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GENERAL TWO-STEP MODELS FOR PROJECTION METHODS AND THEIR APPLICATIONS

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Abstract: This paper concerns the general two-step model for projection methods and its applications to the approximation solvability of a system of nonlinear variational problems in general Hilbert spaces. First, a general model for two-step projection methods is introduced and second, it is applied to the approximation solvability of a system of nonlinear variational inequalities in a Hilbert space setting. Let H be a real Hilbert space and K be a nonempty closed convex subset of H. For arbitrarily chosen initial points x^0, $y^0 \in K$, update sequences $\{x^k\}$ and $\{y^k\}$ iteratively such that

$$x^{k+1} = (1-a^k)x^k + a^k P_K[y^k - \rho T(y^k)]\,, \quad \text{for} \quad \rho > 0\,,$$
$$y^k = (1-b^k)x^k + b^k P_K[x^k - \eta T(x^k)]\,, \quad \text{for} \quad \eta > 0,$$

where $T: K \longrightarrow H$ is a nonlinear mapping on K, P_K is the projection of H onto K, and sequences $\{a^k\}$and $\{b^k\}$ satisfy $0 \le a^k,\, b^k \le 1$.

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1. Introduction

Recently, the author [12] studied, based on the two-step model for projection methods, the approximation solvability of a system of nonlinear variational inequality problems in a Hilbert space setting. The two-step model for projec-

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tion/ projection type methods contains several known, as well as new projection methods as special cases, while some have been applied vigorously to problems arising, especially from complementarity problems, convex quadratic programming, and variational problems. Later, Nie et al [7] investigated, using the two-step model, the approximation solvability of a system of nonlinear variational inequalities involving a combination of strongly monotone and pseudo-contractive mappings. The two-step nonlinear variational inequality problems are relatively newer than the usual variational inequality problems, which have been studied extensively.

Here in this paper, we intend first to introduce a general two-step model for projection methods, which reduces to the two-step model applied in [12], and then apply its convergence analysis to the approximation solvability of a system of two strongly monotone nonlinear variational inequalities in a Hilbert space setting. The obtained results complement results of Verma [12], Nie et al [7] and others. For a detailed account on general variational inequality problems and related iterative procedures, see [1-19].

Let H be a real Hilbert space with the inner product $\langle x, y\rangle$ and norm $||x||$ for $x, y \in H$. Let $T : K \longrightarrow H$ be any mapping on K and K be a closed convex subset of H. We consider a system of two nonlinear variational inequality (abbreviated as SNVI) problems as follows: determine elements x^*, $y^* \in K$ such that

$$\langle \rho T(y^*) + x^* - y^*, x - x^*\rangle \geq 0\,, \quad \forall x \in K\,, \text{ and for } \rho > 0\,, \tag{1}$$

$$\langle \eta T(x^*) + y^* - x^*, x - y^*\rangle \geq 0\,, \quad \forall x \in K\,, \text{ and for } \eta > 0. \tag{2}$$

The SNVI (1)-(2) problem is equivalent to the following projection formulas

$$x^* = P_K[y^* - \rho T(y^*)] \quad \text{for} \quad \rho > 0\,, \tag{3}$$

$$y^* = P_K[x^* - \eta T(x^*)] \quad \text{for} \quad \eta > 0\,, \tag{4}$$

where P_K is the projection of H onto K.

We note that for $\eta = 0$, the SNVI (1)-(2) problem reduces to the NVI problem: determine an element $x^* \in K$ such that

$$\langle T(x^*), x - x^*\rangle \geq 0\,, \quad \forall x \in K. \tag{5}$$

Let K be a closed convex cone of H. The SNVI (1)-(2) problem is equivalent to a system of nonlinear complementarities (abbreviated as SNC): find the elements $x^*, y^* \in K$ such that $T(x^*), T(y^*) \in K^*$ and,

$$\langle \rho T(y^*) + x^* - y^*, x^*\rangle = 0\,, \quad \text{for} \quad \rho > 0\,, \tag{6}$$

$$\langle \eta T(x^*) + y^* - x^*, y^* \rangle = 0\,, \quad \text{for} \quad \eta > 0\,, \tag{7}$$

where K^* is a polar cone to K defined by

$$K^* = \{f \in H : \langle f, x \rangle \geq 0\,,\ \forall x \in K\}.$$

Now we recall the following auxiliary result, most commonly used in the context of approximation solvability of nonlinear variational inequality problems based on iterative procedures.

