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# CHARACTERIZING THE DOMAINS OF ATTRACTION OF STABLE STATIONARY SOLUTIONS OF SEMILINEAR PARABOLIC EQUATIONS

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**Abstract:** We discuss the initial-Neumann-boundary value problem $u_t = D\Delta u + f(u)$, with $u_\nu|_{\partial\Omega} = 0$ and $u(x,0) = \phi(x)$, in a bounded domain $\Omega \in R^m$, with $m \geq 1$, D> 0. Under mild assumptions on the nonlinear term $f$, we prove that if $0 \leq \phi < \lambda v$ with $\lambda < 1$, the global solutions of this problem decay exponentially, while if $\phi > \lambda v$ with $\lambda > 1$, then the solutions must grow at least exponentially, and may blow up in finite time, where the $v$ are positive solutions of $D\Delta v + f(v) = 0$, with $v_\nu|_{\partial\Omega} = 0$. The method used in this paper provides an easily verifiable sufficient condition for the initial function to belong to the domain of attraction of a stable constant stationary solution of the original problem.

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## 1. Introduction

In this paper we study the initial-Neumann-condition reaction-diffusion problem

$$u_t = D\Delta u + f(u), \qquad x \in \Omega, \quad t > 0, \tag{1}$$

$$u_\nu(x,t) = 0, \qquad x \in \partial\Omega, \quad t > 0, \tag{2}$$

$$u(x,0) = \phi(x), \qquad x \in \Omega, \quad \phi \in C^\alpha(\bar{\Omega}), \quad 0 < \alpha < 1, \tag{3}$$

where $D$ is a positive diffusion coefficient, $f$ is continuously differentiable with $f(0) = 0$, $\Omega$ is a bounded domain in $R^m$ with smooth boundary, $\nu$ is the unit outward normal to the boundary $\partial\Omega$, and $m \geq 1$. Any root of $f(u) = 0$ is a solution of (1) – (2), so the original problem (1) – (3) may have multiple stationary solutions. The issue then, for problem (1) – (3), is given the initial shape $\phi(x)$, what will happen immediately as $t$ increases.

Observe that if $D$ is zero, (1) – (2) becomes a first order differential equation with multiple constant solutions $u_j$ corresponding to roots of $f(u)$. These solutions are stable when $f'(u_j) < 0$, unstable when $f'(u_j) > 0$, and the stability properties of the steady states with $f'(u_j) = 0$ can be determined by analyzing the higher order terms in the Taylor series expansion of $f(u)$ centered at $u_j$. An alternate way of determining stability is by using any antiderivative $F(u)$ of $f(u)$: the maxima of $F$ correspond to unstable constant solutions, while the minima of $F$ correspond to the stable constant solutions. In this case the domains of attraction of the stable constant solutions are given by the troughs between maxima. If the initial condition function $\phi(x)$ has values in two or more of these troughs, then the solution $u$ in different regions of $\Omega$ will tend to different stable constant solutions. The attraction, in these regions, to different constant solutions leads to the formation of stationary fronts.

For large positive $D = D(\Omega)$, diffusion will average nearby values of the initial condition $\phi(x)$, and the solution $u(x,t)$ will tend to flatten out on $\Omega$. As $t$ increases, $u(x,t)$ may eventually lie in one of the troughs mentioned above; once that happens the solution quickly tends to the stable constant solution in that domain.

For small positive $D = D(\Omega)$, the averaging effect of diffusion is muted so it might seem (based on the discussion for $D = 0$) that fronts would *always* appear if the initial condition $\phi(x)$ lies in different troughs of the stable constant solutions. Figure 1 shows the formation and the motion of a traveling front with $D = 0.008$ and a particular choice of initial condition $\phi(x)$. In this case, the traveling front moves left and eventually collapses to the stable constant solution $u \equiv 0$.

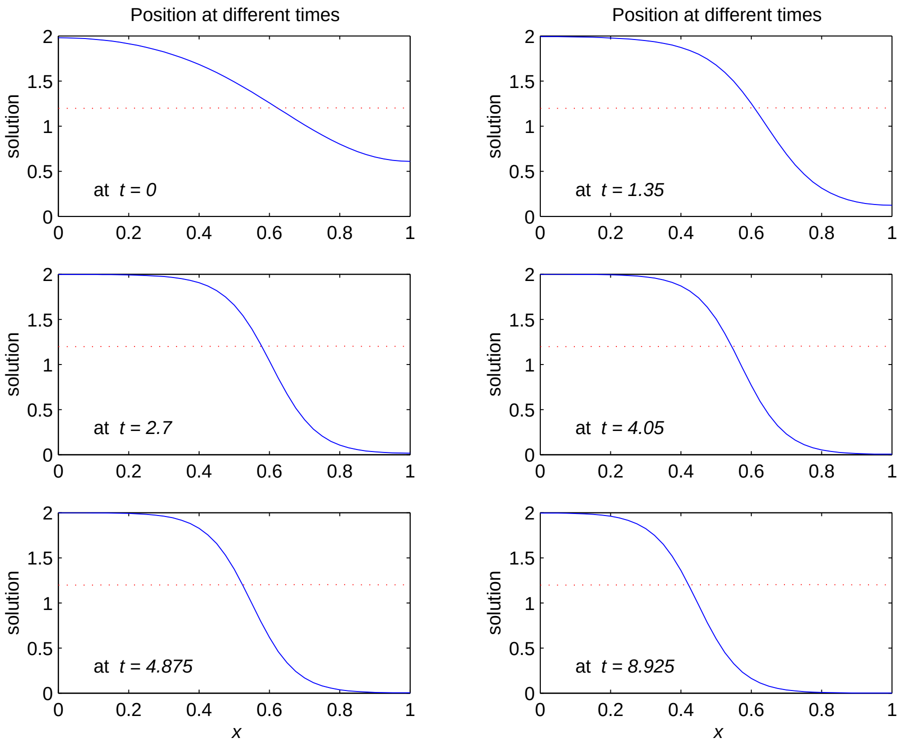


Figure 1

However, even if the initial condition $\phi(x)$ lies in different troughs of the stable constant solutions, traveling fronts need not occur. While such fronts do appear in the limiting case (as $D \to 0$), *the intuitive result that traveling fronts always appear for initial conditions in two or more troughs and small but finite $D$ is, in general, false* as we will prove in this paper. For small but finite $D$ the domains of attraction for the stable constant solutions are not given simply by the troughs between maxima of $F(u)$. Instead, the domains of attraction consist not only of the troughs between stable constant solutions, but also of the regions below or above the *nodal solutions* about an unstable constant solution of the steady-state problem

$$D\Delta v + f(v) = 0, \qquad x \in \Omega, \tag{4}$$

$$v_\nu(x) = 0, \qquad x \in \partial\Omega. \tag{5}$$

These nodal solutions are other nonconstant solutions of problem (4) – (5) that may exist for small values of $D$. If $\phi(x)$ lies below (above) one of these nodal solutions, then $u(x,t)$ will be immediately attracted to the stable constant solution below (above) that unstable constant solution even when $\phi(x)$ lies in different troughs of the stable constant solutions.

