International Journal of Pure and Applied Mathematics

Volume 11 No. 2 2004, 119-149

ON SOME ROBUST CONTROL PROBLEMS FOR NONLINEAR PARABOLIC EQUATIONS

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Abstract: In this paper, we study a robust control problem for a class of systems governed by nonlinear parabolic equations. Firstly the control is the forcing. We formulate the robust control problem, we prove the existence and the uniqueness of the solution and we give the necessary conditions of optimality. Secondly an initial perturbation problem is considered. A robust control problem is reformulated and the existence and the conditions of the uniqueness of the optimal solution are derived. First order necessary conditions of optimality are obtained.

AMS Subject Classification: 49J20, 49J50, 35K55, 49K35, 35K57
Key Words: robust control, nonlinear parabolic equations, existence and uniqueness of optimal solution, necessary optimality conditions

1. Introduction and Mathematical Setting of the Problem

1.1. Motivation

This paper is devoted to the study of the robust control problem for a class of nonlinear parabolic systems with disturbances (perturbation or noise) and controls. The objective of a robust control is to compensate the undesirable effects of system disturbances through control actions such that a cost function achieves its minimum for the worst disturbances, i.e. to find the best control

Received: September 18, 2003

which takes into account the worst-case disturbance.

A solution of a robust control problems was proposed at first for finite-dimensional linear time-invariant systems in terms of the so-called gap-metric by Zames et al [25]. More recently a robust control framework has been object of numerous studies either from a theoritical or from a numerical point of view: to some classes of infinite (or finite)-dimensional linear systems (see, e.g. [10, 11, 13, 16, 23, 26] and the references therein) and to the case of nonlinear systems (see, e.g. [14, 15, 17] and the references therein). The objective of our study is to develop a robust control problem, for some nonlinear parabolic equations, in order to take into account the influence of noise in data.

1.2. The Study Equations and the Outline of the Paper

In this paper we will consider the nonlinear parabolic partial differential equations of the form

$$\begin{aligned} &\frac{\partial U}{\partial t} + AU + F(U) + K(V,U) = f \text{ on } \mathcal{Q} = (0,T) \times \Omega\,, \\ &U(t=0) = U_0 \text{ on } \Omega, \end{aligned} \tag{1.1}$$

where Ω is a bounded subset of $\mathbb{R}^m$, $m \geq 1$, A is an elliptic, selfadjoint operator, $K(V,.)$ is a linear operator, V is given sufficiently regular and $F : \mathcal{Q} \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Nemytsky operator on $L^2(\mathcal{Q})$. We state the following hypothesis for the operator F:

1) $F(.,0) = 0$.

2) F satisfies Carathodory conditions and a one-sided Lipschitz condition :
$$-2(F(.,u) - F(.,v))(u-v) \leq \gamma_0 \mid u - v \mid^2 \;\; \forall (u,v) \in \mathbb{R}^2. \tag{1.2}$$

3) F is differentiable with $G(.,u) = F'(.,u)$, Lipschitz continuous :
$\mid G(.,u) - G(.,v) \mid \leq \lambda \mid u - v \mid \;\; \forall (u,v) \in \mathbb{R}^2$.

Remark 1.1. According to (1.2) we have then $-2G(.,u) \leq \gamma_0 \; \forall u \in \mathbb{R}$.

We assume that we can introduce a Banach space $\mathcal{D}$ of functions on Ω satisfying the boundary conditions such that $\mathcal{D} \subset L^2(\Omega) \subset \mathcal{D}'$ and the operator A: $\mathcal{D} \longrightarrow \mathcal{D}'$ with $< Av, v > \geq \nu \parallel v \parallel_{\mathcal{D}}^2 \; \forall v \in \mathcal{D}$ ($<,>$ is the duality pairing on $\mathcal{D}'$ and $\mathcal{D}$, $\parallel . \parallel_{\mathcal{D}}$ is the norm on $\mathcal{D}$, $\parallel . \parallel_*$ is the dual norm on $\mathcal{D}'$ and $\nu > 0$ is a constant). Moreover we impose the condition that $\mathcal{D}$ is embeds in $L^4(\Omega)$, i.e.

$$\mathcal{D} \subset L^4(\Omega) \text{ and there exists } c_e > 0 \text{ such that } \quad \| v \|_{L^4(\Omega)} \leq c_e \| v \|_{\mathcal{D}} \; \forall v \in \mathcal{D}. \tag{1.3}$$

We introduce the following spaces: $\mathcal{H} = L^\infty(0, T, L^2(\Omega))$, $\mathcal{V} = L^2(0, T, \mathcal{D})$, $\mathcal{W} = \mathcal{H} \cap \mathcal{V}$ and we denote $\| \,.\, \|_{\mathcal{W}} = \max(\| \,.\, \|_{\mathcal{H}}, \nu \| \,.\, \|_{\mathcal{V}})$ and by c_I the constant of the embedding map: $\mathcal{D} \longrightarrow L^2(\Omega)$, i.e.

$$\| v \|_{L^2(\Omega)} \leq c_I \| v \|_{\mathcal{D}} \; \forall v \in \mathcal{D}. \tag{1.4}$$

Moreover we assume that there exists a constant $\gamma_\infty \geq 0$ such that:

$$\| K(V, v) \|_{L^2(\Omega)} \leq \sqrt{\nu \gamma_\infty} \| v \|_{\mathcal{D}} \; \forall v \in \mathcal{D}. \tag{1.5}$$

Remark 1.2. (i) γ_∞ is depending on the norm of a given function V.

(ii) we denote by γ the value of the sum: $\gamma_0 + \gamma_\infty$.

(iii) we denote by K^* the adjoint of K, i.e.

$$< K^*(V, u), v > = < K(V, v), u >, \quad \forall (u, v) \in \mathcal{D}^2.$$

In the present paper, the cost function J describing the control problem is depending on the disturbance ψ, the control ϕ and the perturbation $u(\phi, \psi)$ in the domain Ω over the time interval under consideration $[0, T]$. The robust control is to obtain the saddle point of the J which measures the distance between the pronostic variables u and the observation (u_{obs}, v_{obs}). Precisely we will study the following robust control problem: find (u, ϕ, ψ) such that the cost function

$$J(\phi, \psi) = \frac{1}{2} \| \mathcal{C}(u - u_{obs}) \|^2_{L^2(\mathcal{Q})} + \frac{\mu}{2} \| u(T) - v_{obs} \|^2_{L^2} + \frac{\alpha}{2} \| \phi \|^2_{Rh} - \frac{\beta}{2} \| \psi \|^2_{Rs}$$

is minimized with respect to ϕ and maximized with respect to ψ subject to the perturbation of the problem (1.1), where the spaces Rs and Rh are two subsets of some Banach spaces and $\mathcal{C}$ is unbounded operator on $L^2(\Omega)$.

The main result of the paper includes the existence, the uniqueness and the first order necessary conditions of optimality for the worst disturbance and optimal controllers. Our approach is a generalization of the work of Seidman et al [24] (the authors studied the optimal control for a class of quasilinear parabolic equations) to the robust control problems.

The plan of the paper is as follows: in the next section we prove the existence and the uniqueness of the problem (1.1) and obtain some a priori estimates. In Section 3 we formulate the robust control problem in the case when the forcing f is decomposed into a disturbance ψ and a control ϕ. We prove the existence, the uniqueness and we give the appropriate optimality system. The optimality system is corresponding to identify the gradient of the cost function that is necessary to develop a numerical scheme (for example gradient method, quasi-Newton method) in order to solve the robust control problem. In Section 4 we consider an initial perturbation problem because in many situations (for example in data assimilation) the initial condition is not well-know. We reformulate the robust control problem in two cases: firstly the control is the initial condition and the disturbance is the forcing f, secondly the initial condition is decomposed into a disturbance ψ and a control ϕ. As in previous section we study the existence, the uniqueness and we give the optimality conditions. In Section 5 we present an example of convection-diffusion in the case of the pollutant in the liquid or atmospheric system (this type of model has been presented for example by Ahmed et al [2]).

The notations, the fonctional spaces and some definitions are basically standard; see for example [1], [19] and [22]. Recall that $\mathcal{L}^*$ always denotes the adjoint operator of a linear operator $\mathcal{L}$ between Banach spaces.

2. Existence and Uniqueness Solution

Proposition 2.1. *Assume that $U_0 \in L^2(\Omega)$ and $f \in L^2(\mathcal{Q})$. Then the problem (1.1) admits an unique solution U such that $U \in \mathcal{W}$ and $\parallel U \parallel_{\mathcal{W}}^2 \leq \exp((\gamma+1)T)(\parallel U_0 \parallel_{L^2}^2 + \parallel f \parallel_{L^2(\mathcal{Q})}^2)$.*

Proof. We will just sketch the proof based on suitable a priori estimates. Multiplying (1.1) by U and integrating over Ω give

$$\frac{1}{2}\frac{\partial \parallel U \parallel_{L^2}^2}{\partial t} + < AU, U > + < F(U), U > + < K(V,U), U > = < f, U > .$$

Since $U \in \mathcal{D}$ and, by using the definition of the norm in $\mathcal{D}$ and the assumption (1.2.1), we obtain

$$\frac{\partial \parallel U \parallel_{L^2}^2}{\partial t} + 2 < AU, U > \\ = -2 < F(U) - F(0), U - 0 > -2 < K(V,U), U > +2 < f, U > .$$

According to the assumptions (1.2.2) and (1.5) we have

$$\frac{\partial \parallel U \parallel_{L^2}^2}{\partial t} + \nu \parallel U \parallel_{\mathcal{D}}^2 \leq \gamma \parallel U \parallel_{L^2}^2 + 2 \parallel f \parallel_{L^2} \parallel U \parallel_{L^2} .$$

Integrating over $(0,t)$, it gives

$$\parallel U \parallel_{L^2}^2 + \nu \int_0^t \parallel U \parallel_{\mathcal{D}}^2$$
$$\leq \gamma \int_0^t \parallel U \parallel_{L^2}^2 + 2 \parallel f \parallel_{L^2(\mathcal{Q})} (\int_0^t \parallel U \parallel_{L^2}^2)^{1/2} + \parallel U_0 \parallel_{L^2}^2 ,$$

and then (according to Corollary of Lemman of Gronwall, see Appendix)

$$\parallel U \parallel_{\mathcal{W}}^2 \leq \exp((\gamma+1)T)(\parallel U_0 \parallel_{L^2}^2 + \parallel f \parallel_{L^2(\mathcal{Q})}^2). \tag{2.1}$$

The proof of Theorem can be completed by implementing the Galerkin method and by taking advantage of the above estimate.

Uniqueness of the solution of (1.1) follows easily in view of the assumption (1.2.2). □

Proposition 2.2. *Let U_0, V_0 be two functions of $L^2(\Omega)$ and let f_1, f_2 be two functions of $L^2(\mathcal{Q})$. If U_1 (resp. U_2) is solution of (1.1), where the forcing-initial condition is (f_1, U_0) (resp. (f_2, V_0)) then*

$$\parallel U_1 - U_2 \parallel_{\mathcal{W}} \leq \exp((\gamma+1)T/2)(\parallel U_0 - V_0 \parallel_{L^2}^2 + \parallel f_1 - f_2 \parallel_{L^2(\mathcal{Q})}^2)^{1/2}.$$

Proof. By using the same way to obtain the estimation (2.1), we obtain the result of the proposition.

In the following, this solution U will be treated as the target function. We are then interesting in the robust regulation of the deviation of the problem from the desired targed U. We analyse the full nonlinear equation which models large perturbations u to the target U. Hence we consider the equation:

$$\begin{aligned} &\frac{\partial u}{\partial t} + Au + F(u+U) - F(U) + K(V,u) = g \quad \text{on} \quad \mathcal{Q}, \\ &u(t=0) = u_0 \quad \text{on} \quad \Omega. \end{aligned} \tag{2.2}$$

If we set $\tilde{F}(.,y) = F(.,y+U) - F(.,U)$ then (2.2) reduce to

$$\begin{aligned} &\frac{\partial u}{\partial t} + Au + \tilde{F}(u) + K(V,u) = g \quad \text{on} \quad \mathcal{Q}, \\ &u(t=0) = u_0 \quad \text{on} \quad \Omega. \end{aligned} \tag{2.3}$$

Remark 2.1. (i) We verify easily that $\tilde{F}$ satisfies the same hypothesis that F, i.e. (1.2).

(ii) For simplicity of future reference, we omit the "$\tilde{}$" on $\tilde{F}$ for (2.3).

The problem (2.3) is the same that (1.1) so we have then the following proposition.