Lemma 1.1. (see [4]) *For an element $z \in H$, we have*

$$x \in K \text{ and } \langle x - z, y - x \rangle \geq 0\,,\ \forall y \in K \text{ if and only if } x = P_K(z).$$

Next, we recall some basic notions, some with examples, crucial to our problem on hand.

A mapping $T : H \longrightarrow H$ is called monotone if for each $x, y \in H$, we have

$$\langle T(x) - T(y), x - y \rangle \geq 0.$$

A mapping $T : H \longrightarrow H$ is called r-strongly monotone if for each $x, y \in H$, we have

$$\langle T(x) - T(y), x - y \rangle \geq r||x - y||^2 \quad \text{for a constant} \quad r > 0.$$

This implies that

$$||T(x) - T(y)|| \geq r||x - y||,$$

that is, T is r-expansive, and when $r = 1$, it is expansive.

The mapping T is called s-Lipschitz continuous (or Lipschitzian) if there exists a constant $s \geq 0$ such that

$$||T(x) - T(y)|| \leq s||x - y||\,, \quad \forall x, y \in H.$$

T is called μ-cocoercive (see [1, 4]) if for each $x, y \in H$, we have

$$\langle T(x) - T(y), x - y \rangle \geq \mu||T(x) - T(y)||^2 \quad \text{for a constant} \quad \mu > 0.$$

Clearly, every μ-cocoercive mapping T is $(1/\mu)$-Lipschitz continuous.

We can easily see that the following implications on monotonicity, strong monotonicity and expansiveness hold:

$$\begin{array}{ccc} \text{strong monotonicity} & \Rightarrow & \text{monotonicity} \\ \Downarrow & & \\ \text{expansiveness} & & \end{array}$$

Example 1.1. Consider a function $f : R^n \longrightarrow R^n$ defined by

$$f(x) = cI(x) + v,$$

where $c > 0$, $x, v \in R^n$ with v fixed, and I is the $n \times n$ identity matrix. Then f is r-strongly monotone for $0 < r \leq c$.

A mapping $T : H \longrightarrow H$ is called a generalized pseudocontraction if there exists a positive constant α such that

$$\langle T(x) - T(y), x - y\rangle \leq \alpha \|x - y\|^2 , \ \forall\ x, y \in H.$$

A mapping $T : H \longrightarrow H$ is said to be a generalized pseudocontraction (see [8]), if, for all $x, y \in H$, there exists positive constant r such that

$$\|T(x) - T(y)\|^2 \leq r^2\|x - y\|^2 + \|T(x) - T(y) - r(x - y)\|^2 ,$$

or, equivalently,

$$\langle T(x) - T(y), x - y\rangle \leq r\|x - y\|^2.$$

Example 1.2. (see [8]) Let $T : H \longrightarrow H$ be a generalized pseudocontraction on H. Then $I - T$ is $(1 - r)$-strongly monotone for $0 < r \leq 1$.

T is called strongly r-pseudomonotone if there exists a constant $r > 0$ such that

$$\langle T(y), x - y\rangle \geq 0 \Rightarrow \langle T(x), x - y\rangle \geq r\|x - y\|^2 , \quad \forall x, y \in H.$$

2. Projection Methods

This section deals with an introduction of general two-step models for projection methods and its special forms that can be applied to convergence analyses for projection methods in the context of the approximation solvability of the SNVI (1)-(2) problem.

