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THE BIRKHOFF-KELLOGG THEOREM CONCERNING 0-EPI MAPS

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Abstract: Considering fundamental properties of 0-epi maps in a topological vector space setting, we give a perturbation theorem for 0-epi maps and a generalization of the Birkhoff-Kellogg Theorem concerning 0-epi maps.

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1. Introduction

The concept of a 0-epi map was introduced by M. Furi, M. Martelli, and A. Vignoli [4]. Since then the theory of 0-epi maps has developed by many researchers (see [5], [7], [16]). 0-epi maps have analogous properties to degree theory, such as existence, normalization, homotopy invariance, and boundary dependence. It turns out that under suitable conditions the 0-epi maps are precisely those maps with nonzero degree (see [6], [18]). Moreover, 0-epi maps can be used to study the solvability of boundary value problems and a spectral theory for nonlinear operators (see [3], [4], [15]).

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In this regard, it is natural to investigate the theory of 0-epi maps in a more general situation. The purpose of this paper is twofold. Firstly, we establish some results on 0-epi maps in a topological vector space setting, including a perturbation theorem for 0-epi maps, which were initiated by M. Furi, M. Martelli, and A. Vignoli [4]. Secondly, we give a generalization of the Birkhoff-Kellogg Theorem concerning 0-epi maps due to M. Furi and A. Vignoli [5].

Throughout this paper we assume that E, F are Hausdorff topological vector spaces and Ω is a nonempty bounded open subset of E unless otherwise specified.

The closure and the boundary of Ω in E are denoted by $\overline{\Omega}$ and $\partial\Omega$, respectively.

A continuous map $f : \overline{\Omega} \to F$ is said to be *compact* if its range $f(\overline{\Omega})$ is contained in a compact subset of F. A continuous map $f : \overline{\Omega} \to F$ is said to be *proper* if $f^{-1}(K)$ is compact for every compact subset K of F.

A continuous map $f : \overline{\Omega} \to F$ is said to be 0-*epi* if:

(1) $f(x) \neq 0$ for all $x \in \partial\Omega$; and

(2) for any compact map $h : \overline{\Omega} \to F$ with $h(x) = 0$ for all $x \in \partial\Omega$, the equation $f(x) = h(x)$ has a solution in Ω.

A continuous map $f : \overline{\Omega} \to F$ is called *p-epi* if the map $f - p$ defined by $(f - p)(x) := f(x) - p$ for $x \in \overline{\Omega}$ is 0-epi.

A nonempty subset X of a topological vector space E is said to be *admissible* provided that for every compact subset K of X and for every neighborhood V of the origin in E, there exists a continuous map $h : K \to X$ such that $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a finite dimensional subspace of E. If $X = E$, then the topological vector space E is called *admissible* (see [9], [11]).

It is well-known that every nonempty convex subset of a locally convex topological vector space is admissible. The spaces $L^p(0, 1)$ for $0 < p < 1$ and $S(0, 1)$ are admissible topological vector spaces (see [12], [13]). Here $S(0, 1)$ is the space of measurable real-valued functions defined on $[0, 1]$ with the metric given by

$$d(f, g) = \int_0^1 \frac{|f(t) - g(t)|}{1 + |f(t) - g(t)|}\, dt \qquad \text{for } f, g \in S(0, 1).$$

2. A Perturbation Theorem for 0-epi Maps

We begin with the following fixed point theorem which is a particular form of [8, Satz 4.2.5].

Lemma 1. *Let X be an admissible closed convex set in a Hausdorff topological vector space E. If $f : X \to X$ is a compact map, then f has a fixed point.*

In a topological vector space setting we consider some basic properties of 0-epi maps which originate from M. Furi, M. Martelli, and A. Vignoli [4].

Lemma 2. *For Hausdorff topological vector spaces E, F and a nonempty bounded open subset Ω of E, the class of p-epi maps has the following properties:*

(a) (Existence) If $f : \overline{\Omega} \to F$ is p-epi, then the equation $f(x) = p$ has a solution in Ω.