Much has been written on the formation and motion of fronts for small $D = D(\Omega)$. By following the motion of these fronts, called traveling waves, it is possible to determine how a solution develops in time. The first instances where traveling waves were investigated is in the celebrated papers of Fisher [7] and Kolmogorov et al [12]. Further development of traveling waves were studied by Aronson and Weinberger [3], Fife and Hsiao [6], Friedlin [8], Rubinstein et al [15], and many others. Fife [5] reviews much of the literature on traveling waves. Numerous applications exist in genetics, activator-inhibitor chemical systems, predator-prey waves of pursuit, etc. (see Murray [13]). Our result does not contradict any of this work: fronts do occur when the initial condition $\phi(x)$ lies in different basins of attractions, but the basins of attraction are much more complicated than the troughs between stable constant solutions.

In this paper we will assume that the function $f(u)$ satisfies the condition that $f(u)/u$ is nondecreasing for $u \geq 0$. We choose this condition because it makes all the results and proofs very transparent. Our condition is not quite as general as the condition

$$f(u_1) - f(u_2) \geq -c(u_1 - u_2), \quad \text{for} \quad \hat{u} \leq u_2 \leq u_1 < \infty, \quad (c > 0),$$

where $\hat{u}$ is a lower solution, used by Pao ([14], Theorem 4.2) to prove the existence of a solution to (1) – (3). Indeed, if $g(u) = f(u)/u$ is nondecreasing and $u_1 > u_2$, then

$$\frac{f(u_1) - f(u_2)}{u_1 - u_2} = \frac{u_1}{u_1 - u_2}[g(u_1) - g(u_2)] + g(u_2) > g(u_1) > g(0) = -c.$$

Hence, our condition guarantees the existence of a solution by Pao result [14]. Our goal is to determine the immediate behavior of the solution $u(x,t)$, that is , whether a traveling wave will form, not the long term behavior. Thus, our results differ from those of Fujita [10] (where $f(u)$ is positive, $C^2$, strictly convex and increasing) and Wolfrum [16] (where $f(u)/u < 0$).

Most of our results will apply to arbitrary regions $\Omega$ in $R^m$, $m \geq 1$, with smooth boundaries. In some of our work we have reduced consideration to $m = 1$ to avoid having to discuss the nature of the boundaries in higher dimensional space between subregions of $\Omega$ that are attracted to different constant solutions of (1) – (2). The purpose here is to establish easily defined conditions that will be easy to test, instead of opting for full generality.

In Section 2 we compare a solution $v$ of problem (4) – (5) with the initial condition $\phi(x)$. The idea of comparing such solutions was first suggested by Gel'fand [11], but the method of comparison used here was first done in Chen and Derrick [4]. We prove that if $\phi(x) < v(x)$, then $u(x,t) < v(x)$ for all $t > 0$ and $x \in \Omega$. Similarly, if $\phi(x) > v(x)$, it follows that $u(x,t) > v(x)$ for all $t > 0$ and $x \in \Omega$. Further, we show that the solution $v(x)$ repels the solutions to problem (1) – (3). We also prove results on the rate of repulsion, which may be exponential or may even blow up in finite time.

In Section 3 we study what happens if the initial condition $\phi(x)$ lies partly below one solution $v_1(x)$ of problem (4) – (5) over some subregion of $\Omega$ and partly below a second solution $v_2(x)$ of the same problem on the complementary region. To simplify our hypotheses and the discussion, we limit the Section 3 discussion to the situation where $m = 1$. We show that if a certain condition is satisfied, then similar results as those in Section 2 apply in this situation. The construction can clearly be carried over to cases where $\phi(x)$ lies below or above several different solutions $v_i$ of problem (4) – (5), provided appropriate conditions hold at the boundaries of the individual subregions.

In Section 4 we discuss some numerical computations for the problem

$$u_t = Du_{xx} + u(1.2 - u)(u - 2), \qquad 0 < x < 1, \quad t > 0, \tag{6}$$

$$u_x(0,t) = u_x(1,t) = 0, \qquad t > 0, \tag{7}$$

$$u(x,0) = \phi(x), \qquad 0 \le x \le 1, \tag{8}$$

for various values of $D > 0$ and $\phi(x)$. Here the steady-state problem for comparison will be the boundary-value problem

$$Dv_{xx} + v(1.2 - v)(v - 2) = 0, \qquad 0 < x < 1, \tag{9}$$

$$v_x(0) = v_x(1) = 0\,. \tag{10}$$

Observe that (9) – (10) has at least three constant solutions $v \equiv 0$, 1.2, 2, and may have other nonconstant solutions for appropriate values of $D$. We show that if the given $\phi(x)$ is below some solution $v(x)$ of (9) – (10), then the solution $u(x,t)$ decays exponentially to zero, while if $\phi(x)$ is above $v(x)$ the solution grows to $u \equiv 2$.

We should point out that a similar problem was considered in Pao [14] (Section 5.7), where it is stated that the stability regions are the troughs of the stable constant solutions. Our numerical computations show that this is incomplete, and the stability regions are determined not only by the troughs, but also by the nodal solutions of the steady state problem.

## 2. Domains of Attraction and Rates of Convergence

In what follows we will assume that either a global solution of problem (1) – (3) exists, or that such a solution exists in $\Omega \times [0,T)$ and blows up at $t = T$. This assumption is guaranteed by the results in Pao ([14], Theorem 4.2). Our task is to determine whether the traveling wave will arise, or not, for $\phi(x)$ in different troughs of the stable constant solutions. We begin by showing that even if $\phi(x)$ lies in different troughs, the solution may converge exponentially to one of the stable constant solutions without forming a traveling wave.

Let $u$ be a solution (with $u \geq 0$) to problem (1) – (3) with $\phi(x) \geq 0$ on $\bar{\Omega}$, and let $v(x)$ be a positive solution to problem (4) – (5). Note that the requirements that $\phi(x) \geq 0$ and $v(x) > 0$ on $\bar{\Omega}$ are fairly minor, since the substitutions $U(x,t) = u(x,t)+K$ and $V(x) = v(x)+K$ lead to equivalent problems for $U$ and $V$ with the initial condition or solution raised by $K = \min(\phi(x), v(x) : x \in \bar{\Omega})$.