Algorithm 2.1. For arbitrarily chosen initial points $x^0, y^0 \in K$, compute the sequences $\{x^k\}$ and $\{y^k\}$ such that

$$x^{k+1} = (1 - a^k)x^k + a^k P_K[y^k - \rho T(y^k)],$$

$$y^k = (1 - b^k)x^k + b^k P_K[x^k - \eta T(x^k)],$$

where P_K is the projection of H onto K, $\rho, \eta > 0$ are constants, and

$$0 \leq a^k, b^k \leq 1 \ \text{ for } \ k \geq 0.$$

Algorithm 2.2. For an arbitrarily chosen initial point $x^0 \in K$, compute the sequence $\{x^k\}$ such that

$$x^{k+1} = (1 - a^k)x^k + a^k P_K[x^k - \rho T(x^k)],$$

where

$$0 \leq a^k \leq 1.$$

Algorithm 2.3. For arbitrarily chosen initial points $x^0, y^0 \in K$, sequences $\{x^k\}$ and $\{y^k\}$ are generated by

$$x^{k+1} = (1 - a^k)x^k + a^k P_K[y^k - \rho T(y^k)]\,,$$

$$y^k = P_K[x^k - \eta T(x^k)]\,,$$

where $0 \leq a^k \leq 1$ for $k \geq 0$.

3. 3. Applications

We now present, based on Algorithm 2.1, the approximation-solvability of the SNVI (1)-(2) problem involving strongly monotone and μ-Lipschitz continuous mappings in a Hilbert space setting.

Theorem 3.1. *Let H be a real Hilbert space and K a nonempty closed convex subset of H. Let $T : K \longrightarrow H$ be strongly r-monotone and μ-Lipschitz continuous. Suppose that the following assumptions hold:*

(i) Elements $x^*, y^* \in K$ form a solution to the SNVI (1)-(2) problem.

(ii) Sequences $\{x^k\}$ and $\{y^k\}$ are generated by Algorithm 2.1.

(iii) Sequences $\{a^k\}$ and $\{b^k\}$ satisfy $0 \leq a^k, b^k \leq 1$ and $\Sigma_{k=0}^{\infty} a^k = \infty$.

Then sequences $\{x^k\}$ and $\{y^k\}$, respectively, converge to x^* and y^* for

$$0 < \rho < 2r/\mu^2 \quad \text{and} \quad 0 < \eta < 2r/\mu^2.$$

Proof. Since x^* and y^* form a solution to the SNVI (1)-(2) problem, it follows that

$$x^* = P_K[y^* - \rho T(y^*)],$$

$$y^* = P_K[x^* - \eta T(x^*)].$$

Applying Algorithm 2.1, we have

$$\begin{aligned}\|x^{k+1}-x^*\| &= \|(1-a^k)x^k + a^k P_K[y^k-\rho T(y^k)] \\ &\quad - (1-a^k)x^* - a^k P_K[y^*-\rho T(y^*)]\| \\ &\le (1-a^k)\|x^k-x^*\| + a^k\|P_K[y^k-\rho T(y^k)] - P_K[y^*-\rho T(y^*)]\| \\ &\le (1-a^k)\|x^k-x^*\| + a^k\|y^k-y^*-\rho[T(y^k)-T(y^*)]\| . \quad (8)\end{aligned}$$

Since T is strongly r-monotone and μ-Lipschitz continuous, we have

$$\begin{aligned}&\|y^k-y^*-\rho[T(y^k)-T(y^*)]\|^2 \\ &= \|y^k-y^*\|^2 - 2\rho\langle T(y^k)-T(y^*), y^k-y^*\rangle + \rho^2\|T(y^k)-T(y^*)\|^2 \\ &\le \|y^k-y^*\|^2 - 2\rho r\|y^k-y^*\|^2 + (\rho^2\mu^2)\|y^k-y^*\|^2 \\ &\le \|y^k-y^*\|^2 + (\rho\mu)^2\|y^k-y^*\|^2 - 2\rho r\|y^k-y^*\|^2 \\ &= [1-2\rho r + (\rho\mu)^2]\|y^k-y^*\|^2.\end{aligned}$$

In light of (8), we have

$$\|x^{k+1}-x^*\| \le (1-a^k)\|x^k-x^*\| + a^k\theta\|y^k-y^*\| , \quad (9)$$

where $\theta = [1-2\rho r+(\rho\mu)^2]^{1/2}$.