Proposition 2.3. *(i) Assume that $g \in L^2(\mathcal{Q})$ and $u_0 \in L^2(\Omega)$. Then the problem (2.3) admits a unique solution u such that $u \in \mathcal{W}$ and $\| u \|_{\mathcal{W}}^2 \leq \exp((\gamma+1)T)(\| u_0 \|_{L^2}^2 + \| g \|_{L^2(\mathcal{Q})}^2)$.*

(ii) Let u_0, v_0 be two functions of $L^2(\Omega)$ and let g_1, g_2 be two functions of $L^2(\mathcal{Q})$. If u_1 (resp. u_2) is solution of (2.3), where the forcing-initial condition is (g_1, u_0) (resp. (g_2, v_0)) then

$$\| u_1 - u_2 \|_{\mathcal{W}} \leq \exp((\gamma+1)T/2)(\| u_0 - v_0 \|_{L^2}^2 + \| g_1 - g_2 \|_{L^2(\mathcal{Q})}^2)^{1/2}.$$

3. Study of the Control Framework

In the control framework, the value g is decomposed into the disturbance $\psi \in L^2(\mathcal{Q})$ and the control $\phi \in L^2(\mathcal{Q})$, i.e. $g = B_1\phi + B_2\psi$, where $B_i, i = 1, 2$ are given (linear) bounded operators on $L^2(\Omega)$:

$$\text{there exists } b_i > 0 \text{ such that } \forall h_i \in L^2(\Omega),$$
$$\| B_i h_i \|_{L^2}^2 \leq b_i^2 \| h_i \|_{L^2}^2, \quad i = 1, 2. \quad (3.1)$$

Remark 3.1. If we put $b = \max(b_1, b_2)$ and $\mathbf{h} = (h_1, h_2)$ then $b_1^2 \| h_1 \|_{L^2}^2 + b_2^2 \| h_2 \|_{L^2}^2 \leq b^2 \| \mathbf{h} \|_{L^2}^2$, where $\| \mathbf{h} \|_{L^2}^2 = \| h_1 \|_{L^2}^2 + \| h_2 \|_{L^2}^2$.

The objective in the robust control problem is to find the best control ϕ in the presence of the disturbance ψ which maximally spoils the control objective. The function u is assumed to be related to the disturbance ψ and control ϕ through the problem (2.3):

$$\begin{aligned} &\frac{\partial u}{\partial t} + Au + F(u) + K(V, u) = B_1\phi + B_2\psi \quad \text{on} \quad \mathcal{Q}, \\ &u(t=0) = u_0 \quad \text{(given) on} \quad \Omega. \end{aligned} \quad (3.2)$$

To obtain the regularity of Proposition 2.3, we suppose the following hypothesis: $u_0 \in L^2(\Omega)$, $(\phi, \psi) \in L^2(\mathcal{Q})^2$ and $U \in \mathcal{W}$. Let $\mathbf{U} : (\phi, \psi) \longrightarrow u =$

$\mathbf{U}(\phi,\psi)$ be the map: $(L^2(\mathcal{Q}))^2 \longrightarrow \mathcal{W}$ defined by (3.2). We introduce the cost function defined by

$$J(\phi,\psi) = \frac{1}{2} \parallel \mathcal{C}(u-u_{obs}) \parallel^2_{L^2(\mathcal{Q})} + \frac{\mu}{2} \parallel u(T) - v_{obs} \parallel^2_{L^2} + \frac{\alpha}{2} \parallel \phi \parallel^2_{L^2(\mathcal{Q})} - \frac{\beta}{2} \parallel \psi \parallel^2_{L^2(\mathcal{Q})}, \quad (3.3)$$

where $\mu, \alpha, \beta > 0$ are fixed, $(u_{obs}, v_{obs}) \in \mathcal{V} \times L^2(\Omega)$ is the observation (given) and $\mathcal{C}$ is unbounded, linear operator on $L^2(\Omega)$ satisfying:

$$\parallel \mathcal{C}v \parallel^2_{L^2} \leq \delta_1 \parallel v \parallel^2_{L^2} + \delta_2 \parallel v \parallel^2_{\mathcal{D}}, \quad \forall v \in \mathcal{D}. \quad (3.4)$$

The robust control problem then is to minimize the fonctionnal J with respect to ϕ and maximalize J with respect to ψ, i.e., to find $(\phi^*, \psi^*) \in \mathcal{U}_{ad} \times \mathcal{V}_{ad}$ such that

$$J(\phi^*,\psi) \leq J(\phi^*,\psi^*) \leq J(\phi,\psi^*), \forall (\phi,\psi) \in \mathcal{U}_{ad} \times \mathcal{V}_{ad}, \quad (3.5)$$

with $\mathcal{U}_{ad}$ and $\mathcal{V}_{ad}$ are (given) nonempty, closed, convex, bounded subsets of $L^2(\mathcal{Q})$.

Proposition 3.1. *Let F satisfy the assumptions (1.2). Then the function $\mathbf{U} : (\phi,\psi) \longrightarrow u = \mathbf{U}(\phi,\psi)$ solution of (3.2) is continuously Fréchet differentiable from $(L^2(\mathcal{Q}))^2$ to $\mathcal{W}$ with the derivative $\mathbf{U}'(\phi,\psi) : (h_1,h_2) \longrightarrow w$ given by the linear parabolic problem*

$$\begin{aligned} &\frac{\partial w}{\partial t} + Aw + G(\mathbf{U}(\phi,\psi))w + K(V,w) = B_1 h_1 + B_2 h_2 \quad \text{on} \quad \mathcal{Q}, \\ &w(t=0) = 0 \quad \text{on} \quad \Omega, \end{aligned} \quad (3.6)$$

and satisfies the following estimates ($\forall \Phi_i = (\phi_i,\psi_i) \in (L^2(\mathcal{Q}))^2, i = 1,2$):

(i) $\parallel \mathbf{U}'(\phi_1,\psi_1) \parallel_{\mathcal{L}((L^2(\mathcal{Q}))^2,\mathcal{W})} \leq \exp((\gamma+1)T/2)b,$

(ii) $\parallel \mathbf{U}'(\phi_1,\psi_1) - \mathbf{U}'(\phi_2,\psi_2) \parallel_{\mathcal{L}((L^2(\mathcal{Q}))^2,\mathcal{W})} \leq 2\exp(2(\gamma+1)T)\lambda c_e^2 b^2 \parallel \Phi_1 - \Phi_2 \parallel_{L^2(\mathcal{Q})}.$

Proof. Let $(\phi,\psi,h_1,h_2) \in (L^2(\mathcal{Q}))^4$, let $u = \mathbf{U}(\phi,\psi)$ and $u_h = \mathbf{U}(\phi + h_1, \psi + h_2) = u + w_h$. Let v_h be the solution of

$$\begin{aligned} &\frac{\partial v_h}{\partial t} + Av_h + G(u)v_h + K(V,v_h) = B_1 h_1 + B_2 h_2 \quad \text{on} \quad \mathcal{Q}, \\ &v_h(t=0) = 0 \quad \text{on} \quad \Omega. \end{aligned} \quad (3.7)$$

Multiplying (3.7) by v_h and integrating over $(0,t)\times\Omega$ we obtain (according to $-2G(u)\leq\gamma_0$)

$$\| v_h \|_{L^2}^2 + \nu \int_0^t \| v_h \|_{\mathcal{D}}^2 \leq \gamma \int_0^t \| v_h \|_{L^2}^2 + 2(\| B_1 h_1 \|_{L^2(\mathcal{Q})} + \| B_2 h_2 \|_{L^2(\mathcal{Q})})(\int_0^t \| v_h \|_{L^2}^2)^{1/2}.$$

By using Gronwall formula we have then (according to (3.1)):

$$\| v_h \|_{\mathcal{W}}^2 \leq \exp((\gamma+1)T) b^2 \| \mathbf{h} \|_{L^2(\mathcal{Q})}^2 \quad \text{where } \mathbf{h} = (h_1, h_2). \tag{3.8}$$

This shows that the map: $\mathbf{h}=(h_1,h_2)\longrightarrow v_h$ solution of (3.7) is continuous. Moreover by using Proposition 2.3 and (3.1) we deduce easily that:

$$\| w_h \|_{\mathcal{W}}^2 \leq \exp((\gamma+1)T) b^2 \| \mathbf{h} \|_{L^2(\mathcal{Q})}^2 \quad \text{where } \mathbf{h} = (h_1, h_2). \tag{3.9}$$

Let $v = w_h - v_h$ and according to the equations satisfy by u, u_h and v_h we have:

$$\begin{aligned} &\frac{\partial v}{\partial t} + Av + G(u)v + K(V,v) = g \quad \text{on} \quad \mathcal{Q}, \\ &v(t=0) = 0 \quad \text{on} \quad \Omega, \end{aligned} \tag{3.10}$$

where $g=-(F(u+w_h)-F(u))+G(u)w_h$. Multiplying (3.10) by v and integrating over $(0,t)\times\Omega$ we obtain (according to $-2G(u)\leq\gamma_0$)

$$\| v \|_{L^2}^2 + \nu \int_0^t \| v \|_{\mathcal{D}}^2 \leq \gamma \int_0^t \| v \|_{L^2}^2 + 2\int_0^t < g, v > (s) ds. \tag{3.11}$$

We are now to estimate the term $2\int_0^t < g,v > (s)ds$.

By using a simple manipulation we obtain that $g = \int_0^1 (G(u) - G(u + sw_h))w_h ds$. Applying the assumption (1.2.3) we have then $2 \mid g \mid\leq \lambda \mid w_h \mid^2$. So $2\int_0^t < g,v > (s)ds \leq \lambda \int_0^t\int_\Omega \mid w_h \mid^2 \mid v \mid d\Omega ds$ and then $2\int_0^t < g,v > (s)ds \leq \lambda \| w_h \|_{L^2(0,T,L^4(\Omega))}^2 \| v \|_{L^\infty(0,T,L^2(\Omega))}$.

According to the assumption (1.3) we obtain

$$2\int_0^t < g,v > (s)ds \leq \lambda c_e^2 \| w_h \|_{\mathcal{W}}^2 \| v \|_{\mathcal{W}} . \tag{3.12}$$

From (3.11) and (3.12), we deduce that

$$\| v \|_{L^2}^2 + \nu \int_0^t \| v \|_{\mathcal{D}}^2 \leq \gamma \int_0^t \| v \|_{L^2}^2 + \lambda c_e^2 \| w_h \|_{\mathcal{W}}^2 \| v \|_{\mathcal{W}} .$$

Using the Gronwall formula we have then

$$\| v \|_{\mathcal{W}}^2 \leq \exp((\gamma+1)T)\lambda c_e^2 \| w_h \|_{\mathcal{W}}^2 \| v \|_{\mathcal{W}} .$$

According to (3.9) we can deduce that $\| v \|_{\mathcal{W}} \leq \exp(2(\gamma+1)T)\lambda c_e^2 b^2 \| \mathbf{h} \|_{L^2(\mathcal{Q})}^2$ and then

$$\| v \|_{\mathcal{W}} = o(\| \mathbf{h} \|_{L^2(\mathcal{Q})}).$$

Therefore $\mathbf{U}'(\phi,\psi)$ defined by (3.6), is the Fréchet derivative of $\mathbf{U}$ at point (ϕ,ψ) and verifies $\| \mathbf{U}'(\phi,\psi) \|_{\mathcal{L}((L^2(\mathcal{Q}))^2,\mathcal{W})} \leq \exp((\gamma+1)T/2)b$.

Let $\Phi_i = (\phi_i,\psi_i) \in (L^2(\mathcal{Q}))^2$, $i=1,2$ (given) and ($w_i = \mathbf{U}'(\phi_i,\psi_i)\mathbf{h}$, i$=1,2$) solution of the problem (3.6) (we denote by $u_i = \mathbf{U}(\phi_i,\psi_i)$, $i=1,2$ and by $\mathbf{h}(h_1,h_2)$).