For $0 \leq \phi < \lambda v$, $\lambda < 1$, define the functions

$$g_n(t) = \int_\Omega \frac{u^{n+2}(x,t)}{v^n(x)} dx, \qquad \text{for} \quad t > 0, \tag{11}$$

and for $\phi > \lambda v$, $\lambda > 1$, define

$$h_n(t) = \int_\Omega \frac{v^{n+2}(x)}{u^n(x,t)} dx, \qquad \text{for} \quad 0 < t < T. \tag{12}$$

We shall show for (12) that $u \geq v$ in $\Omega \times [0,T)$, with $T$ possibly equal to $\infty$, so that the functions $g_n$ and $h_n$ are well defined, since $\Omega$ is bounded.

**Theorem 1.** *Let $f(z)/z$ be a nondecreasing function for $z \geq 0$. If $0 \leq \phi < \lambda v$, $\lambda < 1$, where $v$ is a solution of (4) – (5), then $g_n'(t) < 0$ and $u(x,t) \leq \lambda v(x)$ for all $t > 0$ and $x$ in $\Omega$, where $u$ is the solution of (1) – (3). Further, if $f(0) = 0$ and $u(x,t) > 0$ for all $t > 0$ and $x$ in $\Omega$, then there exists a constant $c_0 > 0$ such that*

$$u(x,t) \leq \lambda v(x) \exp(-c_0 t), \qquad \textit{for all} \quad t > 0 \quad \textit{and} \quad x \in \Omega, \tag{13}$$

*so that $u$ decays exponentially to the constant steady state solution $u_1 \equiv 0$ of (1) – (2).*

*Proof.* Differentiating (11) with respect to $t$ inside the integral sign and replacing $u_t$ by the right side of (1) we have

$$g_n'(t) = (n+2) \int_\Omega \frac{u^{n+1}(x,t)}{v^n(x)} (D\Delta u + f(u)) dx. \tag{14}$$

Using Green's theorem and the condition $u_\nu = 0$ on $\partial\Omega$ on the first term in parenthesis yields

$$g_n'(t) = (n+2)\left\{\int_\Omega \frac{u^{n+2}}{v^n}\frac{f(u)}{u}dx + Dn\int_\Omega \left(\frac{u}{v}\right)^{n+1} \nabla u \nabla v dx \right.$$
$$\left. -D(n+1)\int_\Omega \left(\frac{u}{v}\right)^n |\nabla u|^2 dx\right\}.$$

But

$$0 = (n+2)\int_\Omega \frac{u^{n+2}}{v^{n+1}}(D\Delta v + f(v))dx$$
$$= (n+2)\left\{\int_\Omega \frac{u^{n+2}}{v^n}\frac{f(v)}{v}dx - D(n+2)\int_\Omega \left(\frac{u}{v}\right)^{n+1} \nabla u \nabla v dx \right.$$
$$\left. +D(n+1)\int_\Omega \left(\frac{u}{v}\right)^{n+2} |\nabla v|^2 dx\right\}.$$

Subtracting, we have

$$g_n'(t) = (n+2)\left\{\int_\Omega \frac{u^{n+2}}{v^n}\left(\frac{f(u)}{u} - \frac{f(v)}{v}\right)dx \right.$$
$$\left. -D(n+1)\int_\Omega \frac{u^n}{v^{n+2}}|v\nabla u - u\nabla v|^2 dx\right\}. \tag{15}$$

Assume there are $t_0'$ for which $u(x, t_0') \geq v(x)$ for some $x$ in $\Omega$ and let $t_0$ be the least of these numbers. Since $v > u \geq 0$ in $\Omega \times [0, t_0)$, it follows that $(f(u)/u) - (f(v)/v) < 0$, so that $g_n'(t) < 0$ and $g_n(t) \leq g_n(0)$ for $t \in (0, t_0]$. Taking the $(n+2)$-nd roots of this inequality,

$$\left[\int_\Omega \left(\frac{u(x,t)}{v(x)}\right)^{n+2} v^2(x)dx\right]^{\frac{1}{n+2}}$$
$$\leq \left[\int_\Omega \left(\frac{\phi(x)}{v(x)}\right)^{n+2} v^2(x)dx\right]^{\frac{1}{n+2}}, \tag{16}$$

and letting $n \to \infty$, we have an inequality for the $L^\infty(v^2(x)dx)$ norm:

$$\frac{u(x,t)}{v(x)} \leq \sup_\Omega \frac{u}{v} \leq \sup_\Omega \frac{\phi}{v} \leq \lambda, \qquad \text{for} \quad t \in (0, t_0], \tag{17}$$

so that $u(x, t_0) \leq \lambda v(x)$. This contradicts the definition of $t_0$. Hence $u(x, t) \leq \lambda v(x)$ for all $t > 0$ and $x$ in $\Omega$.

Now assume $0 < u \leq \lambda v < v$ on $\Omega$. Then by continuity the inequality holds on the closure $\bar{\Omega}$ of $\Omega$, so a constant $c_0$ exists such that

$$\left(\frac{f(v)}{v} - \frac{f(u)}{u}\right) \geq c_0 > 0 \qquad \text{on} \quad \bar{\Omega}.$$

Then, by (15),

$$g_n'(t) \leq -(n+2) \int_\Omega \frac{u^{n+2}}{v^n} \left(\frac{f(v)}{v} - \frac{f(u)}{u}\right) dx \leq -c_0(n+2) g_n(t).$$

Integrating this inequality, we obtain

$$g_n(t) \leq g_n(0) \exp(-c_0(n+2)t).$$

Again taking the $(n+2)$-nd root of both sides, and letting $n \to \infty$, we have the $L^\infty(v^2(x)dx)$ inequality

$$\frac{u(x,t)}{v(x)} \leq \sup_\Omega \frac{u}{v} \leq \sup_\Omega \frac{\phi}{v} \exp(-c_0 t) \leq \lambda \exp(-c_0 t),$$

for all $t > 0$ and $x$ in $\Omega$. This completes the theorem.

**Remark 1.** Note that we have not made use of the second integral in equation (15). If, for example, we knew that there were values $x_0$ in $\Omega$ and $t_0 > 0$ such that

$$|v(x_0) \nabla u(x_0, t_0) - u(x_0, t_0) \nabla v(x_0)| > 0,$$

we would be able to use this term to produce the result without the use of the first integral. However, there are examples to show that this inequality need not hold.

