + (\mu e)^2. \end{aligned}$$

Notice that g is both strongly monotone with constant m and Lipschitz constinuous with constant h. It follows that $h \geq m$. Hence $S \leq 0$.

Theorem 3.5. *Let $k, S, f, g, a, b, c, d, h, N, M, W$ be as in Theorem 3.1. Let $as \geq \lambda$,*

$$\begin{aligned} j &= h + \sqrt{1 + 2\tau + \zeta^2 q^2} + \sqrt{1 - 2\lambda + a^2 s^2}, \\ L &= (\xi p + \eta r)^2 - j^2, \quad T = l - \omega - (1-k)j. \end{aligned}$$

If there exists a constant $\rho > 0$ satisfying (3.5) *and one of* (3.6)-(3.8)*, then the nonlinear quasivariational inequality* (2.1) *has a solution $u \in H$ with $g(u) \in$* $\mathrm{dom}(W(\cdot, hu))$ *and the sequence $\{u_n\}_{n\geq 0}$ generated by Algorithm 3.1 converges strongly to u.*

Proof. Observe that

$$\begin{aligned} &\|u_n - u_{n-1} + \rho(N(au_n, bu_n) - N(au_{n-1}, bu_n) \\ &\quad - M(cu_n, du_n) + M(cu_{n-1}, du_n))\|^2 \\ &= \|u_n - u_{n-1}\|^2 \\ &\quad + 2\rho\langle N(au_n, bu_n) - N(au_{n-1}, bu_n), u_n - u_{n-1}\rangle \\ &\quad - 2\rho\langle M(cu_n, du_n) - M(cu_{n-1}, du_n), u_n - u_{n-1}\rangle \\ &\quad + \rho^2\|N(au_n, bu_n) - N(au_{n-1}, bu_n) - M(cu_n, du_n) \\ &\quad + M(cu_{n-1}, du_n)\|^2 \\ &\leq [1 - 2\rho(l - \omega) + \rho^2(\xi q + \eta r)^2]\|u_n - u_{n-1}\|^2. \end{aligned} \tag{3.22}$$

According to (3.1), (3.2), (3.10), (3.12), (3.13), (3.22) and Lemma 2.1, we conclude that

$$\begin{aligned} &\|u_{n+1} - u_n\| \\ &\leq (1-t)\|u_n - u_{n-1}\| + 2t\|u_n - u_{n-1} - (gu_n - gu_{n-1})\| \\ &\quad + t\rho\|gu_n - gu_{n-1}\| + t\|u_n - u_{n-1} + \rho(N(au_n, bu_n) - N(au_{n-1}, bu_n) \\ &\quad - M(cu_n, du_n) + M(cu_{n-1}, du_n))\| \\ &\quad + t\rho\|N(au_{n-1}, bu_n) - N(au_{n-1}, bu_{n-1}) - (u_n - u_{n-1})\| \\ &\quad + t\rho\|u_n - u_{n-1} - M(cu_{n-1}, du_n) + M(cu_{n-1}, du_{n-1})\| \\ &\quad + t\mu\|hu_n - hu_{n-1}\| \\ &\leq (1 - (1-\theta)t)\|u_n - u_{n-1}\|, \end{aligned}$$

where

$$\theta = k + \sqrt{1 - 2\rho(l - \omega) + \rho^2(\xi p + \eta r)^2} + \rho j.$$

The rest now follows as in Theorem 3.1. This completes the proof. □

Remark 3.3. Theorem 3.3 and Theorem 3.5 improve and unify Theorem 3.1 in [5], Theorem 4.1 in [6]-[8] and Theorem 4.2 in [6].