Set $w = w_1 - w_2$ and according to the equations satisfy by w_1 and w_2 we have:

$$\begin{aligned} &\frac{\partial w}{\partial t} + Aw + G(u_1)w + K(V,w) = (G(u_2) - G(u_1))w_2 \quad \text{on} \quad \mathcal{Q}, \\ &w(t=0) = 0 \quad \text{on} \quad \Omega. \end{aligned} \tag{3.13}$$

Multiplying (3.13) by w and integrating over $(0,t)\times\Omega$ we obtain (according to $-2G(u) \leq \gamma_0$ and (1.2))

$$\begin{aligned} &\| w \|_{L^2}^2 + \nu \int_0^t \| w \|_{\mathcal{D}}^2 \\ &\qquad \leq \gamma \int_0^t \| w \|_{L^2}^2 + 2\lambda \int_0^t \int_\Omega | u_2 - u_1 || w_2 || w | \, d\Omega ds. \end{aligned}$$

By using Hölder inequality and the relationship (1.3) we have

$$\begin{aligned} &\| w \|_{L^2}^2 + \nu \int_0^t \| w \|_{\mathcal{D}}^2 \\ &\qquad \leq \gamma \int_0^t \| w \|_{L^2}^2 + 2\lambda c_e^2 \| u_2 - u_1 \|_{\mathcal{W}} \| w_2 \|_{\mathcal{W}} \| w \|_{\mathcal{W}} . \end{aligned}$$

Using the Gronwall formula we have then

$$\| w \|_{\mathcal{W}} \leq 2\exp((\gamma+1)T)\lambda c_e^2 \| u_2 - u_1 \|_{\mathcal{W}} \| w_2 \|_{\mathcal{W}} .$$

According to Proposition 2.3 and (3.8) we can deduce that

$$\| w \|_{\mathcal{W}} \leq 2\exp(2(\gamma+1)T)\lambda c_e^2 b^2 \| \Phi_1 - \Phi_2 \|_{(L^2(\mathcal{Q}))^2} \| \mathbf{h} \|_{(L^2(\mathcal{Q}))^2} .$$

Therefore

$$\| \mathbf{U}'(\phi_1,\psi_1) - \mathbf{U}'(\phi_2,\psi_2) \|_{\mathcal{L}((L^2(\mathcal{Q}))^2,\mathcal{W})} \leq 2\exp(2(\gamma+1)T)\lambda c_e^2 b^2 \| \Phi_1 - \Phi_2 \|_{(L^2(\mathcal{Q}))^2} . \quad \square$$

Proposition 3.2. *Let F satisfy the assumptions (1.2). Then for each $t \in [0,T]$, the function $\mathbf{V}_t : (\phi,\psi) \longrightarrow u(t) = \mathbf{V}_t(\phi,\psi)$ solution of (3.2) is continuously Fréchet differentiable from $(L^2(\mathcal{Q}))^2$ to $L^2(\Omega)$ with the derivative $\mathbf{V}_t'(\phi,\psi) : (h_1,h_2) \longrightarrow w(t)$ given by the linear parabolic problem*

$$\begin{aligned} &\frac{\partial w}{\partial t} + Aw + G(\mathbf{V}_t(\phi,\psi))w + K(V,w) = B_1 h_1 + B_2 h_2 \quad on \quad \mathcal{Q}\,, \\ &w(t=0) = 0 \quad on \quad \Omega\,, \end{aligned} \tag{3.14}$$

and satisfies the estimates ($\forall \Phi_i = (\phi_i,\psi_i) \in (L^2(\mathcal{Q}))^2, i = 1,2$):

(i) $\| \mathbf{V}_t'(\phi_1,\psi_1) \|_{\mathcal{L}((L^2(\mathcal{Q}))^2,L^2(\Omega))} \leq \exp((\gamma+1)T/2)b$,

(ii) $\| \mathbf{V}_t'(\phi_1,\psi_1) - \mathbf{V}_t'(\phi_2,\psi_2) \|_{\mathcal{L}((L^2(\mathcal{Q}))^2,L^2(\Omega))} \leq 2\exp(2(\gamma+1)T)\lambda c_e^2 b^2 \| \Phi_1 - \Phi_2 \|_{L^2(\mathcal{Q})}$.

Proof. The fonctionnal $(\phi,\psi) \longrightarrow u(t)$ is continuous from $L^2(0,T,L^2(\Omega))$ to $L^2(\Omega)$ (corollary of Proposition 2.3). The rest of the proposition is a corollary of Proposition 3.1. $\square$

Proposition 3.3. *Let F satisfies the assumptions (1.2). Then the maps $\mathbf{U}$ and $\mathbf{V}_t$ defined by (3.2) are continuous from the weak topology of $(L^2(\mathcal{Q}))^2$ to the strong topology of $L^2(\mathcal{Q})$ and the weak topology of $L^2(\Omega)$, respectively.*

Proof. Let $\Phi = (\phi,\psi)$ be given in $(L^2(\mathcal{Q}))^2$ and let the sequence $\Phi_k = (\phi_k,\psi_k)$ such that Φ_k is weakly convergent in $(L^2(\mathcal{Q}))^2$ to Φ.

Set $u = \mathbf{U}(\phi,\psi)$, $u_k = \mathbf{U}(\phi_k,\psi_k)$ and $v_k = u - u_k$. Since $\Phi_k \rightharpoonup \Phi$ weakly in $(L^2(\mathcal{Q}))^2$ then the sequence Φ_k is uniformly bounded in $(L^2(\mathcal{Q}))^2$ and therefore (according to Proposition 2.3) u_k is uniformly bounded in $\mathcal{W}$. By using the assumption (1.2.2) we deduce that $F(u_k)$ is uniformly bounded in $L^2(\mathcal{Q})$.

Using this result and the equation (3.2) we obtain easily that $\frac{\partial u_k}{\partial t}$ is uniformly bounded in $L^1(0,T,\mathcal{D}')$. Let us introduce the space $\mathcal{Y} = \{v \in L^2(0,T,\mathcal{D}), \frac{\partial v}{\partial t} \in L^1(0,T,\mathcal{D}')\}$. According to Lions [18] (for example), the injection of $\mathcal{Y}$ into $L^2(0,T,L^2(\Omega))$ is compact. Therefore u_k is uniformly bounded in $\mathcal{Y}$. This result makes it possible to extract from $(u_k,\phi_k,\psi_k,F(u_k))$ a subsequence also denoted by $(u_k,\phi_k,\psi_k,F(u_k))$ and such that:

$$\begin{aligned} &(\phi_k,\psi_k) \rightharpoonup (\phi,\psi) \quad \text{weakly in} \quad (L^2(\mathcal{Q}))^2, \\ &u_k \rightharpoonup \tilde{u} \quad \text{weakly in} \quad L^2(0,T,\mathcal{D}), \\ &u_k \longrightarrow \tilde{u} \quad \text{weakly in} \quad L^2(\mathcal{Q}), \\ &F(u_k) \longrightarrow F(\tilde{u}) \quad \text{strongly in} \quad L^2(\mathcal{Q})(^1)\,. \end{aligned} \tag{3.15}$$

We proof easily that $\tilde{u}$ is solution of (3.2) with a forcing (ϕ, ψ) and according to the uniqueness of the solution of (3.2), we have then $\tilde{u} = u = \mathbf{U}(\phi, \psi)$.

In the same way we prove that for each $t \in [0, T]$, $\mathbf{V}_t(\phi_k, \psi_k) \rightharpoonup \mathbf{V}_t(\phi, \psi)$ weakly in $L^2(\Omega)$. □

Theorem 3.1. *Let F satisfy the assumptions (1.2). Then, for α and β sufficiently large, there exists $(\phi^*, \psi^*) \in (L^2(\mathcal{Q}))^2$ and $u^* \in \mathcal{W}$ such that (ϕ^*, ψ^*) is defined by (3.5) and $u^* = \mathbf{U}(\phi^*, \psi^*)$ solution of (3.2).*

Proof. Let P_ψ be the map: $\phi \longrightarrow J(\phi, \psi)$ and Q_ϕ be the map: $\psi \longrightarrow J(\phi, \psi)$. To obtain the exitence of the robust control problem we prove that P_ψ is convex and lower semicontinuous for all $\psi \in \mathcal{V}_{ad}$, and P_ϕ is concave and upper semicontinuous for all $\phi \in \mathcal{U}_{ad}$ and we use the classical minimax theorem in infinite dimensions (see, e.g., [3] and [12]).

Firstly we prove that for α and β sufficiently large we have the convexity of the map P_ψ and the concavity of the map Q_ϕ. In order to prove the convexity, it is sufficient to show that for all $(\phi_1, \phi_2) \in \mathcal{U}_{ad}$ we have: $(P'_\psi(\phi_1) - P'_\psi(\phi_2)).\phi \geq 0$, where $\phi = \phi_1 - \phi_2$. According to the definition of J, we have that:

$$\begin{aligned}
&(P'_\psi(\phi_1) - P'_\psi(\phi_2)).\phi \\
&\qquad = \int_0^T < C(u_1 - u_2), Cw_2 > dt + \mu < u_1(T) - u_2(T), w_{2T} > \\
&+ \int_0^T < C(u_1 - u_{obs}), C(w_1 - w_2) > dt + \mu < u_1(T) - v_{obs}, w_{1T} - w_{2T} > \\
&\qquad\qquad + \alpha \parallel \phi \parallel^2_{L^2(\mathcal{Q})}, \quad (3.16)
\end{aligned}$$

where $u_i = U(\phi_i, \psi)$, $w_i = U'(\phi_i, \psi).(\phi, 0)$ and $w_{iT} = V'_T(\phi_i, \psi).(\phi, 0)$, for $i = 1, 2$. According to (1.4), (3.4) and the result of Proposition 2.3 and Proposition 3.1 we have

$$\begin{aligned}
&\int_0^T < C(u_1 - u_2), Cw_2 > dt + \mu < u_1(T) - u_2(T), w_{2T} > dt \\
&\quad \leq \parallel C(u_1 - u_2) \parallel_{L^2(\mathcal{Q})} \parallel Cw_2 \parallel_{L^2(\mathcal{Q})} + \mu \parallel u_1(T) - u_2(T) \parallel_{L^2} \parallel w_{2T} \parallel_{L^2} \\
&\quad \leq (\delta_1 c_I^2 + \delta_2) \parallel u_1 - u_2 \parallel_{\mathcal{V}} \parallel w_2 \parallel_{\mathcal{V}} + \mu \parallel u_1(T) - u_2(T) \parallel_{L^2} \parallel w_{2T} \parallel_{L^2} \\
&\qquad \leq (\mu + \delta_1 c_I^2 + \delta_2) \exp((\gamma + 1)T) b \parallel \phi \parallel^2_{L^2(\mathcal{Q})} \\
&\qquad\qquad \leq C_0(\mu, \delta_1, c_1, \delta_2, \gamma, T, b) \parallel \phi \parallel^2_{L^2(\mathcal{Q})} \quad (3.17)
\end{aligned}$$

[1]The operator F is continuous then $F(u_k) \longrightarrow F(\tilde{u})$ strongly in $L^2(\mathcal{Q})$ (according to (3.15.3)).

and

$$\int_0^T < C(u_1 - u_{obs}), C(w_1 - w_2) > dt + \mu < u_1(T) - v_{obs}, w_{1T} - w_{2T} >$$
$$\leq \| C(u_1 - u_{obs}) \|_{L^2(\mathcal{Q})} \| C(w_1 - w_2) \|_{L^2(\mathcal{Q})}$$
$$+ \mu \| u_1(T) - v_{obs} \|_{L^2} \| w_{1T} - w_{2T} \|_{L^2}$$
$$\leq (\delta_1 c_I^2 + \delta_2) \| u_1 - u_{obs} \|_{\mathcal{V}} \| w_1 - w_2 \|_{\mathcal{V}}$$
$$+ \mu \| u_1(T) - v_{obs} \|_{L^2} \| w_{1T} - w_{2T} \|_{L^2}$$
$$\leq 2((\delta_1 c_I^2 + \delta_2) \| u_1 - u_{obs} \|_{\mathcal{V}}$$
$$+ \mu \| u_1(T) - v_{obs} \|_{L^2}) \exp(2(\gamma + 1)T) \lambda c_e^2 b^2 \| \phi \|^2_{L^2(\mathcal{Q})}$$
$$\leq C_1(\mu, \delta_1, c_I, c_e, \lambda, \delta_2, \gamma, T, b) C_2(u_{obs}, v_{obs}, u_0) \| \phi \|^2_{L^2(\mathcal{Q})} \quad (3.18)$$

From (3.16)-(3.18) we deduce that for $\alpha \geq \alpha_l = C_0 + C_1 C_2$ we have $(P'_\psi(\phi_1) - P'_\psi(\phi_2)).\phi \geq 0$ and then the convexity of P_ψ. In the same way, we can find β_l such that for $\beta \geq \beta_l$ we have the concavity of Q_ϕ.

We prove now that P_ψ is lower semicontinuous for all $\psi \in \mathcal{V}_{ad}$, and P_ϕ is upper semicontinuous for all $\phi \in \mathcal{U}_{ad}$.