**Theorem 2.** *Let $f(z)/z$ be a nondecreasing function for $z \geq 0$. If $\phi > \lambda v$, $\lambda > 1$, where $v$ is a positive solution of (4) − (5), then $h_n'(t) < 0$ and $u(x, t) \geq \lambda v(x)$ for all $t > 0$ and $x$ in $\Omega$, where the solution $u$ to (1) − (3) exists. If $f(z)/z$ increases to the next stable constant solution $u_i$ then $u(x, t)$ converges exponentially to $u_i$.*

*Proof.* Differentiating with respect to $t$ inside the integral (12) we have

$$h_n'(t) = -n \int_\Omega \frac{v^{n+2}}{u^n} \frac{f(u)}{u} dx$$

$$+Dn\left\{(n+2)\int_\Omega\left(\frac{v}{u}\right)^{n+1}\nabla u\nabla v dx-(n+1)\int_\Omega\left(\frac{v}{u}\right)^{n+2}|\nabla u|^2dx\right\}.$$

But

$$0=n\int_\Omega\frac{v^{n+2}}{u^n}\left(\frac{D\Delta v+f(v)}{v}\right)dx$$

$$=n\int_\Omega\frac{v^{n+2}}{u^n}\frac{f(v)}{v}dx-Dn\left\{(n+1)\int_\Omega\left(\frac{v}{u}\right)^n|\nabla v|^2dx\right.$$

$$\left.-n\int_\Omega\left(\frac{v}{u}\right)^{n+1}\nabla u\nabla v dx\right\}.$$

Adding the two equations we get

$$\begin{aligned} h_n'(t)&=-n\left\{\int_\Omega\frac{v^{n+2}}{u^n}\left(\frac{f(u)}{u}-\frac{f(v)}{v}\right)dx\right\}\\ &-Dn(n+1)\int_\Omega\frac{v^n}{u^{n+2}}|v\nabla u-u\nabla v|^2dx.\end{aligned}\tag{18}$$

Assume that there are $t_1'$ such that $u(x,t_1')\geq v(x)$ for some $x$ in $\Omega$, and let $t_1$ be the least of these numbers. Since $u>v$ in $\Omega\times[0,t_1)$, both integrals in (18) are positive, so $h_n'(t)<0$ in $(0,t_1)$. Then $h_n(t)\leq h_n(0)$ on $t\in[0,t_1]$. Taking the $n$-th roots of this inequality

$$\left[\int_\Omega\left(\frac{v(x)}{u(x,t)}\right)^n v^2(x)dx\right]^{\frac{1}{n}}\leq\left[\int_\Omega\left(\frac{v(x)}{\phi(x)}\right)^n v^2(x)dx\right]^{\frac{1}{n}},\tag{19}$$

and letting $n\to\infty$, we have the $L^\infty(v^2dx)$ inequality

$$\frac{v(x)}{u(x,t)}\leq\sup_\Omega\frac{v}{u}\leq\sup_\Omega\frac{v}{\phi}\leq\frac{1}{\lambda},\qquad\text{for}\quad t\in(0,t_1].\tag{20}$$

This contradicts the definition of $t_1$. Thus, $\lambda v(x)\leq u(x,t_1)$ for all $t>0$, $x$ in $\Omega$, where the solution exists. If $f(z)/z$ increases only to the next higher root $u_i$ of $f$, then by substituting $U=u_i-u$ in (1) – (3) and $V=u_i-v$ in (4) – (5) we recast the two problems so that $U(x,t)$ decays exponentially to zero by Theorem 1, implying that $u(x,t)$ converges exponentially to $u_i$.

It is possible for $u(x,t)$ to blow-up for finite $t$. Assume that

$$\frac{f(u)}{u^{n+1}}\left(1-\frac{f(v)/v}{f(u)/u)}\right)\geq c_2>0,$$

then

$$h_n'(t) \le -n \int_\Omega \frac{v^{n+2}}{u^n} \left( \frac{f(u)}{u} - \frac{f(v)}{v} \right) dx \le -nc_2 \int_\Omega v^{n+2} dx.$$

Integrating, we have

$$0 \le h_n(t) \le h_n(0) - nc_2 t \int_\Omega v^{n+2} dx,$$

so that

$$t \le \frac{h_n(0)}{nc_2 \int_\Omega v^{n+2} dx} < \infty. \tag{21}$$

Of course this assumes that $f(z)/z$ is increasing for all $z > \inf_\Omega v$.

Thus, the nodal solutions $v(x)$, when they occur, play a significant role in determining the domains of attraction of solutions to problem (1) – (3).

## 3. Several Regions

In this section we extend the results of Section 2 to the case, where $\phi(x)$ is below one solution $v_1$ of problem (4) – (5) on a subregion of $\Omega$ and below some other solution $v_2$ of this problem on the subregion's complement. To simplify the statement of the hypotheses and to avoid considering the nature of the boundaries of the subregions, we assume that $\Omega$ is the interval (0, 1) in $R^1$, an interval that can always be obtained by rescaling the $x$ variable. Further, if $v_1$ or $v_2$ is a nodal solution, we classify it by the number of times it crosses an unstable constant solution.

Let $0 < a < 1$ and suppose for $\lambda < 1$ that

$$0 \le \phi(x) < \lambda V(x) \equiv \begin{cases} \lambda v_1, & \text{on} \quad [0, a], \\ \lambda v_2, & \text{on} \quad [a, 1]. \end{cases} \tag{22}$$

Define

$$g_n(t) = \int_0^a \frac{u^{n+2}(x,t)}{v_1^n(x)} dx + \int_a^1 \frac{u^{n+2}(x,t)}{v_2^n(x)} dx, \tag{23}$$

then

$$g_n'(t) = (n+2) \left\{ \int_0^a \frac{u^{n+1} u_t}{v_1^n} dx + \int_a^1 \frac{u^{n+1} u_t}{v_2^n} dx \right\}.$$