Employing the methods of proofs of Theorems 3.1-3.5, we have the following theorem:

Theorem 3.6. *Let $k, S, L, T, f, g, a, b, c, d, h, N, M, W$ be as in Theorem 3.3. Let $\zeta q \geq \tau$,*

$$j = \sqrt{1 - 2\tau + (\zeta q)^2} + \sigma s - \sqrt{1 - 2m + i^2} \geq 0.$$

If there exists a constant $\rho \in (0, 1]$ satisfying (3.5) *and one of* (3.6)-(3.8)*, then the nonlinear quasivariational inequality* (2.1) *has a solution $u \in H$ with $g(u) \in \mathrm{dom}(W(\cdot, hu))$ and the sequence $\{u_n\}_{n\geq 0}$ generated by Algorithm 3.1 converges strongly to u.*

Theorem 3.7. *Let $f, g, a, b, c, d, i, N, M, W$ be as in Theorem 3.2, k, L, T and S be as in Theorem 3.4, where $\xi p + \eta r \geq l - w$,*

$$j = \sqrt{1 - 2(l - \omega) + (\xi p + \eta r)^2} + \zeta q + \sigma s.$$

If there exists a constant $\rho > 0$ satisfying (3.5) *and one of* (3.6)-(3.8)*, then the nonlinear quasivariational inequality* (2.1) *has a solution $u \in H$ with $g(u) \in \mathrm{dom}(W(\cdot, hu))$ and the sequence $\{u_n\}_{n\geq 0}$ generated by Algorithm 3.1 converges strongly to u.*

Theorem 3.8. *Let $k, S, f, g, a, b, c, d, h, N, M, W$ be as in Theorem 3.1, B and D be as in Theorem 3.2, L and T be as in Theorem 3.5, where $j = h + \zeta q + \sigma s$. If there exists a constant $\rho > 0$ satisfying* (3.5) *and one of* (3.6)-(3.8)*, then the nonlinear quasivariational inequality* (2.1) *has a solution $u \in H$ with $g(u) \in \mathrm{dom}(W(\cdot, hu))$ and the sequence $\{u_n\}_{n\geq 0}$ generated by Algorithm 3.1 converges strongly to u.*

Remark 3.4. Theorem 3.8 extends Theorem 4.3 in [8], Theorem 4.1 in [11], [12], [16] and [17] and Theorem 3.1 in [18].

4. Resolvent Equations Technique

Now we show that the nonlinear quasivariational inequality (2.1) is equivalent to the nonlinear resolvent equation (2.2).

Lemma 4.1. *Let ρ be a positive parameter. Then* (a) *is equivalent to the following condition:*

(d) *the nonlinear resolvent equation* (2.2) *has a solution* $P, u \in H$ *with* $P = (1-\rho)gu + \rho N(au, bu) - \rho M(cu, du) + \rho f$.

Proof. (a)⇒(d). It follows from Lemma 3.1 and the definition of $R_{W(\cdot,hu)}$ that

$$\begin{aligned} &R_{W(\cdot,hu)}((1-\rho)gu + \rho N(au,bu) - \rho M(cu,du) + \rho f) \\ &= (1-\rho)gu + \rho N(au,bu) - \rho M(cu,du) + \rho f \\ &\quad - J_{W(\cdot,hu)}((1-\rho)gu + \rho N(au,bu) - \rho M(cu,du) + \rho f) \\ &= -\rho gu + \rho N(au,bu) - \rho M(cu,du) + \rho f, \end{aligned}$$

which yields that

$$gu + \rho^{-1} R_{W(\cdot,hu)}(P) = N(au,bu) - M(cu,du) + f,$$

where $P = (1-\rho)gu + \rho N(au,bu) - \rho M(cu,du) + \rho f$. That is, (d) holds.