(b) (Normalization) Let E be an admissible topological vector space. Then the inclusion map $i : \overline{\Omega} \to E$ is p-epi if and only if $p \in \Omega$.

(c) (Homotopy invariance) Suppose $f : \overline{\Omega} \to F$ is a 0-epi map such that one of the following conditions holds:

(1) f is proper.

(2) $\overline{\Omega}$ is normal.

Let $H : \overline{\Omega} \times [0,1] \to F$ be a compact homotopy such that $H(x,0) = 0$ for each $x \in \overline{\Omega}$. If $f(x) + H(x,t) \neq 0$ for all $x \in \partial\Omega$ and for all $t \in [0,1]$, then the map $f(\cdot) + H(\cdot,1) : \overline{\Omega} \to F$ is 0-epi.

(d) (Localization) Let $f : \overline{\Omega} \to F$ be a 0-epi map such that $f^{-1}(0)$ is contained in an open set $\Omega_1 \subset \Omega$. Then the restriction of f to $\overline{\Omega}_1$, $f|_{\overline{\Omega}_1} : \overline{\Omega}_1 \to F$, is 0-epi.

(e) (Boundary dependence) If $f : \overline{\Omega} \to F$ is a 0-epi map and $g : \overline{\Omega} \to F$ is a compact map such that $g(x) = 0$ for all $x \in \partial\Omega$, then $f + g : \overline{\Omega} \to F$ is 0-epi.

Proof. Statements (a) and (e) follow directly from the definition of p-epi maps.

(b) If i is p-epi, the existence property (a) implies that $p \in \Omega$. Conversely, it is sufficient to show that i is 0-epi if $0 \in \Omega$. Let $h : \overline{\Omega} \to E$ be a compact map such that $h(x) = 0$ for all $x \in \partial\Omega$. Extend h to a compact map $\hat{h} : E \to E$ by setting $\hat{h}(x) = 0$ for each $x \notin \Omega$. Since $0 \in \Omega$, the equation $i(x) = h(x)$ has a solution in Ω if and only if the map $\hat{h} : E \to E$ has a fixed point. Since $\hat{h}$ is a compact map defined on an admissible Hausdorff topological vector space E, we conclude by Lemma 1 that $\hat{h}$ has a fixed point. Hence, the inclusion map i is 0-epi.

(c) Let $k : \overline{\Omega} \to F$ be a compact map such that $k(x) = 0$ for all $x \in \partial\Omega$. Consider the set

$$S := \{x \in \overline{\Omega} : f(x) + H(x,t) = k(x) \quad \text{for some } t \in [0,1]\}.$$

(1) Suppose f is proper. Then S is a closed set which is contained in some compact set because H and k are compact and f is proper. Since S is compact in a completely regular topological space E and $S \cap \partial\Omega = \emptyset$, there is a continuous function $\varphi : \overline{\Omega} \to [0,1]$ such that $\varphi(x) = 1$ for every $x \in S$ and $\varphi(x) = 0$ for every $x \in \partial\Omega$. Now observe the equation

$$f(x) = k(x) - H(x, \varphi(x)).$$

A map $h : \overline{\Omega} \to F$ defined by $h(x) := k(x) - H(x, \varphi(x))$ for $x \in \overline{\Omega}$ is compact and vanishes on $\partial\Omega$. Since f is 0-epi, the above equation $f(x) = h(x)$ has a solution x_0 in Ω. Clearly, we have $x_0 \in S$ which implies $\varphi(x_0) = 1$ and hence $f(x_0) + H(x_0, 1) = k(x_0)$. Thus, the map $f(\cdot) + H(\cdot, 1)$ is 0-epi.

(2) Suppose $\overline{\Omega}$ is normal. Note that the set S is closed in $\overline{\Omega}$ because the maps H and k are continuous and $[0,1]$ is compact. Since $\overline{\Omega}$ is normal and $S \cap \partial\Omega = \emptyset$, there exists a continuous function $\varphi : \overline{\Omega} \to [0,1]$ such that $\varphi(x) = 1$ for every $x \in S$ and $\varphi(x) = 0$ for every $x \in \partial\Omega$. An analogous argument to the first part (1) of the proof establishes the rest of this proof.