Replacing $u_t$ by the term $Du_{xx} + f(u)$ and integrating by parts, we obtain

$$g_n'(t) = (n+2) \int_0^a \frac{u^{n+2}}{v_1^n} \frac{f(u)}{u} dx + D(n+2) \left\{ \frac{u^{n+1} u_x}{v_1^n} \right|_0^a$$

$$-\int_0^a \left(\frac{u^{n+1}}{v_1^n}\right)_x u_x dx\Bigg\} + (n+2)\int_a^1 \frac{u^{n+2}}{v_2^n}\frac{f(u)}{u}dx$$

$$+D(n+2)\left\{\frac{u^{n+1}u_x}{v_2^n}\Bigg|_a^1 - \int_a^1 \left(\frac{u^{n+1}}{v_2^n}\right)_x u_x dx\right\},$$

or

$$g_n'(t) = (n+2)\left\{\int_0^a \frac{u^{n+2}}{v_1^n}\frac{f(u)}{u}dx + \int_a^1 \frac{u^{n+2}}{v_2^n}\frac{f(u)}{u}dx\right\} \tag{24}$$

$$+D(n+2)\Bigg\{n\int_0^a \left(\frac{u}{v_1}\right)^{n+1} u_x v_1' dx - (n+1)\int_0^a \left(\frac{u}{v_1}\right)^{n+1} u_x^2 dx$$

$$+\frac{u^{n+1}(a,t)u_x(a,t)}{v_1^n(a)}\Bigg\} + D(n+2)\Bigg\{n\int_a^1 \left(\frac{u}{v_2}\right)^{n+1} u_x v_2' dx$$

$$-(n+1)\int_a^1 \left(\frac{u}{v_2}\right)^{n+1} u_x^2 dx - \frac{u^{n+1}(a,t)u_x(a,t)}{v_2^n(a)}\Bigg\}.$$

But

$$0 = (n+2)\left\{\int_0^a \frac{u^{n+2}[Dv_1'' + f(v_1)]}{v_1^{n+1}}dx + \int_a^1 \frac{u^{n+2}[Dv_2'' + f(v_2)]}{v_2^{n+1}}dx\right\}$$

$$= (n+2)\int_0^a \frac{u^{n+2}}{v_1^n}dx + (n+2)\int_a^1 \frac{u^{n+2}}{v_2^n}\frac{f(v_2)}{v_2}dx$$

$$+\int_0^a \frac{f(v_1)}{v_1}dx + D(n+2)\left\{\frac{u^{n+2}v_1'}{v_1^{n+1}}\Bigg|_0^a - \int_0^a \left(\frac{u^{n+2}}{v_1^{n+1}}\right)_x v_1' dx\right\}$$

$$+D(n+2)\left\{\frac{u^{n+2}v_2'}{v_2^{n+1}}\Bigg|_a^1 - \int_a^1 \left(\frac{u^{n+2}}{v_2^{n+1}}\right)_x v_2' dx\right\},$$

or

$$0 = (n+2)\left\{\int_0^a \frac{u^{n+2}}{v_1^n}\frac{f(v_1)}{v_1}dx + \int_a^1 \frac{u^{n+2}}{v_2^n}\frac{f(v_2)}{v_2}dx\right\} \tag{25}$$

$$+D(n+2)\Bigg\{(n+1)\int_0^a \left(\frac{u}{v_1}\right)^{n+2} (v_1')^2 dx$$

$$-(n+2)\int_0^a \left(\frac{u}{v_1}\right)^{n+1} u_x v_1' dx + \frac{u^{n+2}(a,t)v_1'(a)}{v_1^{n+1}(a)}\Bigg\}$$

$$+D(n+2)\left\{(n+1)\int_a^1 \left(\frac{u}{v_2}\right)^{n+2} (v_2')^2 dx\right.$$

$$\left.-(n+2)\int_a^1 \left(\frac{u}{v_2}\right)^{n+1} u_x v_2' dx - \frac{u^{n+2}(a,t)v_2'(a)}{v_2^{n+1}(a)}\right\}.$$

Subtracting (25) from (24), we have

$$g_n'(t) = (n+2)\left\{\int_0^a \frac{u^{n+2}}{v_1^n}\left(\frac{f(u)}{u} - \frac{f(v_1)}{v_1}\right) dx\right.$$

$$\left.+\int_a^1 \frac{u^{n+2}}{v_2^n}\left(\frac{f(u)}{u} - \frac{f(v_2)}{v_2}\right) dx\right\} \tag{26}$$

$$-D(n+1)(n+2)\left\{\int_0^a \frac{u^n}{v_1^{n+2}}(v_1 u_x - u v_1')^2 dx\right.$$

$$\left.+\int_a^1 \frac{u^n}{v_2^{n+2}}(v_2 u_x - u v_2')^2 dx\right\}$$

$$+D(n+2)\left\{\frac{u^{n+1}(a,t)u_x(a,t)}{v_1^n(a)} - \frac{u^{n+1}(a,t)u_x(a,t)}{v_2^n(a)}\right.$$

$$\left.-\frac{u^{n+2}(a,t)v_1'(a)}{v_1^{n+1}(a)} + \frac{u^{n+2}(a,t)v_2'(a)}{v_2^{n+1}(a)}\right\}.$$

Since $u$ lies below $v_1$ and $v_2$ on the corresponding intervals, the first two terms in the right hand side of (26) are negative. Thus, for $g_n'$ to be negative it is sufficient to have the last term in the right hand side of (26) negative or zero. Observe that if the last term (in brackets) is non-positive, we have

$$\frac{u^{n+1}(a,t)u_x(a,t)}{v_1^n(a)} - \frac{u^{n+2}(a,t)v_1'(a)}{v_1^{n+1}(a)}$$

$$\leq \frac{u^{n+1}(a,t)u_x(a,t)}{v_2^n(a)} - \frac{u^{n+2}(a,t)v_2'(a)}{v_2^{n+1}(a)}.$$

Dividing both sides by $u^{n+1}(a,t) > 0$, we get

$$\frac{1}{v_1^{n-1}(a)}\left(\frac{u}{v_1}\right)_x(a) \leq \frac{1}{v_2^{n-1}(a)}\left(\frac{u}{v_2}\right)_x(a). \tag{27}$$

When $v_1(a) = v_2(a)$, the inequality reduces to $v_2'(a) \le v_1'(a)$. Hence we have proved the following theorem.