(d)⇒(a). It follows from (2.2) and (d) that

$$\begin{aligned} gu &= N(au,bu) - M(cu,du) + f - \rho^{-1} R_{W(\cdot,hu)}(P) \\ &= N(au,bu) - M(cu,du) + f - \rho^{-1}(P - J_{W(\cdot,hu)}(P)) \\ &= N(au,bu) - M(cu,du) + f - \rho^{-1}((1-\rho)gu + \rho N(au,bu) \\ &\quad - \rho M(cu,du) + \rho f - J_{W(\cdot,hu)}((1-\rho)gu + \rho N(au,bu) \\ &\quad - \rho M(cu,du) + \rho f)) \\ &= -\rho^{-1} gu + gu \\ &\quad + \rho^{-1} J_{W(\cdot,hu)}((1-\rho)gu + \rho N(au,bu) - \rho M(cu,du) + \rho f), \end{aligned}$$

which means that

$$gu = J_{W(\cdot,hu)}((1-\rho)gu + \rho N(au,bu) - \rho M(cu,du) + \rho f).$$

That is, (a) holds by Lemma 3.1. This completes the proof. □

Lemma 4.1 enable to suggest the following iterative method.

Algorithm 4.1. Let $g, a, b, c, d, h : H \to H$, $N, M : H \times H \to H$. For given $P_0, u_0 \in H$, compute the sequences $\{P_n\}_{n\geq 0}$, $\{u_n\}_{n\geq 0}$ by the iterative schemes

$$gu_n = J_{W(\cdot,hu_n)}(P_n), \tag{4.1}$$

$$P_{n+1} = (1-\rho)gu_n + \rho N(au_n, bu_n) - \rho M(cu_n, du_n) + \rho f, \tag{4.2}$$

for all $n \geq 0$, where $\rho > 0$ is a parameter.

Theorem 4.1. *Let $g, a, b, c, d, h : H \to H$ be Lipschitz continuous with constants i, p, q, r, s, e respectively, g be strongly monotone with constant m and $f \in H$. Let $N, M : H \times H \to H$ be Lipschitz continuous with respect to the first and second arguments with constants ξ, ζ, η, a, respectively. Assume that a is relaxed Lipschitz with constant l with respect to the first argument of N, and c is relaxed monotone with constant ω with respect to the first argument of M. Let $W : H \times H \to 2^H$ be a multivalued mapping such that for each $x \in H$, $W(\cdot, x) : H \to 2^H$ is a maximal monotone mapping with $g(H) \cap \mathrm{dom}(W(\cdot, x)) \neq \emptyset$ and (3.2) holds and*

$$\begin{aligned} &k = 2\sqrt{1-2m+i^2} + \mu e, \quad j = h + \zeta q + \sigma s, \\ &L = (\xi p + \eta r)^2 - j^2, \quad T = l - \omega - (1-k)j, \quad S = k^2 - 2k. \end{aligned} \tag{4.3}$$

If there exists a constant $\rho > 0$ satisfying (3.5) and one of (3.6)-(3.8), then the nonlinear resolvent equation (2.2) has a solution $P, u \in H$ with $P = (1-\rho)gu + \rho N(au, bu) - \rho M(cu, du) + \rho f$ and the sequences $\{P_n\}_{n\geq 0}$, $\{u_n\}_{n\geq 0}$ generated by Algorithm 4.1 converge strongly to P, u respectively.

Proof. In light of (4.2), (4.3), (3.10), (3.18) and (3.22), we infer that

$$\begin{aligned} &\|P_{n+1} - P_n\| \\ &= \|(1-\rho)(gu_n - gu_{n-1}) + \rho(N(au_n, bu_n) - N(au_{n-1}, bu_{n-1}) \\ &- M(cu_n, du_n) + M(cu_{n-1}, du_{n-1}))\| \\ &\leq \|gu_n - gu_{n-1} - (u_n - u_{n-1})\| + \rho\|gu_n - gu_{n-1}\| \\ &\quad + \|u_n - u_{n-1} + \rho(N(au_n, bu_n) - N(au_{n-1}, bu_n) \\ &\quad - M(cu_n, du_n) + M(cu_{n-1}, du_n))\| + \rho\|N(au_{n-1}, bu_n) \\ &\quad - N(au_{n-1}, bu_{n-1})\| + \rho\|M(cu_{n-1}, du_n) - M(cu_{n-1}, du_{n-1})\| \\ &\leq (\sqrt{1-2m+i^2} + \rho i + \sqrt{1 - 2\rho(l-\omega) + \rho^2(\xi p + \eta r)^2} \\ &\quad + \rho(\zeta q + \sigma s))\|u_n - u_{n-1}\|, \end{aligned} \tag{4.4}$$