(d) Let $h : \overline{\Omega}_1 \to F$ be a compact map that vanishes on $\partial\Omega_1$. Extend h to a compact map $\hat{h} : \overline{\Omega} \to F$ by putting $\hat{h}(x) = 0$ if $x \in \overline{\Omega} \setminus \Omega_1$. Since f is 0-epi, the equation $f(x) = \hat{h}(x)$ has a solution x_0 in Ω. The relation $f^{-1}(0) \subset \Omega_1$ implies that $x_0 \in \Omega_1$ and $f(x_0) = h(x_0)$. Therefore, the map $f|_{\overline{\Omega}_1}$ is 0-epi. This completes the proof. □

Remark 3. (1) In the admissible topological vector space $L^p(0,1)$ for $0 < p < 1$, $\Omega = \{f \in L^p(0,1) : \int_0^1 |f(t)|^p \, dt < 1\}$ is a bounded open set which contains the origin 0. The normalization property implies that the inclusion map $i : \overline{\Omega} \to L^p(0,1)$ is 0-epi.

(2) If Ω is a subset of a metrizable topological vector space E, then $\overline{\Omega}$ is normal. For example, every closed subset of the metrizable topological vector space $S(0,1)$ is normal.

Now we present the following two results in a metrizable topological vector space setting which extend [4, Theorem 1.1] and [4, Theorem 1.2].

Lemma 4. *Let E and F be metrizable topological vector spaces. Let $f : \overline{\Omega} \to F$ be a proper 0-epi map. If C is the connected component of $F \setminus f(\partial\Omega)$ which contains the origin 0, then $C \subset f(\Omega)$. This means that $f(\Omega)$ contains a neighborhood of 0.*

Proof. Since f is proper, the set $f(\partial\Omega)$ is closed. Hence C is path connected and open. Notice that in a topological vector space the connected component and the path connected component coincide (see e.g. [2, Theorem 5.19]). For each $p \in C$, let $\varphi : [0,1] \to C$ be a continuous map such that $\varphi(0) = 0$ and $\varphi(1) = p$. Consider a compact homotopy $H : \overline{\Omega} \times [0,1] \to F$ defined by

$$H(x,t) := -\varphi(t) \qquad \text{for } (x,t) \in \overline{\Omega} \times [0,1].$$

From $f(\partial\Omega) \cap C = \emptyset$ and $\varphi([0,1]) \subset C$ it follows that $f(x) + H(x,t) \neq 0$ for all $x \in \partial\Omega$ and for all $t \in [0,1]$. By the homotopy invariance property of Lemma 2, the map $f(\cdot) + H(\cdot,1) : \overline{\Omega} \to F$ is 0-epi, which implies that $f(x_0) = p$ for some $x_0 \in \Omega$ and hence $p \in f(\Omega)$. We conclude that $C \subset f(\Omega)$. This completes the proof. □

Theorem 5. *Let E and F be metrizable topological vector spaces such that F is admissible. Let F be an admissible topological vector space and $f : \overline{\Omega} \to F$ a proper, injective, continuous map. Then $f(\Omega)$ is open if and only if f is p-epi for every $p \in f(\Omega)$.*

Proof. Suppose $p \in f(\Omega)$ and f is p-epi. In view of Lemma 4, $(f-p)(\Omega)$ contains a neighborhood of 0 and hence $f(\Omega)$ is open. Conversely, suppose $f(\Omega)$ is open. Since f is injective and proper, it is obvious that f is invertible on its image and f^{-1} is continuous. Recall that any proper operator f in metric spaces maps closed sets into closed sets (see e.g. [17]). It suffices to show that if $0 \in f(\Omega)$, then f is 0-epi. Consider a compact map $h : \overline{\Omega} \to F$ such that $h(x) = 0$ for all $x \in \partial\Omega$ and define a map $\psi : F \to F$ by

$$\psi(y) := \begin{cases} h(f^{-1}(y)) & \text{for } y \in \overline{f(\Omega)}, \\ 0 & \text{for } y \notin f(\Omega). \end{cases}$$