Let ϕ_k be a minimizing sequence of J, i.e.

$$\liminf_k J(\phi_k, \psi) = \min_{\phi \in L^2(\mathcal{Q})} J(\phi, \psi) \ (\forall \psi \in \mathcal{V}_{ad}).$$

Then ϕ_k is uniformly bounded in $\mathcal{U}_{ad}$ and we can extract from ϕ_k a subsequence also denoted by ϕ_k such that $\phi_k \rightharpoonup \phi_\psi$ weakly in $\mathcal{U}_{ad}$. By using Proposition 3.3 we have then

$$\begin{array}{l} \mathbf{U}(\phi_k, \psi) \rightharpoonup u_\psi = \mathbf{U}(\phi_\psi, \psi) \ \text{ weakly in } \ L^2(0, T, \mathcal{D}), \\ \mathbf{U}(\phi_k, \psi) \longrightarrow u_\psi \ \text{ strongly in } \ L^2(\mathcal{Q}), \\ \mathbf{V}_t(\phi_k, \psi) \rightharpoonup u_\psi(t) = \mathbf{V}_t(\phi_\psi, \psi) \ \text{ weakly in } \ L^2(\Omega), \forall t \in [0, T]. \end{array}$$

Therefore, since the norm is lower semicontinuous we have that the map $P_\psi : \phi \longrightarrow J(\phi, \psi)$ is lower semicontinuous for all $\psi \in \mathcal{V}_{ad}$. By using the same technique we obtain then P_ϕ is upper semicontinuous for all $\phi \in \mathcal{U}_{ad}$. □

In order to characterize the solution of the robust control problem, we introduce the "adjoint" problem corresponding to the primal problem (3.2) (we denote by $u = \mathbf{U}(\phi, \psi)$ the solution of problem (3.2) with the forcing is (ϕ, ψ)):

$$\begin{array}{l} -\dfrac{\partial \tilde{u}}{\partial t} + A\tilde{u} + (G(u))^* \tilde{u} + K^*(V, \tilde{u}) = \mathcal{C}^* \mathcal{C}(u - u_{obs}) \ \text{ on } \ \mathcal{Q}, \\ \tilde{u}(t = T) = \mu(u(t = T) - v_{obs}) \ \text{ on } \ \Omega, \end{array} \quad (3.19)$$

where $\mathcal{C}^*$ (resp. $(G(u))^*$) is the adjoint of the operator $\mathcal{C}$ (resp. $G(u)$).

Proposition 3.4. *Let F satisfy the assumptions (1.2), $u \in \mathcal{W}$ and $\nu > \delta_2$ then the solution of (3.19) is in $\mathcal{W}$ and satisfies the following estimate:* $\| \tilde{u} \|^2_{\mathcal{H}} + (\nu - \delta_2) \| \tilde{u} \|^2_{\mathcal{V}} \leq \exp((\delta_1 + \gamma + 1)T)(\mu^2 \| u(T) - v_{obs} \|^2_{L^2} + \| \mathcal{C}(u - u_{obs}) \|^2_{L^2(\mathcal{Q})})$.

Proof. Multiplying (3.19) by $\tilde{u}$ and integrating over $(t,T)\times\Omega$, we obtain (according to $-2G(u) \leq \gamma_0$)

$$\| \tilde{u} \|^2_{L^2} + \nu \int_t^T \| \tilde{u} \|^2_{\mathcal{D}} \leq \mu^2 \| u(T) - v_{obs} \|^2_{L^2} + \gamma \int_t^T \| \tilde{u} \|^2_{L^2} + \int_t^T \| \mathcal{C}(u - u_{obs}) \|^2_{L^2} + \int_t^T \| \mathcal{C}\tilde{u} \|^2_{L^2} .$$

According to (3.4) we obtain

$$\| \tilde{u} \|^2_{L^2} + \nu \int_t^T \| \tilde{u} \|^2_{\mathcal{D}} \leq \mu^2 \| u(T) - v_{obs} \|^2_{L^2} + (\delta_1 + \gamma) \int_t^T \| \tilde{u} \|^2_{L^2} + \delta_2 \int_t^T \| \tilde{u} \|^2_{\mathcal{D}} + \| \mathcal{C}(u - u_{obs}) \|^2_{L^2(\mathcal{Q})} ,$$

and then

$$\| \tilde{u} \|^2_{L^2} + (\nu - \delta_2) \int_t^T \| \tilde{u} \|^2_{\mathcal{D}} \leq \mu^2 \| u(T) - v_{obs} \|^2_{L^2} + \| \mathcal{C}(u - u_{obs}) \|^2_{L^2(\mathcal{Q})} + (\delta_1 + \gamma) \int_t^T \| \tilde{u} \|^2_{L^2} .$$

By using Gronwall formula we have then:

$$\| \tilde{u} \|^2_{\mathcal{H}} + (\nu - \delta_2) \| \tilde{u} \|^2_{\mathcal{V}} \leq \exp((\delta_1 + \gamma + 1)T)(\mu^2 \| u(T) - v_{obs} \|^2_{L^2} + \| \mathcal{C}(u - u_{obs}) \|^2_{L^2(\mathcal{Q})}). \quad \square$$

We will denote in the sequel by $\tilde{\mathbf{U}} : (\phi, \psi) \longrightarrow \tilde{u} = \tilde{\mathbf{U}}(\phi, \psi)$ the map defined by (3.19).

We can now give the optimality system for the robust control problem (3.5).

Theorem 3.2. *Let F satisfy the assumptions (1.2), $\nu > \delta_2$, and $(\phi^*, \psi^*, u^*) \in (L^2(\mathcal{Q}))^2 \times \mathcal{W}$ such that (ϕ^*, ψ^*) is defined by (3.5) and $u^* = \mathbf{U}(\phi^*, \psi^*)$ is solution of (3.2). Then (for α and β sufficiently large) $\int_0^T \int_\Omega (\alpha\phi^* + B_1^*\tilde{u})(\phi - \phi^*) d\Omega dt \geq 0$, and $\int_0^T \int_\Omega (-\beta\psi^* + B_2^*\tilde{u})(\psi - \psi^*) d\Omega dt \leq 0$ $\forall (\phi, \psi) \in \mathcal{U}_{ad} \times \mathcal{V}_{ad}$, where $\tilde{u} = \tilde{\mathbf{U}}(\phi^*, \psi^*)$ is solution of adjoint problem (3.19).*

Proof. The cost fonctionnal J is a composition of Fréchet differentiable maps then J is Fréchet differentiable and we have $(\forall \mathbf{h} = (h_1, h_2) \in (L^2(\mathcal{Q}))^2)$:

$$\begin{aligned} J'(\phi,\psi)\mathbf{h} \\ &= \int_0^T < C(u - u_{obs}), C\mathbf{U}'(\phi,\psi)\mathbf{h} > + < \mu(u(T) - v_{obs}), \mathbf{V}'_T(\phi,\psi)\mathbf{h} > \\ &\qquad + \int_0^T (< \alpha\phi, h_1 > - < \beta\psi, h_2 >), \end{aligned}$$

and then

$$\begin{aligned} J'(\phi,\psi)\mathbf{h} \\ &= \int_0^T < \mathcal{C}^*\mathcal{C}(u - u_{obs}), \mathbf{U}'(\phi,\psi)\mathbf{h} > + < \mu(u(T) - v_{obs}), \mathbf{V}'_T(\phi,\psi)\mathbf{h} > \\ &\qquad + \int_0^T (< \alpha\phi, h_1 > - < \beta\psi, h_2 >). \end{aligned}$$

Multiplying (3.6) by $\tilde{u}$, integrating over $\mathcal{Q}$ and (integrating) by parts in time t, we obtain $(\forall \mathbf{h} = (h_1, h_2) \in (L^2(\mathcal{Q}))^2)$:

$$\begin{aligned} \int_0^T (< B_1^*\tilde{u}, h_1 > + < B_2^*\tilde{u}, h_2 >) &=< \mathbf{V}'_T(\phi^*,\psi^*)\mathbf{h}, \tilde{u}(T) > \\ &+ \int_0^T < -\frac{\partial \tilde{u}}{\partial t} + A\tilde{u} + (G(u))^*\tilde{u} + K^*(V,\tilde{u}), \mathbf{U}'(\phi^*,\psi^*)\mathbf{h} >. \end{aligned}$$

Since $\tilde{u}$ is solution of adjoint problem (3.19) we obtain then $((\forall \mathbf{h} = (h_1, h_2) \in (L^2(\mathcal{Q}))^2)$:

$$\begin{aligned} \int_0^T (< B_1^*\tilde{u}, h_1 > + < B_2^*\tilde{u}, h_2 >) &=< \mathbf{V}'_T(\phi^*,\psi^*)\mathbf{h}, \mu(u^*(T) - v_{obs}) > \\ &+ \int_0^T < \mathcal{C}^*\mathcal{C}(u^* - u_{obs}), \mathbf{U}'(\phi^*,\psi^*)\mathbf{h} >. \end{aligned} \tag{3.20}$$

As (ϕ^*, ψ^*) is solution of (3.5) then $J'(\phi^*,\psi^*)(\phi - \phi^*, 0) \geq 0$ and $J'(\phi^*,\psi^*)(0, \psi - \psi^*) \leq 0$, $\forall(\phi,\psi) \in \mathcal{U}_{ad} \times \mathcal{V}_{ad}$, and therefore (according to (3.20) and to the expression of J') we deduce that $\int_0^T \int_\Omega (\alpha\phi^* + B_1^*\tilde{u})(\phi - \phi^*) d\Omega dt \geq 0$, and $\int_0^T \int_\Omega (-\beta\psi^* + B_2^*\tilde{u})(\psi - \psi^*) d\Omega dt \leq 0$ $\forall(\phi,\psi) \in \mathcal{U}_{ad} \times \mathcal{V}_{ad}$. So the proof is complete. □

We will assume in the sequel that there exists (ϕ^*, ψ^*, u^*) such that (ϕ^*, ψ^*) is defined by (3.5), $u^* = \mathbf{U}(\phi^*, \psi^*)$ is solution of (3.2) and $\int_0^T \int_\Omega (\alpha\phi^* + B_1^*\tilde{u})(\phi - \phi^*) d\Omega dt \geq 0$, $\int_0^T \int_\Omega (-\beta\psi^* + B_2^*\tilde{u})(\psi - \psi^*) d\Omega dt \leq 0 \ \forall (\phi, \psi) \in \mathcal{U}_{ad} \times \mathcal{V}_{ad}$ (where $\tilde{u} = \tilde{\mathbf{U}}(\phi^*, \psi^*)$ is solution of (3.19)). We give now some conditions to obtain the uniqueness of the solution (ϕ^*, ψ^*).

Theorem 3.3. *Suppose that F satisfies the assumptions (1.2), $\nu > \gamma_2$ and $\mu < 1$ holds. Then if:*

(i) $\theta = (\nu - \delta_2 - c_I^2(\gamma + \delta_1)) - 2b^2 c_I^2(\frac{1}{\alpha} + \frac{1}{\beta}) > 0$,

(ii) $\lambda c_e^2 \exp((\delta_1 + \gamma + 1)T/2)(\| u^*(T) - v_{obs} \|_{L^2}^2 + \| \mathcal{C}(u^* - u_{obs}) \|_{L^2(\mathcal{Q})}^2)^{1/2} < \theta$,

then, the solution (ϕ^*, ψ^*, u^*) is unique.

Proof. Suppose $({\phi^*}_1, {\psi^*}_1, u_1^*)$ is another solution, then $({\phi^*}_1, {\psi^*}_1)$ satisfies (3.5), $u_1^* = \mathbf{U}({\phi^*}_1, {\psi^*}_1)$ solution of (3.2) and $\int_0^T \int_\Omega (\alpha{\phi^*}_1 + B_1^*\tilde{u}_1)(\phi - {\phi^*}_1) d\Omega dt \geq 0$, $\int_0^T \int_\Omega (-\beta{\psi^*}_1 + B_2^*\tilde{u}_1)(\psi - {\psi^*}_1) d\Omega dt \leq 0 \ \forall (\phi, \psi) \in \mathcal{U}_{ad} \times \mathcal{V}_{ad}$ (where $\tilde{u}_1 = \tilde{\mathbf{U}}({\phi^*}_1, {\psi^*}_1)$ is solution of (3.19)).