from (4.1), (4.3), (3.2), (3.10) and Lemma 2.1, we have

$$\begin{aligned} &\|u_n - u_{n-1}\| \\ &\leq \|u_n - u_{n-1} - (gu_n - gu_{n-1})\| + \|J_{W(\cdot, hu_n)}(P_n) - J_{W(\cdot, hu_n)}(P_{n-1})\| \\ &\quad + \|J_{W(\cdot, hu_n)}(P_{n-1}) - J_{W(\cdot, hu_{n-1})}(P_{n-1})\| \\ &\quad \leq \sqrt{1-2m+i^2}\|u_n - u_{n-1}\| + \|P_n - P_{n-1}\| + \mu\|hu_n - hu_{n-1}\| \\ &\quad \leq (\sqrt{1-2m+i^2} + \mu e)\|u_n - u_{n-1}\| + \|P_n - P_{n-1}\|, \end{aligned}$$

from which and (3.5) it follows that

$$\|u_n - u_{n-1}\| \\ \leq (1 - \sqrt{1 - 2m + i^2} - \mu e)^{-1}\|P_n - P_{n-1}\|, \quad n \geq 1. \quad (4.5)$$

Substituting (4.5) into (4.4), we get that

$$\|P_{n+1} - P_n\| \leq \theta\|P_n - P_{n-1}\|, \quad n \geq 1, \quad (4.6)$$

where

$$\theta = (\sqrt{1 - 2m + i^2} + \sqrt{1 - 2\rho(l - \omega) + \rho^2(\xi p + \eta r)^2} \\ + \rho j)(1 - \sqrt{1 - 2m + i^2} - \mu e)^{-1},$$

as $n \to \infty$. It follows from (4.3) and (3.5) that

$$\begin{aligned} \theta < 1 \quad &\Leftrightarrow \quad \sqrt{1 - 2\rho(l - \omega) + \rho^2(\xi p + \eta r)^2} < 1 - k - \rho j \\ &\Leftrightarrow \quad L\rho^2 - 2\rho T < S. \end{aligned}$$

Observe that one of (3.6)-(3.8) yields that $\theta < 1$. Therefore (4.6) ensures that $\{P_n\}_{n\geq 0}$ is a Cauchy sequence in H, that is, $P_n \to P \in H$ as $n \to \infty$. Clearly, (4.5) and (4.6) mean that $\{u_n\}_{n\geq 0}$ is also a Cauchy sequence in H, that is $u_n \to u \in H$ as $n \to \infty$. Notice that g, a, b, c, d, N and M are continuous. Thus (4.2) reduces to

$$P = (1 - \rho)gu + \rho N(au, bu) - \rho M(cu, du) + \rho f. \quad (4.7)$$

Using Lemma 4.1 and (4.7), we know that $P, u \in H$ are a solution of the nonlinear resolvent equation (2.2). This completes the proof. □

Remark 4.1. Theorem 4.1 extends, improves and unifies Theorem 5.2 in [11] and [12], Theorem 3.2 in [13], Theorem 3.1 in [14], Theorem 4.1 in [15] and Theorems 5.2-5.4 in [16].