Note that $\partial f(\Omega) = f(\partial\Omega)$ and $\psi(y) = 0$ for all $y \in \partial f(\Omega)$. Since ψ is compact and F is admissible, Lemma 1 implies that there exists a point $y_0 \in F$ such that $y_0 = \psi(y_0)$. From $0 \in f(\Omega)$ it follows that $y_0 \in f(\Omega)$ and $y_0 = h(f^{-1}(y_0))$. Hence $x_0 := f^{-1}(y_0)$ is a solution of the equation $f(x) = h(x)$. Thus, f is 0-epi. This completes the proof. □

The following result is a generalization of [4, Theorem 1.3].

Theorem 6. *Let E and F be Hausdorff topological vector spaces. Let $f : \overline{\Omega} \to F$ be a 0-epi map and $Q \subset F$ a star-shaped set with respect to the origin such that $Q \cap f(\partial\Omega) = \emptyset$. Assume that one of the following conditions holds:*

(1) f is proper.

(2) $\overline{\Omega}$ is normal.

Then the equation $f(x) = h(x)$ has a solution in Ω for any compact map $h : \overline{\Omega} \to F$ such that $h(\partial\Omega) \subset Q$. In particular, $Q \subset f(\Omega)$.

Proof. Let $h : \overline{\Omega} \to F$ be any compact map such that $h(\partial\Omega) \subset Q$. For every $x \in \partial\Omega$ and for every $t \in [0,1]$, since $th(x) \in Q$ and $Q \cap f(\partial\Omega) = \emptyset$, we have $f(x) \neq th(x)$. By the homotopy invariance property, $f - h$ is 0-epi. The existence property implies that there exists a point $x_0 \in \Omega$ such that $f(x_0) = h(x_0)$. Thus, the first part is proved. Next let $p \in Q$ be an arbitrary element. By taking h as the constant map $h(x) = p$ for all $x \in \overline{\Omega}$, the first part shows that $p \in f(\Omega)$. Consequently, we have $Q \subset f(\Omega)$. This completes the proof. □

We give the following perturbation theorem for 0-epi maps in topological vector spaces. For the case of normed spaces, we refer to [4, Theorem 1.5] or [7, Theorem 4.2.9].

Theorem 7. *Let E and F be Hausdorff topological vector spaces and Ω a nonempty bounded open subset of E. Suppose $f : \overline{\Omega} \to F$ is a proper 0-epi map and $H : \overline{\Omega} \times [-1, 1] \to F$ is a compact map such that $H(x, 0) = 0$ for each $x \in \overline{\Omega}$. Then there exists a real number $\varepsilon > 0$ such that $f(\cdot) - H(\cdot, t)$ is 0-epi for every $t \in (-\varepsilon, \varepsilon)$.*

Proof. According to the homotopy invariance property of Lemma 2, it suffices to show that there exists an $\varepsilon > 0$ such that $f(x) \neq H(x, t)$ for all $x \in \partial\Omega$ and for all $t \in (-\varepsilon, \varepsilon)$. Assume to the contrary that there exists a net $\{(x_\iota, t_\iota)\}$ in $\partial\Omega \times [-1, 1]$ such that $t_\iota \to 0$ and $f(x_\iota) = H(x_\iota, t_\iota)$ for each ι. Since H is compact and f is proper, the net $\{x_\iota\}$ is contained in a compact set. Without loss of generality we may suppose that the net $\{x_\iota\}$ converges to some point x_0 in $\partial\Omega$. From the continuity of f and H it follows that $f(x_0) = H(x_0, 0) = 0$ which contradicts the fact that $0 \notin f(\partial\Omega)$. This completes the proof. □

3. A Generalized Birkhoff-Kellogg Theorem

In this section we give a generalized Birkhoff-Kellogg Theorem concerning 0-epi maps, based on the fundamental properties of 0-epi maps stated in Section 2.

For our aim we need a modification of [5, Lemma 1] (see also [7, Lemma 4.2.11]). A set K in a topological vector space F is said to be *radially bounded* if the intersection of K with each straight line through the origin is bounded (see [14]).