**Theorem 3.** *Let $f(z)/z$ be a nondecreasing function for $z \ge 0$ and let $\phi(x)$ satisfy* (22), *where $v_1$ and $v_2$ are solutions of $Dv_{xx} + f(v) = 0$, $v_x(0) = v_x(1) = 0$. Define $g_n(t)$ as in* (23) *and assume* (27) *holds. Then $g_n'(t) < 0$ and $u(x,t) \le \lambda V(x)$ for all $t > 0$ and $x \in (0,1)$.*

**Remark 2.** Assume $f(z)/z$ is nondecreasing for $z \ge 0$, that $0 < a < 1$, and suppose for $\lambda > 1$ that

$$\phi > \lambda V \equiv \begin{cases} \lambda v_1, & \text{on} \quad [0,a], \\ \\ \lambda v_2, & \text{on} \quad [a,1], \end{cases} \tag{28}$$

where $v_1$ and $v_2$ are solutions of problem (4) – (5) as above. Define

$$h_n(t) = \int_0^a \frac{v_1^{n+2}(x)}{u^n(x,t)}dx + \int_a^1 \frac{v_2^{n+2}(x)}{u^n(x,t)}dx, \tag{29}$$

and suppose that

$$v_2^{n+3}\left(\frac{u}{v_2}\right)_x(a) \le v_1^{n+3}\left(\frac{u}{v_1}\right)_x(a), \tag{30}$$

or, when $v_1(a) = v_2(a)$, that $v_1'(a) \le v_2'(a)$. Then $h_n'(t) < 0$ and $u(x,t) \ge \lambda V(x)$ for all $t > 0$ and $x$ in $(0,1)$.

**Remark 3.** Since the values of $u(a,t)$ and $u_x(a,t)$ vary in time, conditions (27) and (30) are difficult to apply. However, when $v_1(a) = v_2(a)$, it is quite easy to check the slopes of the nodal solutions. It can easily be checked that Theorem 3 and Remark 2 provide additional information only when one of the solutions is 3-nodal and the other is at least 2-nodal. If this is not the case, then the result follows by either Theorem 1 or Theorem 2, because $\phi$ lies completely below (above) a single solution of (4) – (5). In the next section we will illustrate this with an example (see Figure 6).

## 4. Numerical Calculations

In this section we consider the problem

$$\begin{aligned} &u_t = Du_{xx} + u(1.2-u)(u-2), && 0 < x < 1, \quad t > 0, \\ &u_x(0,t) = u_x(1,t) = 0, && t > 0, \\ &u(x,0) = \phi(x), && 0 \le x \le 1, \end{aligned} \tag{31}$$

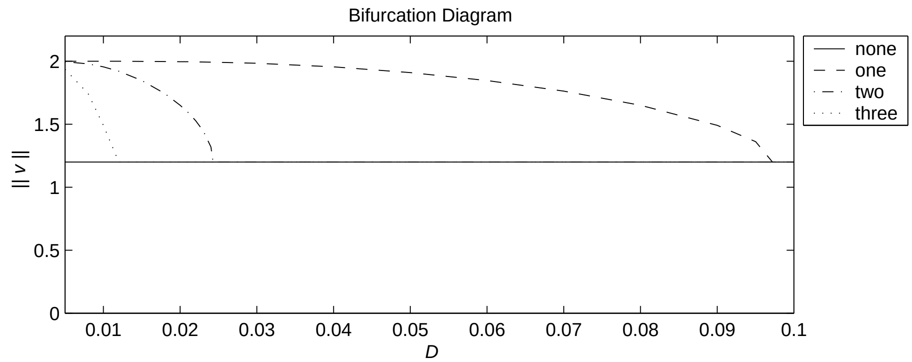


Figure 2

for various values of $D \geq 0$ and $\phi(x)$. Here the steady-state solutions for comparisons will be obtained from the boundary-value problem

$$\begin{aligned} &Dv_{xx} + v(1.2 - v)(v - 2) = 0, \quad 0 < x < 1, \\ &v_x(0) = v_x(1) = 0. \end{aligned} \tag{32}$$

Observe that (32) has at least three constant solutions $v \equiv 0$, 1.2, 2, and may have other nonconstant solutions for appropriate values of $D$. Writing (32) as a system, we obtain its Jacobian matrix:

$$J = \begin{pmatrix} 0 & 1 \\ g(v) & 0 \end{pmatrix}, \tag{33}$$

where $g(v) = (3v^2 - 6.4v + 2.4)/D$, so that $(0,0)$ and $(2,0)$ are saddles and $(1.2,0)$ is a center. Thus, nonconstant solutions crossing the solution line $v = 1.2$ may exist for appropriate $D$. These *nodal solutions* are classified by the number of crossings (nodes) of the line $v = 1.2$. This can also be inferred by examining the concavity of the direction field. The bifurcation diagram for these nodal solutions is shown in Figure 2.

In the computations that follow we have used the von Neumann-Eddy procedure to approximate the solution $u(x,t)$ to problem (31). This is an implicit method that uses a predictor-corrector procedure to determine the nonlinear term $f(u)$. The procedure is known to be stable with an error of order $O(h^2+k^2)$, where $h$ and $k$ are the step sizes in the $t$ and $x$ directions respectively (see [2], p. 359).

Let us select $D = 0.05$. Then we have two single-node solutions of (32). All of the solutions (constant and single-node) are shown as dotted lines in Figure

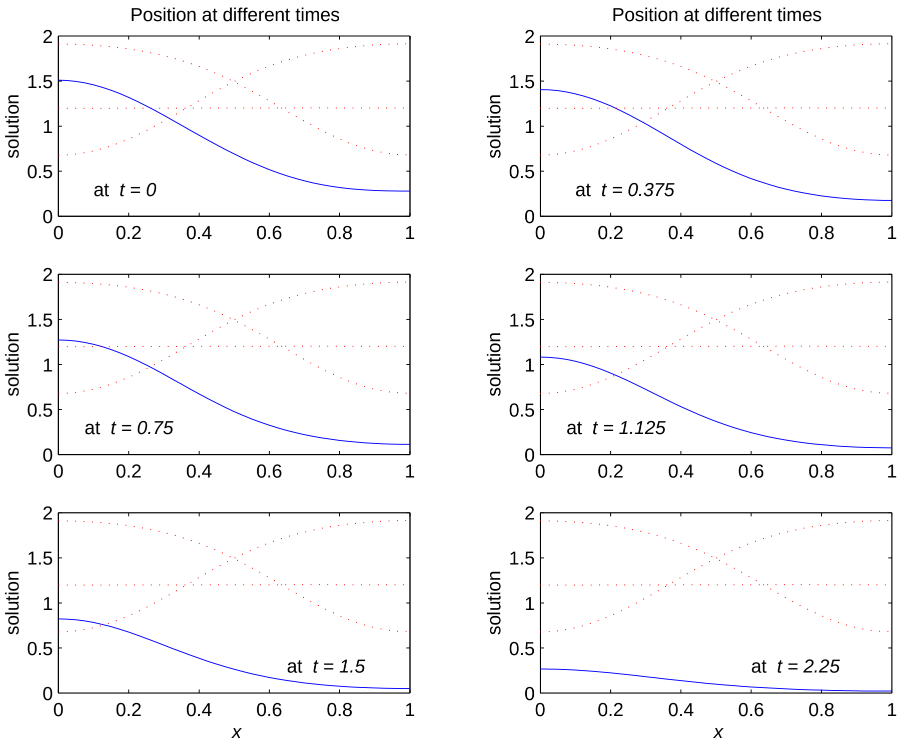


Figure 3

3. Suppose $\phi(x)$ is the solid line shown in the $t = 0$ frame in Figure 3. Then $\phi$ lies below one of the single-node solutions which we will call $v$. Note that $f(z)/z$ is increasing for $0 \leq z \leq 1.6$, so Theorem 1 and Theorem 2 apply since $\phi \leq 1.6$. Observe in the succeeding frames that $u(x,t)$ decays exponentially to zero as predicted.