Theorem 4.2. *Let $f, g, a, b, c, d, h, N, M, W$ be as in Theorem 4.1. Let*

$$\begin{aligned} &k = \sqrt{1 - 2m + i^2} + \mu e, \qquad \xi p + \eta r \geq l - \omega, \\ &j = \sqrt{1 - 2(l - \omega) + (\xi p + \eta r)^2} + \zeta q + \sigma s, \\ &L = h^2 + 2m + 1 - j^2, \quad T = i^2 + m - (1 - k)j, \quad S = (1 - k)^2 - i^2. \end{aligned}$$

If there exists a constant $\rho > 0$ satisfying (3.5) and one of (3.6)-(3.8), then the nonlinear resolvent equation (2.2) has a solution $P, u \in H$ with $P = (1-\rho)gu + \rho N(au, bu) - \rho M(cu, du) + \rho f$ and the sequences $\{P_n\}_{n\geq 0}$, $\{u_n\}_{n\geq 0}$ generated by Algorithm 4.1 converges strongly to P, u, respectively.

Proof. From (4.2), (4.3), (3.11), (3.18), (3.20) and (3.21), we have

$$\begin{aligned}
&\|P_{n+1} - P_n\| \\
&\leq \|(1-\rho)(gu_n - gu_{n-1}) - \rho(u_n - u_{n-1})\| + \rho\|u_n - u_{n-1} \\
&\quad + N(au_n, bu_n) \\
&\quad - N(au_{n-1}, bu_n) - M(cu_n, du_n) + M(cu_{n-1}, du_n)\| \\
&\quad + \rho\|N(au_{n-1}, bu_n) - N(au_{n-1}, bu_{n-1})\| + \rho\|M(cu_{n-1}, du_n) \\
&\quad - M(cu_{n-1}, du_{n-1})\| \leq [\sqrt{(1-\rho)^2 i^2 - 2\rho(1-\rho)m + \rho^2} \\
&\quad + \rho(\sqrt{1 - 2(l-\omega) + (\xi p + \eta r)^2} + \zeta q + \sigma s)]\|u_n - u_{n-1}\|.
\end{aligned}$$

From which and (4.5) it follows that (4.6) holds, where

$$\theta = (\sqrt{(1-\rho)^2 i^2 - 2\rho(1-\rho)m + \rho^2} + \rho j)(1-k)^{-1}.$$

The rest of the argument is now essentialy the same as in the proof of Theorem 4.1 and is therefore omitted. This completes the proof. □

Theorem 4.3. *Let $k, L, T, S, f, g, a, b, c, d, h, N, M, W$ be as in Theorem 4.1, where $\zeta q \geq \tau$ and*

$$j = \sqrt{1 - 2\tau + (\zeta q)^2} + \sigma s - \sqrt{1 - 2m + i^2} \geq 0.$$

Suppose that b is strongly monotone with constant $\tau$with respect to the second argument of N. If there exists a constant $\rho \in (0, 1]$ satisfying (3.5) and one of (3.6)-(3.8), then the nonlinear resolvent equation (2.2) has a solution P, $u \in H$ with $P = (1-\rho)gu + \rho N(au, bu) - \rho M(cu, du) + \rho f$ and the sequences $\{P_n\}_{n\geq 0}$, $\{u_n\}_{n\geq 0}$ generated by Algorithm 4.1 converges strongly to P, u, respectively.

Proof. Using (4.2), (4.3), (3.10), (3.12), (3.18), (3.22) and (4.5), we know that

$$\begin{aligned}
&\|P_{n+1} - P_n\| \\
&\leq (1-\rho)\|gu_n - gu_{n-1} - (u_n - u_{n-1})\| \\
&\quad + \|u_n - u_{n-1} + \rho(N(au_n, bu_n) - N(au_{n-1}, bu_n) \\
&\quad - M(cu_n, du_n) + M(cu_{n-1}, du_n))\| \\
&\quad + \rho\|M(cu_{n-1}, du_n) - M(cu_{n-1}, du_{n-1})\| \\
&\quad + \rho\|N(au_{n-1}, bu_n) - N(au_{n-1}, bu_{n-1}) - (u_n - u_{n-1})\| \\
&\leq [(1-\rho)\sqrt{1-2m+i^2} + \sqrt{1-2\rho(l-\omega)+\rho^2(\xi p+\eta r)^2} \\
&\quad + \rho\sqrt{1-2\tau+(\zeta q)^2} + \rho\sigma s]\|u_n - u_{n-1}\| \\
&\leq \theta\|u_n - u_{n-1}\|,
\end{aligned}$$

where

$$\theta = (\sqrt{1-2m+i^2} + \sqrt{1-2\rho(l-\omega)+\rho^2(\xi p+\eta r)^2} + \rho j) \times (1 - \sqrt{1-2m+i^2} - \mu e)^{-1}.$$