Lemma 8. *Let E and F be Hausdorff topological vector spaces and Ω be a nonempty bounded open subset of E. Let $f : \overline{\Omega} \to F$ be a 0-epi map and $p \in F$ a nonzero vector such that f is proper or $\overline{\Omega}$ is normal. If $f(\overline{\Omega})$ is radially bounded, then there exist a real number $\mu_0 > 0$ and a vector $x_0 \in \partial\Omega$ such that $f(x_0) = \mu_0 p$.*

Proof. Since $f(\overline{\Omega})$ is radially bounded, there exists a real number $\mu > 0$ such that $f(x) - \mu p \neq 0$ for all $x \in \overline{\Omega}$. By the existence property of Lemma 2, the map $f - \mu p$ is not 0-epi. Consider a compact homotopy $H : \overline{\Omega} \times [0,1] \to F$ defined by

$$H(x,t) := -t\mu p \qquad \text{for } (x,t) \in \overline{\Omega} \times [0,1].$$

Since f is 0-epi and $f(\cdot)+H(\cdot,1)$ is not 0-epi, the homotopy invariance property of Lemma 2 implies that there are a vector $x_0 \in \partial\Omega$ and a real number $t_0 \in [0,1]$ such that $f(x_0) = t_0\mu p$. From $0 \notin f(\partial\Omega)$ it follows that $t_0 \neq 0$. Setting $\mu_0 = t_0\mu$, the proof is completed. □

Now we can prove a generalization of the Birkhoff-Kellogg Theorem concerning 0-epi maps which was initiated by M. Furi and A. Vignoli (see [5, Theorem 1] or [7, Theorem 4.2.12]).

Theorem 9. *Let Ω be a nonempty bounded open subset of a Hausdorff topological vector space E and F an infinite-dimensional normed space. Let $f : \overline{\Omega} \to F$ be a 0-epi map such that one of the following properties holds:*

(1) f is proper.

(2) $\overline{\Omega}$ is normal.

Suppose $g : \overline{\Omega} \to F$ is a compact map that satisfies the following boundary condition:

$$\|g(x)\| \geq k > 0 \quad \text{for all } x \in \partial\Omega \text{ and for some } k > 0.$$

If $f(\Omega)$ is bounded, then there exist a real number $\lambda_0 > 0$ and a vector $x_0 \in \partial\Omega$ such that $f(x_0) = \lambda_0 g(x_0)$.

Proof. Let $r : F \setminus \{0\} \to S(0,1)$ be the radial projection given by

$$r(y) := y/\|y\| \qquad \text{for } y \in F \setminus \{0\},$$

where $S(0,1)$ is the boundary of the open unit ball $B(0,1)$ of F. Since g is compact, r is continuous and $\dim F = \infty$, we see that the set $r(\overline{g(\Omega)})$ is compact and $S(0,1)$ is not compact (see [10, Theorem 2.5-5]). Hence we can choose a vector $p \in F$ such that

$$\|p\| = 1 \quad \text{and} \quad -p \notin r(\overline{g(\Omega)}).$$

Now we will claim that there is a sufficiently large real number $\lambda > 0$ such that the map $f - \lambda g$ is not 0-epi. Observe that $f - \lambda g$ is proper whenever f is proper. Indeed, if it is not the case, by Lemma 8, for every $n \in \mathbb{N}$, there exist an $x_n \in \partial\Omega$ and a real number $t_n > 0$ such that $f(x_n) - ng(x_n) = t_n p$, that is, $\frac{f(x_n)}{n} - g(x_n) = (\frac{t_n}{n})p$. Since $\{f(x_n)\}$ is bounded, we have $\frac{f(x_n)}{n} \to 0$ as $n \to \infty$. Without loss of generality we may suppose that the sequence $\{g(x_n)\}$ converges to some element $z \in \overline{g(\partial\Omega)}$ because g is compact. From the boundary condition of g it follows that there is a real number $\beta > 0$ such that $-\beta p = z$, which implies that $-p = \frac{z}{\|z\|} = r(z) \in r(\overline{g(\partial\Omega)})$. This contradicts our choice of p.