Set $\phi = \phi^* - {\phi^*}_1$, $\psi = \psi^* - {\psi^*}_1$, $v = u^* - u_1^*$ and $\tilde{v} = \tilde{u} - \tilde{u}_1$ we then have:

$$\begin{aligned} &\frac{\partial v}{\partial t} + Av + (F(u^*) - F(u_1^*)) + K(V, v) = B_1\phi + B_2\psi \quad \text{on} \quad \mathcal{Q}, \\ &v(t = 0) = 0 \quad \text{on} \quad \Omega, \end{aligned} \tag{3.21}$$

$$\begin{aligned} &-\frac{\partial \tilde{v}}{\partial t} + A\tilde{v} + (G(u^*))^*\tilde{v} + K^*(V, \tilde{v}) = \mathcal{C}^*\mathcal{C}v - (G(u^*) - G(u_1^*))^*\tilde{u} \text{ on } \mathcal{Q}, \\ &\tilde{v}(t = T) = \mu v(t = T) \text{ on } \Omega \end{aligned} \tag{3.22}$$

and

$$\alpha \| \phi \|_{L^2(\mathcal{Q})}^2 + \int_0^T \int_\Omega B_1^*\tilde{v}\phi d\Omega dt \leq 0, \quad \beta \| \psi \|_{L^2(\mathcal{Q})}^2 - \int_0^T \int_\Omega B_2^*\tilde{v}\psi d\Omega dt \leq 0. \tag{3.23}$$

According to the assumptions (1.2) and (1.4) we have

$$\begin{aligned} &-2 < F(u^*) - F(u_1^*), v > \leq \gamma_0 c_I^2 \| v \|_{\mathcal{D}}^2, \\ &-2 < (G(u^*))^*\tilde{v}, \tilde{v} > \leq \gamma_0 c_I^2 \| \tilde{v} \|_{\mathcal{D}}^2, \\ &|< (G(u^*) - G(u_1^*))^*\tilde{u}, \tilde{v} >| \leq \lambda \int_\Omega | v || \tilde{v} || \tilde{u} | . \end{aligned} \tag{3.24}$$

By multiplying (3.21) by v, (3.22) by $\tilde{v}$ and integrating over $\mathcal{Q}$ gives (according to (3.23), (3.24) and (3.4))

$$\begin{aligned}
&\int_0^T \frac{\partial}{\partial t} \| v \|_{L^2}^2 + \nu \int_0^T \| v \|_{\mathcal{D}}^2 \\
&\quad \leq \gamma c_I^2 \int_0^T \| v \|_{\mathcal{D}}^2 + \frac{2}{\alpha} \| B_1^* \tilde{v} \|_{\mathcal{Q}} \| B_1^* v \|_{\mathcal{Q}} + \frac{2}{\beta} \| B_2^* \tilde{v} \|_{\mathcal{Q}} \| B_2^* v \|_{\mathcal{Q}}, \\
&\qquad - \int_0^T \frac{\partial}{\partial t} \| \tilde{v} \|_{L^2}^2 + \nu \int_0^T \| \tilde{v} \|_{\mathcal{D}}^2 \leq \gamma c_I^2 \int_0^T \| \tilde{v} \|_{\mathcal{D}}^2 \\
&\qquad + (\delta_1 c_I^2 + \delta_2) \int_0^T (\| v \|_{\mathcal{D}}^2 + \| \tilde{v} \|_{\mathcal{D}}^2) + 2\lambda \int_0^T \int_\Omega | \tilde{u} | \; | \tilde{v} | \; | v |, \\
&\qquad\qquad \tilde{v}(T) = \mu v(T) \text{ and } v(0) = 0.
\end{aligned}$$

By using Hölder inequality and the relationship (1.3) we obtain

$$\begin{aligned}
&\int_0^T \frac{\partial}{\partial t} \| v \|_{L^2}^2 + (\nu - \gamma c_I^2) \int_0^T \| v \|_{\mathcal{D}}^2 \\
&\qquad \leq 2\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) b^2 c_I^2 \int_0^T (\| v \|_{\mathcal{D}}^2 + \| \tilde{v} \|_{\mathcal{D}}^2), \\
&\qquad - \int_0^T \frac{\partial}{\partial t} \| \tilde{v} \|_{L^2}^2 + (\nu - \gamma c_I^2) \int_0^T \| \tilde{v} \|_{\mathcal{D}}^2 \\
&\qquad \leq (\delta_1 c_I^2 + \delta_2 + \lambda c_e^2 \| \tilde{u} \|_{\mathcal{H}}) \int_0^T (\| v \|_{\mathcal{D}}^2 + \| \tilde{v} \|_{\mathcal{D}}^2), \\
&\qquad\qquad \tilde{v}(T) = \mu v(T) \text{ and } v(0) = 0.
\end{aligned} \tag{3.25}$$

Adding (3.25.1) and (3.25.2) (according to $\nu > \delta_2$) we obtain

$$\begin{aligned}
&\int_0^T \frac{\partial}{\partial t} (\| v \|_{L^2}^2 - \| \tilde{v} \|_{L^2}^2) + \theta \int_0^T (\| v \|_{\mathcal{D}}^2 + \| \tilde{v} \|_{\mathcal{D}}^2) \\
&\qquad \leq \lambda c_e^2 \| \tilde{u} \|_{\mathcal{H}} \int_0^T (\| v \|_{\mathcal{D}}^2 + \| \tilde{v} \|_{\mathcal{D}}^2),
\end{aligned} \tag{3.26}$$

where $\theta = \nu - \delta_2 - c_I^2(\gamma + \frac{2b^2}{\alpha} + \frac{2b^2}{\beta} + \delta_1) > 0$ (according to the assumption (i)).

According to (3.25.3) we have

$$\begin{aligned}
&(1 - \mu^2) \| v(T) \|_{L^2}^2 + \| \tilde{v}(0) \|_{L^2}^2 + \theta(\| v \|_{\mathcal{V}}^2 + \| \tilde{v} \|_{\mathcal{V}}^2) \\
&\qquad \leq \lambda c_e^2 \| \tilde{u} \|_{\mathcal{H}} (\| v \|_{\mathcal{V}}^2 + \| \tilde{v} \|_{\mathcal{V}}^2).
\end{aligned}$$

Since $1-\mu^2>0$ then $\theta(\| v \|_{\mathcal{V}}^2 + \| \tilde{v} \|_{\mathcal{V}}^2) \le \lambda c_e^2 \| \tilde{u} \|_{\mathcal{H}} (\| v \|_{\mathcal{V}}^2 + \| \tilde{v} \|_{\mathcal{V}}^2)$.

By applying Proposition 3.4 and $\mu < 1$ we have

$$\| \tilde{u} \|_{L^\infty(0,T,L^2(\Omega))}^2 \le \exp((\delta_1+\gamma+1)T)(\| u^*(T) - v_{obs} \|_{L^2}^2 + \| \mathcal{C}(u^*-u_{obs}) \|_{L^2(\mathcal{Q})}^2)$$

and then $\theta^*(\| v \|_{\mathcal{V}}^2 + \| \tilde{v} \|_{\mathcal{V}}^2) \le 0$, where

$$\theta^* = \theta - \lambda c_e^2 \exp((\delta_1+\gamma+1)T/2)(\| u^*(T) - v_{obs} \|_{L^2}^2 + \| \mathcal{C}(u^*-u_{obs}) \|_{L^2(\mathcal{Q})}^2)^{1/2}.$$

Since $\theta^* > 0$ (according to the assumption (i)), we have $\| v \|_{\mathcal{V}}^2 + \| \tilde{v} \|_{\mathcal{V}}^2 = 0$ and then $v=0$ and $\tilde{v}=0$. We obtain then the uniqueness result. □

4. Study the Initial Condition Control

In this section the objective of the robust control problem is to find the best estimate of the initial state u_0 in the presence of the disturbance which maximally spoils the control objective. For this, we study two problems: firstly the case of the worst disturbance is in the initial condition and secondly the case, where the worst disturbance is in the forcing.

4.1. Distributed Disturbance in the Initial Condition

We suppose now that the value u_0 is decomposed into the disturbance $\psi \in L^2(\Omega)$ and the control $\phi \in L^2(\Omega)$, i.e. $u_0 = B_1\psi + B_2\phi$, where $B_i, i=1,2$ are given bounded operators on $L^2(\Omega)$.

So the function u is assumed to be related to the disturbance ψ and control ϕ through the problem (2.3):

$$\begin{aligned} &\frac{\partial u}{\partial t} + Au + F(u) + K(V,u) = g \quad \text{(given) on} \quad \mathcal{Q}, \\ &u(t=0) = B_1\psi + B_2\phi \quad \text{on} \quad \Omega. \end{aligned} \tag{4.1}$$

To obtain the regularity of Proposition 2.3, we suppose the following hypothesis: $g \in L^2(\mathcal{Q})$, $(\phi,\psi) \in L^2(\Omega)^2$ and $U \in \mathcal{W}$. Let $\mathbf{U} : (\phi,\psi) \longrightarrow u = \mathbf{U}(\phi,\psi)$ be the map: $(L^2(\Omega))^2 \longrightarrow \mathcal{W}$ defined by (4.1) and introduce the cost function defined by

$$J(\phi, \psi) = \frac{1}{2} \parallel \mathcal{C}(u - u_{obs}) \parallel^2_{L^2(\mathcal{Q})} + \frac{\mu}{2} \parallel u(T) - v_{obs} \parallel^2_{L^2} + \frac{\alpha}{2} \parallel \phi \parallel^2_{L^2} - \frac{\beta}{2} \parallel \psi \parallel^2_{L^2}, \quad (4.2)$$

where $\mu, \alpha, \beta > 0$ are fixed, $(u_{obs}, v_{obs}) \in \mathcal{V} \times L^2(\Omega)$ is the observation and $\mathcal{C}$ is unbounded, linear operator on $L^2(\Omega)$ satisfying the hypothesis (3.4).

We want to minimize the fonctionnal J with respect to ϕ and maximize J with respect to ψ, i.e., to find $(\phi^*, \psi^*) \in \mathcal{U}_{ad} \times \mathcal{V}_{ad}$ such that

$$J(\phi^*, \psi) \leq J(\phi^*, \psi^*) \leq J(\phi, \psi^*), \forall (\phi, \psi) \in \mathcal{U}_{ad} \times \mathcal{V}_{ad}\,, \quad (4.3)$$

with $\mathcal{U}_{ad}$ and $\mathcal{V}_{ad}$ are (given) nonempty, closed, convex, bounded subsets of $L^2(\Omega)$.

By using the same technique to obtain Proposition 2.3, Proposition 3.1 and the Theorem 3.1, we have the following results (with no further estimates required):

Proposition 4.1. *Let F satisfy the assumptions (1.2). Then the function $\mathbf{U} : (\phi, \psi) \longrightarrow u = \mathbf{U}(\phi, \psi)$ solution of (4.1) is continuously Fréchet differentiable from $(L^2(\Omega))^2$ to $\mathcal{W}$ with the derivative $\mathbf{U}'(\phi, \psi) : (h_1, h_2) \longrightarrow w$ given by the linear parabolic problem*

$$\begin{aligned} &\frac{\partial w}{\partial t} + Aw + G(\mathbf{U}(\phi, \psi))w + K(V, w) = 0 \quad \text{on} \quad \mathcal{Q}\,, \\ &w(t = 0) = B_1 h_1 + B_2 h_2 \quad \text{on} \quad \Omega\,, \end{aligned} \quad (4.4)$$

and satisfies the estimates ($\forall \Phi_i = (\phi_i, \psi_i) \in (L^2(\Omega))^2, i = 1, 2$):

(i) $\parallel \mathbf{U}'(\phi_1, \psi_1) \parallel_{\mathcal{L}((L^2(\Omega))^2, \mathcal{W})} \leq \exp((\gamma + 1)T/2)b,$

(ii) $\parallel \mathbf{U}'(\phi_1, \psi_1) - \mathbf{U}'(\phi_2, \psi_2) \parallel_{\mathcal{L}((L^2(\Omega))^2, \mathcal{W})} \leq 2 \exp(2(\gamma + 1)T) \lambda c_e^2 b^2 \parallel \Phi_1 - \Phi_2 \parallel^2_{L^2(\mathcal{Q})}.$

Proposition 4.2. *Let F satisfy the assumptions (1.2). Then for each $t \in [0, T]$, the function $\mathbf{V}_t : (\phi, \psi) \longrightarrow u(t) = \mathbf{V}_t(\phi, \psi)$ solution of (4.1) is continuously Fréchet differentiable from $(L^2(\Omega))^2$ to $L^2(\Omega)$ with the derivative $\mathbf{V}'_t(\phi, \psi) : (h_1, h_2) \longrightarrow w(t)$ given by the linear parabolic problem*

$$\begin{aligned} &\frac{\partial w}{\partial t} + Aw + G(\mathbf{V}_t(\phi, \psi))w + K(V, w) = 0 \quad \text{on} \quad \mathcal{Q}\,, \\ &w(t = 0) = B_1 h_1 + B_2 h_2 \quad \text{on} \quad \Omega\,, \end{aligned} \quad (4.5)$$

and satisfies the estimates ($\forall \Phi_i = (\phi_i, \psi_i) \in (L^2(\mathcal{Q}))^2, i = 1, 2$):

(i) $\| \mathbf{V}'_t(\phi_1,\psi_1) \|_{\mathcal{L}((L^2(\Omega))^2,L^2(\Omega))} \leq \exp((\gamma+1)T/2)b$,
(ii) $\| \mathbf{V}'_t(\phi_1,\psi_1) - \mathbf{V}'_t(\phi_2,\psi_2) \|_{\mathcal{L}((L^2(\Omega))^2,L^2(\Omega))}$
$$\leq 2\exp(2(\gamma+1)T)\lambda c_e^2 b^2 \| \Phi_1 - \Phi_2 \|^2_{L^2(\mathcal{Q})}.$$

Theorem 4.1. *Let F satisfy the assumptions (1.2). Then, for α and β sufficiently large, there exists $(\phi^*,\psi^*) \in (L^2(\Omega))^2$ and $u^* \in \mathcal{W}$ such that (ϕ^*,ψ^*) is defined by (4.3) and $u^* = \mathbf{U}(\phi^*,\psi^*)$ is solution of (4.1).*

In order to characterize the solution of the robust control problem, we introduce the "adjoint" problem corresponding to the primal problem (4.1) (we denote by $u = \mathbf{U}(\phi,\psi)$ the solution of problem (4.1), where the initial condition is (ϕ,ψ)):

$$\begin{aligned} &-\frac{\partial \tilde{u}}{\partial t} + A\tilde{u} + (G(u))^*\tilde{u} + K^*(V,\tilde{u}) = \mathcal{C}^*\mathcal{C}(u-u_{obs}) \quad \text{on} \quad \mathcal{Q}, \\ &\tilde{u}(T) = \mu(u(T)-v_{obs}) \quad \text{on} \quad \Omega, \end{aligned} \tag{4.6}$$

where $\mathcal{C}^*$ (resp. $(G(u))^*$) is the adjoint of the operator $\mathcal{C}$ (resp. $G(u)$).