The rest of the proof follows precisely as in the proof of Theorem 4.1. This completes the proof. □

We formulate now a few existence theorems of solutions for the nonlinear resolvent equation (2.2) and the convergence results of iterative sequences generated by Algorithm 4.1. Their proofs are similar to that of Theorem 4.1 and Theorems 3.1-3.5 and left to the reader.

Theorem 4.4. *Let $k, L, T, j, S, f, g, a, b, c, d, h, N, M, W$ be as in Theorem 3.1. If there exists a contant $\rho \in (0, 1]$ satisfying (3.5) and one of (3.6)-(3.8), then the nonlinear resolvent equation (2.2) has a solution $P, u \in H$ with $P = (1-\rho)gu + \rho N(au, bu) - \rho M(cu, du) + \rho f$ and the sequences $\{P_n\}_{n\geq 0}$, $\{u_n\}_{n\geq 0}$ generated by Algorithm 4.1 converge strongly to P, u, respectively.*

Theorem 4.5. *Let $k, L, T, j, S, f, g, a, b, c, d, h, N, M, W$ be as in Theorem 3.2. If there exists a constant $\rho \in (0, 1]$ satisfying (3.5) and one of (3.6)-(3.8), then the nonlinear resolvent equation (2.2) has a solution $P, u \in H$ with $P = (1-\rho)gu + \rho N(au, bu) - \rho M(cu, du) + \rho f$ and the sequences $\{P_n\}_{n\geq 0}$, $\{u_n\}_{n\geq 0}$ generated by Algorithm 4.1 converge strongly to P, u, respectively.*

Theorem 4.6. *Let $k, L, T, j, S, f, g, a, b, c, d, h, , W$ be as in Theorem 3.3. If there exists a constant $\rho > 0$ satisfying (3.5) and one of (3.6)-(3.8), then the nonlinear resolvent equation (2.2) has a solution $P, u \in H$ with $P = (1-\rho)gu +$*

$\rho N(au, bu) - \rho M(cu, du) + \rho f$ and the sequences $\{P_n\}_{n\geq 0}$, $\{u_n\}_{n\geq 0}$ generated by Algorithm 4.1 converge strongly to P, u, respectively.

Theorem 4.7. *Let $k, L, T, j, S, f, g, a, b, c, d, h, , W$ be as in Theorem 3.4. If there exists a constant $\rho > 0$ satisfying (3.5) and one of (3.6)-(3.8), then the nonlinear resolvent equation (2.2) has a solution $P, u \in H$ with $P = (1-\rho)gu + \rho N(au, bu) - \rho M(cu, du) + \rho f$ and the sequences $\{P_n\}_{n\geq 0}$, $\{u_n\}_{n\geq 0}$ generated by Algorithm 4.1 converge strongly to P, u, respectively.*

Theorem 4.8. *Let $k, L, T, j, S, f, g, a, b, c, d, h, , W$ be as in Theorem 3.5. If there exists a constant $\rho \in (0, 1]$ satisfying (3.5) and one of (3.6)-(3.8), then the nonlinear resolvent equation (2.2) has a solution $P, u \in H$ with $P = (1-\rho)gu + \rho N(au, bu) - \rho M(cu, du) + \rho f$ and the sequences $\{P_n\}_{n\geq 0}$, $\{u_n\}_{n\geq 0}$ generated by Algorithm 4.1 converge strongly to P, u, respectively.*

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