In Figure 4 the initial function $\phi(x)$ exceeds 1.6 for small values of $x$, even though $\phi$ lies below one of the nodal solutions. Although Theorem 1 and Theorem 2 do not apply directly, because $f(z)/z$ is decreasing for $z$ larger than 1.6, the solution still decays to zero, probably because the first integral in the formula for $g_n'$ is still negative. Note that $u(x,t)$ actually increases for small values of $x$ where it exceeds 1.6 when $t$ is small, but decreases when it is less than 1.6. Eventually diffusion causes the higher values to decay (at $t = 2.25$). Once the entire $u(x,t)$ curve lies below 1.6 (at $t = 3.75$) and Theorem 1 and Theorem 2 again apply, it decays to zero. This increase for small $x$ between $0 < t < 2.5$ may seem like the initial stage of formation of a traveling wave.

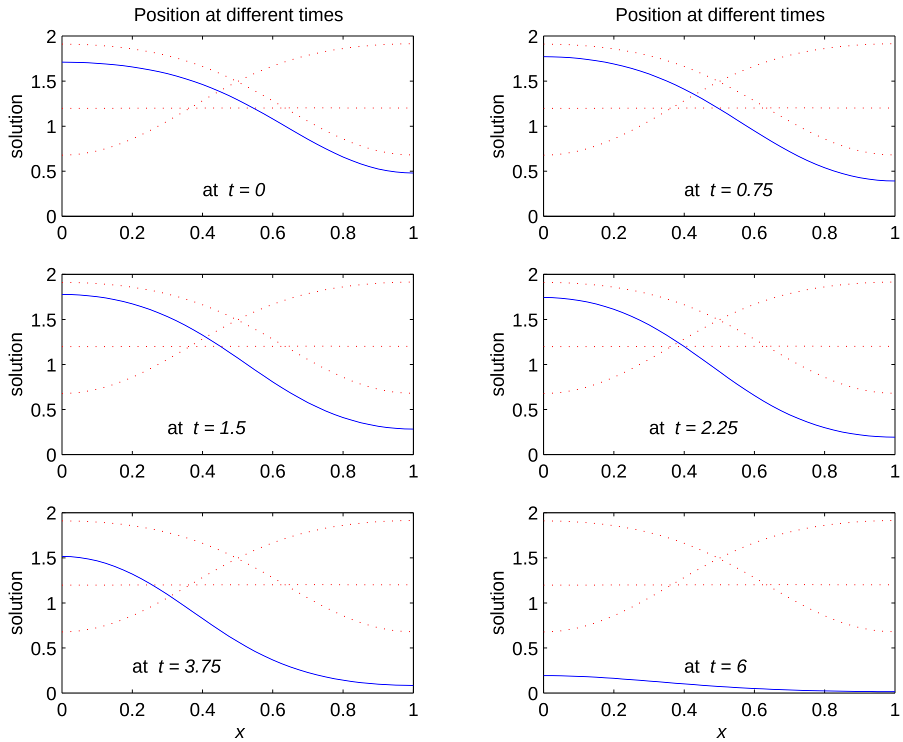


Figure 4

The wave, however, never gets very close to the constant solution $u \equiv 2$ (since it is separated from this constant solution by a nodal solution shown in Figure 4; the fact that $u(x,t)$ cannot cross this nodal solution follows from general maximum principle results for parabolic equations), and moves fairly quickly to the left. By $t = 3.75$, Theorem 1 applies and the decay to zero is swift.

Figure 5 shows the situation where $D = 0.05$ and $\phi(x)$ lies above $v(x)$. Although Theorem 1 and Theorem 2 do not apply (since $f(z)/z$ is increasing only until $z = 1.6$), if we substitute $U = 2 - u$, $V = 2 - v$, and $\Phi = 2 - \phi$, then $f(z)/z = (0.8 - z)(z - 2)$, which increases in $0 < z < 1.4$ and $\Phi(x)$ lies below $V(x)$ with $0.05 \leq \Phi \leq 1.3$. Thus, $U(x,t)$ decays exponentially to 0 with increasing time, so that $u(x,t)$ increases exponentially to 2 as shown in Figure 5.

Figure 6 gives an application of Theorem 3. Here $D$ =0.008 so that two 3-nodal solutions $v_1$ and $v_2$ exist. The given $\phi(x)$ lies below $v_1$ in $0 \leq x \leq 0.5$ and below $v_2$ in $0.5 \leq x \leq 1$, with values of $\phi$ below 1.6. As shown in Figure 6,