Since f is 0-epi and $f - \lambda g$ is not 0-epi, as in the proof of Lemma 8, the homotopy invariance property of Lemma 2 implies that there exist a real number $\lambda_0 > 0$ and a vector $x_0 \in \partial\Omega$ such that $f(x_0) = \lambda_0 g(x_0)$. This completes the proof. □

The following result is just the well-known Birkhoff-Kellogg Theorem when Ω is the open unit ball of a normed space (see [1]).

Corollary 10. *Let Ω be a bounded open subset of an infinite-dimensional normed space E with $0 \in \Omega$. If $g : \overline{\Omega} \to E$ is a compact map such that $\|g(x)\| \geq k > 0$ for all $x \in \partial\Omega$ and for some $k > 0$, then there exist a real number $\mu_0 > 0$ and a vector $x_0 \in \partial\Omega$ such that $g(x_0) = \mu_0 x_0$.*

Proof. Note that E is a metrizable admissible topological vector space, $\overline{\Omega}$ is normal, and $0 \in \Omega$. The normalization property of Lemma 2 implies that the inclusion map $i : \overline{\Omega} \to E$ is 0-epi. Apply Theorem 9 with $f = i$. This completes the proof. □

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References

[1] G.D. Birkhoff, O.D. Kellogg, Invariant points in function space, *Trans. Amer. Math. Soc.*, **23** (1922), 96-115.

[2] F.H. Croom, *Principles of Topology*, Saunders College Publishing, Orlando (1989).

[3] W. Feng, A new spectral theory for nonlinear operators and its applications, *Abstr. Appl. Anal.*, **2** (1997), 163-183.

[4] M. Furi, M. Martelli, A. Vignoli, On the solvability of nonlinear operator equations in normed spaces, *Ann. Mat. Pura Appl.*, **124** (1980), 321-343.

[5] M. Furi, A. Vignoli, Unbounded nontrivial branches of eigenfunctions for nonlinear equations, *Nonlinear Anal.*, **6** (1982), 1267-1270.

[6] E. Giorgieri, M. Väth, A characterization of 0-epi maps with a degree, *J. Funct. Anal.*, **187** (2001), 183-199.

[7] D.H. Hyers, G. Isac, T.M. Rassias, *Topics in Nonlinear Analysis and Applications*, World Scientific, Singapore (1997).

[8] T. Jerofsky, *Zur Fixpunkttheorie mengenwertiger Abbildungen*, Dissertation A, TU Dresden (1983).

[9] V. Klee, Leray-Schauder theory without local convexity, *Math. Ann.*, **141** (1960), 286-296.

[10] E. Kreyszig, *Introductory Functional Analysis with Applications*, Wiley, New York (1989).

[11] M. Landsberg, T. Riedrich, Über positive Eigenwerte kompakter Abbildungen in topologischen Vektorräumen, *Math. Ann.*, **163** (1966), 50-61.

[12] T. Riedrich, Die Räume $L^p(0,1)(0 < p < 1)$ sind zulässig, *Wiss. Z. Techn. Univ. Dresden*, **12** (1963), 1149-1152.

[13] T. Riedrich, Der Raum $S(0,1)$ ist zulässig, *Wiss. Z. Techn. Univ. Dresden*, **13** (1964), 1-6.

[14] T. Riedrich, Das Birkhoff-Kellogg Theorem für lokal radial beschränkte Räume, *Math. Ann.*, **166** (1966), 264-276.

[15] P. Santucci, M. Väth, Grasping the phantom – a new approach to nonlinear spectral theory, *Ann. Mat. Pura Appl.*, **180** (2001), 255-284.

[16] E.U. Tarafdar, H.B. Thompson, On the solvability of nonlinear noncompact operator equations, *J. Austral. Math. Soc.*, Ser. A, **43** (1987), 103-126.

[17] M. Väth, Fixed point theorems and fixed point index for countably condensing maps, *Topol. Methods Nonlinear Anal.*, **13** (1999), 341-363.

[18] M. Väth, On the connection of degree theory and 0-epi map, *J. Math. Anal. Appl.*, **257** (2001), 223-237.