Similarly, we obtain

$$\begin{aligned}\|y^k-y^*\| &= \|(1-b^k)(x^k-x^*) + b^k P_K[x^k-\eta T(x^k)] - P_K[x^*-\eta T(x^*)]\| \\ &\le (1-b^k)\|x^k-x^*\| + b^k\|x^k-x^*-\eta[T(x^k)-T(x^*)]\| \\ &\le (1-b^k)\|x^k-x^*\| + b^k[1-2\eta r+(\eta\mu)^2]^{1/2}\|x^k-x^*\| \\ &\le (1-b^k)\|x^k-x^*\| + b^k\|x^k-x^*\| , \quad (10)\end{aligned}$$

where $\sigma = [1-2\eta r+(\eta\mu)^2]^{1/2} < 1$.

It follows from (9) and (10) that

$$\|x^{k+1}-x^*\| \le (1-a^k)\|x^k-x^*\| + a^k\theta\|x^k-x^*\| \quad (11)$$

$$= [1-(1-\theta)a^k]\|x^k-x^*\|$$

$$\le \Pi_{j=0}^k[1-(1-\theta)a^j]\|x^0-x^*\|, \quad (12)$$

where $\theta = [1-2\rho r+(\rho\mu)^2]^{1/2} < 1$.

Since $\theta < 1$ and $\Sigma_{k=0}^{\infty}a^k$ is divergent, it implies in light of [16] that

$$\lim_{k\longrightarrow\infty}\Pi_{j=0}^k[1-(1-\theta)a^j] = 0.$$

Hence, the sequence $\{x^k\}$ converges to x^* by (12), and the sequence $\{y^k\}$ converges to y^* by (10) for

$$0 < \rho < 2r/\mu^2\,,$$
$$0 < \eta < 2r/\mu^2\,.$$

This concludes the proof. □

Theorem 3.2. (see [12]) *Let H be a real Hilbert space and K a nonempty closed convex subset of H. Let $T : K \longrightarrow H$ be strongly r-monotone and μ-Lipschitz continuous. If x^*, $y^* \in K$ form a solution to SNVI (1)-(2), if Algorithm 2.3 generates sequences $\{x^k\}$ and $\{y^k\}$, and if*

$$0 \leq a^k \leq 1 \;\; \text{and} \;\; \Sigma_{k=0}^{\infty} a^k = \infty\,,$$

then sequences $\{x^k\}$ and $\{y^k\}$, respectively, converge to x^* and y^* for

$$0 < \rho < 2r/\mu^2\,, \quad 0 < \eta < 2r/\mu^2.$$

Theorem 3.3. *Let H be a real Hilbert space and K be a nonempty closed convex subset of H. Let $T : K \longrightarrow H$ be strongly r-monotone and μ-Lipschitz continuous. Suppose that $x^* \in K$ is a solution to the NVI (5) problem, and the sequence $\{x^k\}$ is generated by Algorithm 2.2. Then sequence $\{x^k\}$ converges to x^* for*

$$0 < \rho < 2r/\mu^2.$$

Theorem 3.4. *Let H be a real Hilbert space and K its nonempty closed convex subset. Let the following assumptions hold:*

(i) $T : K \longrightarrow H$ is a strongly r-pseudomonotone and μ-Lipschitz continuous mapping.

(ii) Elements x^*, $y^* \in K$ form a solution to the SNVI (1)-(2) problem.

(iii) Sequences $\{x^k\}$ and $\{y^k\}$ are generated by Algorithm 2.1.

(iv) $\langle T(y^*), y^k - y^*\rangle \geq 0$.

(v) $\langle T(x^*), x^k - x^*\rangle \geq 0$.

(vi) Sequences $\{a^k\}$ and $\{b^k\}$ satisfy

$$0 \leq a^k, b^k \leq 1 \;\; \text{and} \;\; \Sigma_{k=0}^{\infty} a^k = \infty\,.$$

Then sequences $\{x^k\}$ and $\{y^k\}$, respectively, converge to x^* and y^* for

$$0 < \rho < 2r/\mu^2 \;\; \text{and} \;\; 0 < \eta < 2r/\mu^2.$$

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