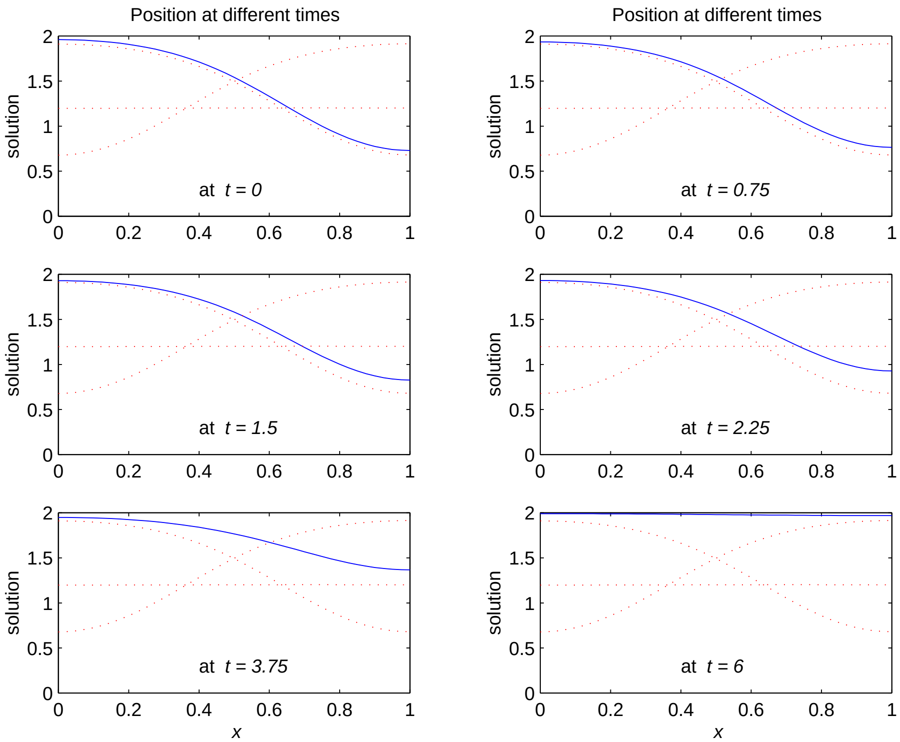


Figure 5

the solution $u(x,t)$ decays exponentially to 0. Note that the solution shown in Figure 6 is produced for the same value of the diffusion coefficient $(D = 0.008)$ as the solution shown in Figure 1. The initial condition $\phi(x)$ in the case of Figure 6 also lies in two different troughs of the stable constant solutions, $u = 0$ and $u = 2$, of (31). However, because of Theorem 3, moving fronts do not form in this situation.

In this paper we have required $f(z)/z$ to be a nondecreasing function in the interval $0 \leq z \leq \max_\Omega v$ for Theorem 1 to apply, and in the interval $z > \min_\Omega v$ for Theorem 2 to apply. In a subsequent paper we will show that these results also hold for $f(z)/z$ bounded when $f$ is analytic.

## References

[1] H. Amann, *Nonlinear Analysis: A Collection of Papers in Honor of Erich Rothe*, Academic Press, New York (1978), 1-29.

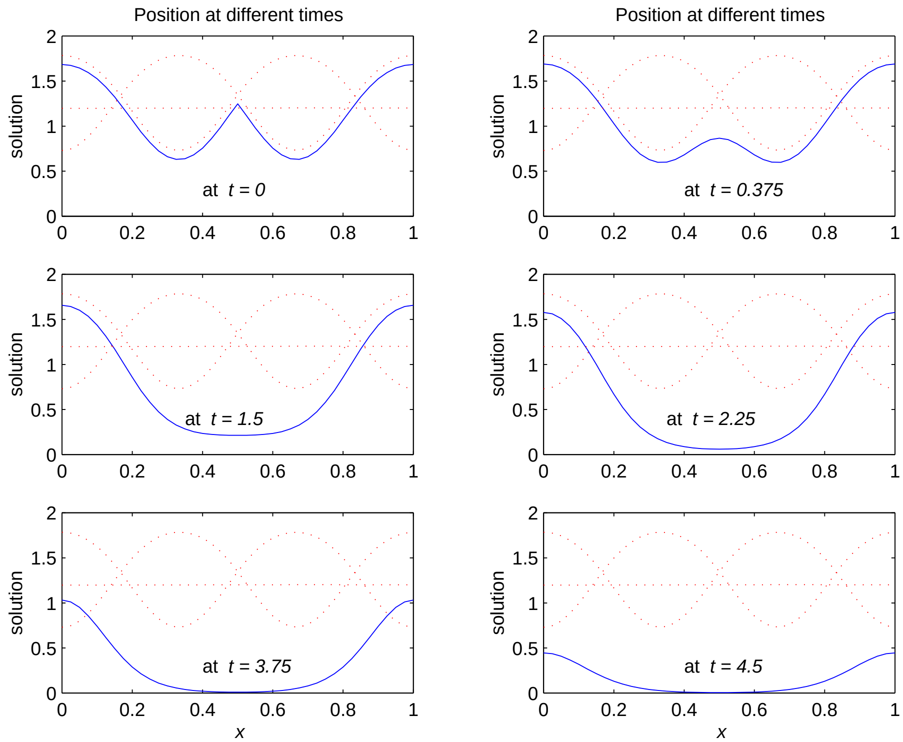


Figure 6

[2] W.F. Ames, *Nonlinear Partial Differential Equations in Engineering*, Vol. 18, Academic Press, New York (1965).

[3] D. Aronson, H. Weinberger, Multidimensional nonlinear diffusion arising in population genetics, *Adv. in Math.*, **30** (1978), 33-76.

[4] S. Chen, W.R. Derrick, Global existence and blow-up of solutions for a semilinear parabolic system, *Rocky Mountain J. Math.*, **29** (1999), 449-457.

[5] P.C. Fife, *Dynamics of Internal Layers and Diffusive Interfaces*, CBMS-NSF Regional Conference Series in Applied Mathematics, No. 53, SIAM, Philadelphia (1988).

[6] P.C. Fife, L. Hsiao, The generation and propagation of internal layers, *J. Nonlinear Analysis*, TMA, **12** (1988), 19 - 41.

[7] R.A. Fisher, The wave of advance of advantageous genes, *Ann. Eugenics*, **7** (1937), 353-369.

[8] M.I. Freidlin, Geometric optics approach to reaction-diffusion equations, *SIAM J. Appl. Math.*, **46** (1986), 222-232.

[9] A. Friedman, *Partial Differential Equations*, Holt, New York (1969).

[10] H. Fujita, On the asymptotic stability of solutions of the equation $v_t = \Delta v + e^v$, In: *Proc. Internat. Conf. on Functional Analysis and Related Topics*, Tokyo 1969, Univ. of Tokyo Press, Tokyo (1970), 252–259 .

[11] I.M. Gel'fand, Some problems in the theory of quasilinear equations, *Uspehi Mat. Nauk. (NS)*, **14** (1959) 87-158 (in Russian), *Amer. Math. Soc. Transl. Ser. 2*, **29** (1963), 295-381.

[12] A. Kolmogoroff, I. Petrovsky, N. Piscounoff, Étude de l'èquation de la diffusion avec croissance de la quantité de matiére et son application à un problème biologique, *Moscow Univ. Bull. Math.*, **1** (1937), 1-25.

[13] J.D. Murray, *Mathematical Biology*, Springer-Verlag (1989).

[14] C.V. Pao, *Nonlinear Parabolic and Elliptic Equations*, Plenum Press, New York (1992).

[15] J. Rubinstein, P. Sternberg, J.B. Keller, Fast reaction, slow diffusion, and curve shortening, *SIAM J. Appl. Math.*, **49** (1989), 116-133.

[16] M. Wolfram, A sequence of order relation: encoding heteroclinic connections in scalar parabolic PDE, *J. Diff. Equations*, **183** (2002), 56 - 78.