Proposition 4.3. *Let F satisfy the assumptions (1.2), $u \in \mathcal{W}$ and $\nu > \delta_2$ then the solution of (4.6) is in $\mathcal{W}$ and satisfies the following estimate: $\| \tilde{u} \|^2_{\mathcal{H}} + (\nu-\delta_2) \| \tilde{u} \|^2_{\mathcal{V}} \leq \exp((\delta_1+\gamma+1)T)(\mu^2 \| u(T)-v_{obs} \|^2_{L^2} + \| \mathcal{C}(u-u_{obs}) \|^2_{L^2(\mathcal{Q})})$.*

Proof. We use the same technique that to obtain Proposition 3.4. So we omit the details. □

We can now give the optimality system for the robust control problem (4.3).

Theorem 4.2. *Let F satisfy the assumptions (1.2), $\nu > \delta_2$ and $(\phi^*,\psi^*,u^*) \in \mathcal{U}_{ad} \times \mathcal{V}_{ad} \times \mathcal{W}$ such that (ϕ^*,ψ^*) is defined by (4.3) and $u^* = \mathbf{U}(\phi^*,\psi^*)$ is solution of (4.1). Then $\int_\Omega(\alpha\phi^* + B_1^*\tilde{u}(0))(\phi-\phi^*)d\Omega \geq 0$, and $\int_\Omega(-\beta\psi^* + B_2^*\tilde{u})(\psi-\psi^*)d\Omega \leq 0$ $\forall(\phi,\psi) \in \mathcal{U}_{ad} \times \mathcal{V}_{ad}$, where $\tilde{u} = \tilde{\mathbf{U}}(\phi^*,\psi^*)$ is solution of the adjoint problem (4.6).*

Proof. The cost fonctionnal J is a composition of Fréchet differentiable maps then J is Fréchet differentiable and we have ($\forall \mathbf{h} = (h_1,h_2) \in (L^2(\Omega))^2$):

$$\begin{aligned} J'(\phi,\psi)\mathbf{h} = \int_0^T &< C(u-u_{obs}), C\mathbf{U}'(\phi,\psi)\mathbf{h} > \\ &+ < \mu(u(T)-v_{obs}), \mathbf{V}'_T(\phi,\psi)\mathbf{h} > + < \alpha\phi, h_1 > - < \beta\psi, h_2 > \end{aligned}$$

and then

$$\begin{aligned} J'(\phi,\psi)\mathbf{h} = \int_0^T &< \mathcal{C}^*\mathcal{C}(u-u_{obs}), \mathbf{U}'(\phi,\psi)\mathbf{h} > \\ &+ < \mu(u(T)-v_{obs}), \mathbf{V}'_T(\phi,\psi)\mathbf{h} > + < \alpha\phi, h_1 > - < \beta\psi, h_2 > . \end{aligned}$$

Multiplying (4.4) by $\tilde{u}$, integrating over $\mathcal{Q}$ and integrating by parts in time t, we obtain ($\forall \mathbf{h} = (h_1, h_2) \in (L^2(\mathcal{Q}))^2$):

$$< B_1 h_1 + B_2 h_2, \tilde{u}(0) > = < \mathbf{V}'_T(\phi^*, \psi^*)\mathbf{h}, \tilde{u}(T) > \\ + \int_0^T < -\frac{\partial \tilde{u}}{\partial t} + A\tilde{u} + (G(u))^* \tilde{u} + K^*(V, \tilde{u}), \mathbf{U}'(\phi^*, \psi^*)\mathbf{h} > .$$

Since $\tilde{u}$ is solution of adjoint problem we obtain then ($\forall \mathbf{h} = (h_1, h_2) \in (L^2(\Omega))^2$):

$$< B_1 h_1 + B_2 h_2, \tilde{u}(0) > = < \mathbf{V}'_T(\phi^*, \psi^*)\mathbf{h}, \mu(u^*(T) - v_{obs}) > \\ + \int_0^T < \mathcal{C}^*\mathcal{C}(u^* - u_{obs}), \mathbf{U}'(\phi^*, \psi^*)\mathbf{h} > . \quad (4.7)$$

As (ϕ^*, ψ^*) is solution of (4.3) then $J'(\phi^*, \psi^*)(\phi - \phi^*, 0) \geq 0$ and $J'(\phi^*, \psi^*)(0, \psi - \psi^*) \leq 0$, $\forall (\phi, \psi) \in \mathcal{U}_{ad} \times \mathcal{V}_{ad}$ and we can then deduce the result of the theorem (according to the expression of J'). So the proof is complete. □

We will assume in the sequel that there exists (ϕ^*, ψ^*, u^*) such that (ϕ^*, ψ^*) is defined by (4.3), $u^* = \mathbf{U}(\phi^*, \psi^*)$ is solution of (4.1) and $\int_\Omega (\alpha\phi^* + B_1^*\tilde{u}(0))(\phi - \phi^*) d\Omega \geq 0$, $\int_\Omega (-\beta\psi^* + B_2^*\tilde{u})(\psi - \psi^*) d\Omega \leq 0$ $\forall (\phi, \psi) \in \mathcal{U}_{ad} \times \mathcal{V}_{ad}$ (where $\tilde{u} = \tilde{\mathbf{U}}(\phi^*, \psi^*)$ is solution of (4.6)). We are now given some conditions to obtain the uniqueness of the solution (ϕ^*, ψ^*).

Theorem 4.3. *Suppose that F satisfies the assumptions (1.2), $\nu > \gamma_2$ and $\mu < 1$ holds. Then if:*
(i) $\theta = (\nu - \delta_2 - c_I^2(\gamma + \delta_1)) > 0$ and $1 - \frac{b^4}{\alpha^2} - \frac{b^4}{\beta^2} \geq 0$,
(ii) $\lambda c_e^2 \exp((\delta_1 + \gamma + 1)T/2)(\| u^*(T) - v_{obs} \|_{L^2}^2 + \| \mathcal{C}(u^* - u_{obs}) \|_{L^2(\mathcal{Q})}^2)^{1/2} < \theta$,
then, the solution (ϕ^*, ψ^*, u^*) is unique.

Proof. Suppose $({\phi^*}_1, {\psi^*}_1, u_1^*)$ is another solution, then $({\phi^*}_1, {\psi^*}_1)$ satisfies (4.3), $u_1^* = \mathbf{U}({\phi^*}_1, {\psi^*}_1)$ is solution of (4.1) and $\int_\Omega (\alpha{\phi^*}_1 + B_1^*\tilde{u}_1(0))(\phi - {\phi^*}_1) d\Omega \geq 0$, $\int_\Omega (-\beta{\psi^*}_1 + B_2^*\tilde{u}_1)(\psi - {\psi^*}_1) d\Omega \leq 0$ $\forall (\phi, \psi) \in \mathcal{U}_{ad} \times \mathcal{V}_{ad}$ (where $\tilde{u}_1 = \tilde{\mathbf{U}}({\phi^*}_1, {\psi^*}_1)$ is solution of (4.6)).

Set $\phi = \phi^* - {\phi^*}_1$, $\psi = \psi^* - {\psi^*}_1$, $v = u^* - u_1^*$ and $\tilde{v} = \tilde{u} - \tilde{u}_1$ we then have:

$$\begin{aligned} &\frac{\partial v}{\partial t} + Av + (F(u^*) - F(u_1^*)) + K(V, v) = 0 \quad \text{on} \quad \mathcal{Q}, \\ &v(t = 0) = B_1\phi + B_2\psi \quad \text{on} \quad \Omega, \end{aligned} \quad (4.8)$$

$$-\frac{\partial\tilde{v}}{\partial t}+A\tilde{v}+(G(u^*))^*\tilde{v}+K^*(V,\tilde{v})=\mathcal{C}^*\mathcal{C}v-(G(u^*)-G(u_1^*))^*\tilde{u} \text{ on } \mathcal{Q},$$
$$\tilde{v}(t=T)=\mu v(t=T) \quad \text{on} \quad \Omega \tag{4.9}$$

and

$$\alpha \parallel \phi \parallel^2_{(\Omega)}+\int_\Omega B_1^*\tilde{v}(0)\phi d\Omega\le 0, \quad \beta\parallel \psi\parallel^2_{(\Omega)} - \int_\Omega B_2^*\tilde{v}(0)\psi d\Omega dt\le 0. \tag{4.10}$$

By multiplying (4.8) by v, (4.9) by $\tilde{v}$ and integrating over Ω give (according to (4.10), (3.4) and (3.24))

$$\frac{\partial}{\partial t}\parallel v\parallel^2_{L^2}+\nu\parallel v\parallel^2_{\mathcal{D}}\le\gamma c_I^2\parallel v\parallel^2_{\mathcal{D}},$$

$$-\frac{\partial}{\partial t}\parallel \tilde{v}\parallel^2_{L^2}+\nu\parallel \tilde{v}\parallel^2_{\mathcal{D}}\le\gamma c_I^2\parallel \tilde{v}\parallel^2_{\mathcal{D}}+(\delta_1c_I^2+\delta_2)(\parallel v\parallel^2_{\mathcal{D}}+\parallel\tilde{v}\parallel^2_{\mathcal{D}}) + 2\lambda\int_\Omega\mid\tilde{u}\mid \quad \mid\tilde{v}\mid \quad \mid v\mid,$$

$$\tilde{v}(T)=\mu v(T) \text{ and } v(0)=B_1\phi+B_2\psi.$$

By using Hölder inequality and the relationship (1.3) we obtain

$$\frac{\partial}{\partial t}\parallel v\parallel^2_{L^2}+(\nu-\gamma c_I^2)\parallel v\parallel^2_{\mathcal{D}}\le 0,$$

$$-\frac{\partial}{\partial t}\parallel \tilde{v}\parallel^2_{L^2}+(\nu-\gamma c_I^2)\parallel \tilde{v}\parallel^2_{\mathcal{D}} \le(\delta_1c_I^2+\delta_2+\lambda c_e^2\parallel\tilde{u}\parallel_{\mathcal{H}})(\parallel v\parallel^2_{\mathcal{D}}+\parallel\tilde{v}\parallel^2_{\mathcal{D}}), \tag{4.11}$$

$$\tilde{v}(T)=\mu v(T) \text{ and } v(0)=B_1\phi+B_2\psi.$$

Adding (4.11.1) and (4.11.2) (according to $\nu>\delta_2$) we obtain

$$\frac{\partial}{\partial t}(\parallel v\parallel^2_{L^2}-\parallel\tilde{v}\parallel^2_{L^2}) +\theta(\parallel v\parallel^2_{\mathcal{D}}+\parallel\tilde{v}\parallel^2_{\mathcal{D}})\le\lambda c_e^2\parallel\tilde{u}\parallel_{\mathcal{H}}(\parallel v\parallel^2_{\mathcal{D}}+\parallel\tilde{v}\parallel^2_{\mathcal{D}}), \tag{4.12}$$

$$\tilde{v}(T)=\mu v(T) \text{ and } v(0)=B_1\phi+B_2\psi\,,$$

where $\theta = \nu - \delta_2 - c_I^2(2\gamma + \delta_1) > 0$ (according to the assumption (i)).

By integrating over $[0, T]$ and according to (4.11.3) we have

$$(1 - \mu^2) \parallel v(T) \parallel_{L^2}^2 + \parallel \tilde{v}(0) \parallel_{L^2}^2 + \theta(\parallel v \parallel_{\mathcal{V}}^2 + \parallel \tilde{v} \parallel_{\mathcal{V}}^2) \\ \leq \lambda c_e^2 \parallel \tilde{u} \parallel_{\mathcal{H}} (\parallel v \parallel_{\mathcal{V}}^2 + \parallel \tilde{v} \parallel_{\mathcal{V}}^2) + (\frac{b^4}{\alpha^2} + \frac{b^4}{\beta^2}) \parallel \tilde{v}(0) \parallel_{L^2}^2 .$$

Since $1 - \mu^2 > 0$ and $1 - \frac{b^4}{\alpha^2} - \frac{b^4}{\beta^2} \geq 0$ then

$$\theta(\parallel v \parallel_{\mathcal{V}}^2 + \parallel \tilde{v} \parallel_{\mathcal{V}}^2) \leq \lambda c_e^2 \parallel \tilde{u} \parallel_{\mathcal{H}} (\parallel v \parallel_{\mathcal{V}}^2 + \parallel \tilde{v} \parallel_{\mathcal{V}}^2).$$

By applying Proposition 4.3 and that $\mu < 1$ we have

$$\parallel \tilde{u} \parallel_{L^\infty(0,T,L^2(\Omega))}^2 \\ \leq \exp((\delta_1 + \gamma + 1)T)(\parallel u^*(T) - v_{obs} \parallel_{L^2}^2 + \parallel \mathcal{C}(u^* - u_{obs}) \parallel_{L^2(\mathcal{Q})}^2)$$

and then $\theta^*(\parallel v \parallel_{\mathcal{V}}^2 + \parallel \tilde{v} \parallel_{\mathcal{V}}^2) \leq 0$, where

$$\theta^* = \theta - \lambda c_e^2 \exp((\delta_1 + \gamma + 1)T/2)(\parallel u^*(T) - v_{obs} \parallel_{L^2}^2 \\ + \parallel \mathcal{C}(u^* - u_{obs}) \parallel_{L^2(\mathcal{Q})}^2)^{1/2}.$$

Since $\theta^* > 0$ (according to the assumption (i)), we have $\parallel v \parallel_{\mathcal{V}}^2 + \parallel \tilde{v} \parallel_{\mathcal{V}}^2 = 0$ and then $v = 0$ and $\tilde{v} = 0$. So the uniqueness result. □

4.2. Distributed Disturbance in the Forcing

In this section, the value g is the disturbance $\psi \in L^2(\Omega)$ and the initial condition u_0 is the control $\phi \in L^2(\Omega)$, i.e. $g = B_2\psi, u_0 = B_1\phi$, where $B_i, i = 1, 2$ are given bounded operators on $L^2(\Omega)$.

The function u is assumed to be related to the disturbance ψ and control ϕ through the problem (2.3):

$$\begin{aligned} &\frac{\partial u}{\partial t} + Au + F(u) + K(V, u) = B_2\psi \quad \text{on} \quad \mathcal{Q}, \\ &u(0) = B_1\phi \quad \text{on} \quad \Omega. \end{aligned} \tag{4.13}$$

To obtain the regularity of Proposition 2.3, we suppose the following hypothesis: $(\phi, \psi) \in L^2(\Omega) \times L^2(\mathcal{Q})$ and $U \in \mathcal{W}$. Let $\mathbf{U} : (\phi, \psi) \longrightarrow u = \mathbf{U}(\phi, \psi)$ be the map: $L^2(\Omega) \times L^2(\mathcal{Q}) \longrightarrow \mathcal{W}$ defined by (4.13) and introduce the cost function defined by

$$J(\phi, \psi) = \frac{1}{2} \parallel \mathcal{C}(u - u_{obs}) \parallel^2_{L^2(\mathcal{Q})} + \frac{\mu}{2} \parallel u(T) - v_{obs} \parallel^2_{L^2} + \frac{\alpha}{2} \parallel \phi \parallel^2_{L^2} - \frac{\beta}{2} \parallel \psi \parallel^2_{L^2(\mathcal{Q})}, \quad (4.14)$$

where $\mu, \alpha, \beta > 0$ are fixed, $(u_{obs}, v_{obs}) \in \mathcal{V} \times L^2(\Omega)$ is the observation (given) and $\mathcal{C}$ is unbounded, linear operator on $L^2(\Omega)$ satisfying the hypothesis (3.4).

We want to minimize the fonctionnal J with respect to ϕ and maximalize J with respect to ψ, i.e., to find $(\phi^*, \psi^*) \in \mathcal{U}_{ad} \times \mathcal{V}_{ad}$ such that

$$J(\phi^*, \psi) \leq J(\phi^*, \psi^*) \leq J(\phi, \psi^*), \forall(\phi, \psi) \in \mathcal{U}_{ad} \times \mathcal{V}_{ad}, \quad (4.15)$$

with $\mathcal{U}_{ad}$ and $\mathcal{V}_{ad}$ are (given) nonempty, closed, convex, bounded subsets of $L^2(\Omega)$ and $L^2(\mathcal{Q})$ respectively.

The arguments of the previous sections extend directly to the present case without further estimates. We have then the following results.

Proposition 4.4. *Let F satisfy the assumptions (1.2). Then the function $\mathbf{U} : (\phi, \psi) \longrightarrow u = \mathbf{U}(\phi, \psi)$ solution of (4.13) is continuously Fréchet differentiable from $L^2(\Omega) \times L^2(\mathcal{Q})$ to $\mathcal{W}$ with the derivative $\mathbf{U}'(\phi, \psi) : (h_1, h_2) \longrightarrow w$ given by the linear parabolic problem*

$$\begin{aligned} &\frac{\partial w}{\partial t} + Aw + G(\mathbf{U}(\phi, \psi))w + K(V, w) = B_2 h_2 \quad \text{on} \quad \mathcal{Q}, \\ &w(t = 0) = B_1 h_1 \quad \text{on} \quad \Omega, \end{aligned} \quad (4.16)$$

and satisfies the estimates ($\forall \Phi_i = (\phi_i, \psi_i) \in L^2(\Omega) \times L^2(\mathcal{Q}), i = 1, 2$):

(i) $\parallel \mathbf{U}'(\phi_1, \psi_1) \parallel_{\mathcal{L}(L^2(\Omega) \times L^2(\mathcal{Q}), \mathcal{W})} \leq \exp((\gamma + 1)T/2)b$,

(ii) $\parallel \mathbf{U}'(\phi_1, \psi_1) - \mathbf{U}'(\phi_2, \psi_2) \parallel_{\mathcal{L}(L^2(\Omega) \times L^2(\mathcal{Q}), \mathcal{W})} \leq 2 \exp(2(\gamma + 1)T) \lambda c_e^2 b^2 \parallel \Phi_1 - \Phi_2 \parallel^2_{L^2(\Omega) \times L^2(\mathcal{Q})}$.

Proposition 4.5. *Let F satisfy the assumptions (1.2). Then for each $t \in [0, T]$, the function $\mathbf{V}_t : (\phi, \psi) \longrightarrow u(t) = \mathbf{V}_t(\phi, \psi)$ solution of (4.13) is continuously Fréchet differentiable from $(L^2(\Omega))^2$ to $L^2(\Omega)$ with the derivative $\mathbf{V}'_t(\phi, \psi) : (h_1, h_2) \longrightarrow w(t)$ given by the linear parabolic problem*

$$\begin{aligned} &\frac{\partial w}{\partial t} + Aw + G(\mathbf{V}_t(\phi, \psi))w + K(V, w) = B_2 h_2 \quad \text{on} \quad \mathcal{Q}, \\ &w(t = 0) = B_1 h_1 \quad \text{on} \quad \Omega, \end{aligned} \quad (4.17)$$

and satisfies the estimates ($\forall \Phi_i = (\phi_i, \psi_i) \in L^2(\Omega) \times L^2(\mathcal{Q}), i = 1, 2$):

(i) $\| \mathbf{V}'_t(\phi_1, \psi_1) \|_{\mathcal{L}(L^2(\Omega)\times L^2(\mathcal{Q}), L^2(\Omega))} \leq \exp((\gamma+1)T/2)b$,
(ii) $\| \mathbf{V}'_t(\phi_1, \psi_1) - \mathbf{V}'_t(\phi_2, \psi_2) \|_{\mathcal{L}(L^2(\Omega)\times L^2(\mathcal{Q}), L^2(\Omega))}$
$\leq 2\exp(2(\gamma+1)T)\lambda c_e^2 b^2 \| \Phi_1 - \Phi_2 \|^2_{L^2(\Omega)\times L^2(\mathcal{Q})}$.

Theorem 4.4. *Let F satisfy the assumptions (1.2). Then, for α and β sufficiently large, there exists $(\phi^*, \psi^*) \in \mathcal{U}_{ad} \times \mathcal{V}_{ad}$ and $u^* \in \mathcal{W}$ such that (ϕ^*, ψ^*) is defined by (4.15) and $u^* = \mathbf{U}(\phi^*, \psi^*)$ is solution of (4.13).*

In order to characterize to solution of the robust control problem, we introduce the "adjoint" problem corresponding to the primal problem (4.13) (we denote by $u = \mathbf{U}(\phi, \psi)$ the solution of problem (4.13), where the forcing-initial condition is (ϕ, ψ)):

$$\begin{aligned} &-\frac{\partial \tilde{u}}{\partial t} + A\tilde{u} + (G(u))^*\tilde{u} + K^*(V, \tilde{u}) = \mathcal{C}^*\mathcal{C}(u - u_{obs}) \quad \text{on} \quad \mathcal{Q}, \\ &\tilde{u}(T) = \mu(u(T) - v_{obs}) \quad \text{on} \quad \Omega, \end{aligned} \tag{4.18}$$

where $\mathcal{C}^*$ (resp. $(G(u))^*$) is the adjoint of the operator $\mathcal{C}$ (resp. $G(u)$).

Proposition 4.6. *Let F satisfy the assumptions (1.2), $u \in \mathcal{W}$ and $\nu > \delta_2$ then the solution of (4.18) is in $\mathcal{W}$ and satisfies the following estimate:*

$$\begin{aligned} &\| \tilde{u} \|^2_{\mathcal{H}} + (\nu - \delta_2) \| \tilde{u} \|^2_{\mathcal{D}} \\ &\qquad \leq \exp((\delta_1 + \gamma + 1)T)(\mu^2 \| u(T) - v_{obs}) \|^2_{L^2} + \| \mathcal{C}(u - u_{obs}) \|^2_{L^2(\mathcal{Q})}). \end{aligned}$$

Proof. We use the same technique to obtain Proposition 4.3. So we omit the details. □

We can now give the optimality system for the robust control problem (4.15).

Theorem 4.5. *Let F satisfy the assumptions (1.2), $\nu > \delta_2$ and $(\phi^*, \psi^*, u^*) \in \mathcal{U}_{ad} \times \mathcal{V}_{ad} \times \mathcal{W}$ such that (ϕ^*, ψ^*) is defined by (4.15) and $u^* = \mathbf{U}(\phi^*, \psi^*)$ solution of (4.13). Then (for α and β sufficiently large)*
$\int_\Omega (\alpha\phi^* + B_1^*\tilde{u}(0))(\phi - \phi^*) d\Omega dt \geq 0$, *and*

$$\int_0^T \int_\Omega (-\beta\psi^* + B_2^*\tilde{u})(\psi - \psi^*) d\Omega dt \leq 0\,, \quad \forall (\phi, \psi) \in \mathcal{U}_{ad} \times \mathcal{V}_{ad},$$

where $\tilde{u} = \tilde{\mathbf{U}}(\phi^*, \psi^*)$ is solution of the adjoint problem (4.18).

Proof. We use the same technique as in Theorem 4.2. We skip the details. □

We will assume in the sequel that there exists (ϕ^*, ψ^*, u^*) such that (ϕ^*, ψ^*) is defined by (4.15), $u^* = \mathbf{U}(\phi^*, \psi^*)$ is solution of (4.13) and $\int_\Omega (\alpha\phi^* + B_1^*\tilde{u}(0))(\phi -$

$\phi^*)d\Omega dt \geq 0$, $\int_0^T \int_\Omega (-\beta\psi^* + B_2^*\tilde{u})(\psi - \psi^*)d\Omega dt \leq 0$ (where $\tilde{u} = \tilde{\mathbf{U}}(\phi^*, \psi^*)$ is solution of (4.18)). We are now given some conditions to obtain the uniqueness of the solution (ϕ^*, ψ^*).

Theorem 4.6. *Suppose that F satisfies the assumptions (1.2), $\nu > \gamma_2$, $\mu < 1$ and $\alpha \geq b^2$ holds. Then if:*

(i) $\theta = (\nu - \delta_2 - c_I^2(\gamma + \delta_1) - \frac{b^2 c_I^2}{\alpha}) > 0$,

(ii) $\lambda c_e^2 \exp((\delta_1 + \gamma + 1)T/2)(\| u^*(T) - v_{obs} \|_{L^2}^2 + \| \mathcal{C}(u^* - u_{obs}) \|_{L^2(\mathcal{Q})}^2)^{1/2} < \theta$,

then, the solution (ϕ^*, ψ^*, u^*) is unique.

Proof. Suppose $({\phi^*}_1, {\psi^*}_1, u_1^*)$ is another solution, then $({\phi^*}_1, {\psi^*}_1)$ satisfies (4.3), $u_1^* = \mathbf{U}({\phi^*}_1, {\psi^*}_1)$ is solution of (4.13) and $\int_\Omega (\alpha\phi^* + B_1^*\tilde{u}_1(0))(\phi - {\phi^*}_1)d\Omega dt \geq 0$, $\int_0^T \int_\Omega (-\beta{\psi^*}_1 + B_2^*\tilde{u}_1)(\psi - {\psi^*}_1)d\Omega dt \leq 0$ $\forall(\phi, \psi) \in \mathcal{U}_{ad} \times \mathcal{V}_{ad}$, (where $\tilde{u}_1 = \tilde{\mathbf{U}}({\phi^*}_1, {\psi^*}_1)$ is solution of (4.18)).

Set $\phi = \phi^* - {\phi^*}_1$, $\psi = \psi^* - {\psi^*}_1$, $v = u^* - u_1^*$ and $\tilde{v} = \tilde{u} - \tilde{u}_1$ we then have:

$$\begin{aligned} &\frac{\partial v}{\partial t} + Av + (F(u^*) - F(u_1^*)) = B_2\psi \quad \text{on} \quad \mathcal{Q}, \\ &v(0) = B_1\phi \quad \text{on} \quad \Omega, \end{aligned} \tag{4.19}$$

$$\begin{aligned} &-\frac{\partial \tilde{v}}{\partial t} + A\tilde{v} + (G(u^*))^*\tilde{v} = \mathcal{C}^*\mathcal{C}v - (G(u^*) - G(u_1^*))^*\tilde{u} \quad \text{on} \quad \mathcal{Q}, \\ &\tilde{v}(T) = \mu v(T) \quad \text{on} \quad \Omega \end{aligned} \tag{4.20}$$

and

$$\begin{aligned} &\alpha \| \phi \|_{L^2(\Omega)}^2 + \int_\Omega B_1^*\tilde{v}(0)\phi d\Omega \leq 0, \\ &\beta \| \psi \|_{L^2(\mathcal{Q})}^2 - \int_0^T \int_\Omega B_2^*\tilde{v}\phi d\Omega dt \leq 0. \end{aligned} \tag{4.21}$$

By multiplying (4.19) by v, (4.20) by $\tilde{v}$ and integrating over $\mathcal{Q}$ give (according to (4.21), (3.24) and (3.4))

$$\int_0^T \frac{\partial}{\partial t} \| v \|_{L^2}^2 + \nu \int_0^T \| v \|_{\mathcal{D}}^2 \leq \gamma c_I^2 \int_0^T \| v \|_{\mathcal{D}}^2 + \frac{2}{\beta} \| B_2^*\tilde{v} \|_{L^2(\mathcal{Q})} \| B_2^* v \|_{L^2(\mathcal{Q})},$$

$$-\int_0^T \frac{\partial}{\partial t} \| \tilde{v} \|_{L^2}^2 + \nu \int_0^T \| \tilde{v} \|_{\mathcal{D}}^2 \leq \gamma c_I^2 \int_0^T \| \tilde{v} \|_{\mathcal{D}}^2 + (\delta_1 c_I^2 + \delta_2) \int_0^T (\| v \|_{\mathcal{D}}^2 + \| \tilde{v} \|_{\mathcal{D}}^2) + 2\lambda \int_0^T \int_\Omega | \tilde{u} | \; | \tilde{v} | \; | v |,$$

$$\tilde{v}(T) = \mu v(T) \text{ and } v(0) = B_1\phi.$$

By using Hölder inequality and the relationship (1.3) we obtain

$$\int_0^T \frac{\partial}{\partial t} \| v \|_{L^2}^2 + (\nu - \gamma c_I^2) \int_0^T \| v \|_{\mathcal{D}}^2 \leq \frac{1}{\beta} b^2 c_I^2 \int_0^T (\| v \|_{\mathcal{D}}^2 + \| \tilde{v} \|_{\mathcal{D}}^2),$$

$$- \int_0^T \frac{\partial}{\partial t} \| \tilde{v} \|_{L^2}^2 + (\nu - \gamma c_I^2) \int_0^T \| \tilde{v} \|_{\mathcal{D}}^2$$
$$\leq (\delta_1 c_I^2 + \delta_2 + \lambda c_e^2 \| \tilde{u} \|_{\mathcal{H}}) \int_0^T (\| v \|_{\mathcal{D}}^2 + \| \tilde{v} \|_{\mathcal{D}}^2), \quad (4.22)$$
$$\tilde{v}(T) = \mu v(T) \text{ and } v(0) = B_1\phi.$$

Adding (4.22.1) and (4.22.2) (according to $\nu > \delta_2$) we obtain

$$\int_0^T \frac{\partial}{\partial t}(\| v \|_{L^2}^2 - \| \tilde{v} \|_{L^2}^2) + \theta \int_0^T (\| v \|_{\mathcal{D}}^2 + \| \tilde{v} \|_{\mathcal{D}}^2)$$
$$\leq \lambda c_e^2 \| \tilde{u} \|_{\mathcal{H}} \int_0^T (\| v \|_{\mathcal{D}}^2 + \| \tilde{v} \|_{\mathcal{D}}^2), \quad (4.23)$$

where $\theta = \nu - \delta_2 - c_I^2(\gamma + \delta_1 + \frac{b^2}{\alpha}) > 0$ (according to the assumption (i)).

According to (4.22.3) we have

$$(1 - \mu^2) \| v(T) \|_{L^2}^2 + \| \tilde{v}(0) \|_{L^2}^2 + \theta(\| v \|_{\mathcal{V}}^2 + \| \tilde{v} \|_{\mathcal{V}}^2)$$
$$\leq \lambda c_e^2 \| \tilde{u} \|_{\mathcal{H}} (\| v \|_{\mathcal{V}}^2 + \| \tilde{v} \|_{\mathcal{V}}^2) + \frac{b^4}{\alpha^2} \| \tilde{v}(0) \|_{L^2}^2.$$

Since $1 - \mu^2 > 0$ and $1 - \frac{b^4}{\alpha^2} > 0$ then

$$\theta(\| v \|_{\mathcal{V}}^2 + \| \tilde{v} \|_{\mathcal{V}}^2) \leq \lambda c_e^2 \| \tilde{u} \|_{\mathcal{H}} (\| v \|_{\mathcal{V}}^2 + \| \tilde{v} \|_{\mathcal{V}}^2).$$

By applying Proposition 4.6 and that $\mu < 1$ we have

$$\| \tilde{u} \|_{\mathcal{H}}^2 \leq \exp((\delta_1 + \gamma + 1)T)(\| u^*(T) - v_{obs} \|_{L^2}^2$$
$$+ \| \mathcal{C}(u^* - u_{obs}) \|_{L^2(\mathcal{Q})}^2),$$

and then $\theta^*(\| v \|_{\mathcal{V}}^2 + \| \tilde{v} \|_{\mathcal{V}}^2) \leq 0$, where

$$\theta^* = \theta - \lambda c_e^2 \exp((\delta_1 + \gamma + 1)T/2)(\| u^*(T) - v_{obs} \|_{L^2}^2$$
$$+ \| \mathcal{C}(u^* - u_{obs}) \|_{L^2(\mathcal{Q})}^2)^{1/2}.$$

Since $\theta^* > 0$ (according to the assumption (i)), we have $\| v \|_{\mathcal{V}}^2 + \| \tilde{v} \|_{\mathcal{V}}^2 = 0$ and then $v = 0$ and $\tilde{v} = 0$. So the uniqueness result. □

5. Example

We presente here a more practical example dealing with a reaction-diffusion-transport system. The system is gouverned by:

$$\begin{aligned}&\frac{\partial U}{\partial t} - \operatorname{div}(D\nabla U) + V.\nabla U + F(U) = f \quad \text{on} \quad \mathcal{Q} = (0,T)\times\Omega\,,\\ &U(.,t) = 0 \quad \text{on} \quad (0,T)\times\partial\Omega\,,\\ &U(t=0) = U_0 \quad \text{on} \quad \Omega.\end{aligned} \tag{5.1}$$

Here, Ω is an open bounded in $\mathbb{R}^3$ with the boundary $\partial\Omega$ is sufficiently regular, for example the liquid (or atmospheric) domain: U represents the concentration of some biochemical pollutant in the studied domain Ω, D denotes the coefficient of eddy diffusivity, V is the 3-dimensional velocity field of the fluid or the air. The second term in the equation accounts for pollutant movement by diffusion and the third term represents the transport of the pollutant by the flow field. The right-hand side of the equation f may consist of agents (for example biological agents capable of producing biodegradation of the pollutant), chimicals extraction or physical extraction.

For the mathematical setting, we take $\mathcal{D} = H_0^1(\Omega)$ and $\mathcal{D}' = H^{-1}(\Omega)$. According to Sobolev Embedding Theorem [1] we have that $\mathcal{D}$ embed in $L^4(\Omega)$. The operator A is $-\operatorname{div}(D\nabla.)$ with Dirichlet boundary condition and K is the operator $V.\nabla$. For the nonlinear operators we suppose that they satisfies the hypothesis (1.2).

If we assume that the coefficient D is positive and bounded function, then the operator A is continuous and coercive and we denote by $\nu = \min_{\Omega}(D)$ the coercivity constant.

Assume now that the velocity field V is known and satisfies $V \in L^\infty(Q)$ and $\operatorname{div}(V) = 0$, so we have easily the estimate (1.5), where $\gamma_\infty = \frac{\|V\|^2_{L^\infty(Q)}}{\nu}$ and that $K^* = -K$.

If we suppose that U is the target function, the equation satisfies the large perturbation u to the target U is:

$$\begin{aligned}&\frac{\partial u}{\partial t} - \operatorname{div}(D\nabla u) + V.\nabla u + F(u+U) - F(U) = g\\ &\qquad\qquad\qquad\qquad \text{on} \quad \mathcal{Q} = (0,T)\times\Omega\,,\\ &u(.,t) = 0 \quad \text{on} \quad (0,T)\times\partial\Omega\,,\\ &u(t=0) = u_0 \quad \text{on} \quad \Omega.\end{aligned} \tag{5.2}$$

In the cost function the operator $\mathcal{C}$ represent the regional and temporal variation in the cost of poluttant extraction. The observation denotes the acceptable pollution level in the studied region. Since all the assumptions of our abstract result are satisfied by the example in this particular case, so our study applies.

6. Conclusion

We have developed a robust control method for problems gouverned by non-linear parabolic equations. Our purpose is to calculate the pronostic variables corresponding to some observation taking into account the influence of the disturbance (noise) in data.

The robust control problem has been studied for two sets of distributed controls: firstly the disturbance and the control in the forcing, secondly the control in the initial condition and the disturbance in the initial condition or in the forcing. Under suitable hypothesis, it is shown that one has existence of solution to a corresponding "minimax" problem, and we obtain the appropriate necessary optimality conditions. Further, for small data or large coefficient ν, α and β, one has uniqueness.

It will be interesting to extend this study to anothers problems. In the future, we want to apply the robust control to oceanographic and atmospheric problems. For example in the tropical Atlantic or Pacific oceans, the initial condition (velocity, pressure, temperature and salinite) is known for each tropical season but the external forcing (for example the windstress) is not well-know (in the realistic case). This problem corresponds to a modelization of tropical instability waves which are illustrated by the El Nino phenomenon. In this case the control problem studied in [4]-[9] consider the external forcing as the control in order to restitute the observer situation (for example the surface pressure) or to minimize the turbulence within the flow. It is interesting to consider this problem with a disturbuted control and a disturbance in the external forcing.

In the case of meteorology, the control problem is the adjustment of the initial condition (the initial condition is not well know) in order to obtain the circulation which agree with observations in situ (see e.g. [20, 21]). It will be desirable to study the robust control with a disturbuted control and a disturbance in the initial condition.

References

[1] R.A. Adams, *Sobolev Spaces*, Academic Press, New-York (1975).

[2] N.U. Ahmed, X. Xiang, Nonlinear uncertain systems and necessary conditions of optimality, *SIAM J. Control Optim.*, **35** (1997), 1755-1772.

[3] V. Barbu, T. Precupanu, *Convexity and Optimization in Banach Spaces,* Reidel, Dordrecht (1986).

[4] A. Belmiloudi, A nonlinear optimal control problem for assimilation of surface data in a Navier-Stokes type equations related to oceanography, *Num. Func. Analy. Opti.*, **20**, No. 1, 2 (1999), 1-26.

[5] A. Belmiloudi, Regularity results and optimal control problems for the perturbation of Boussinesq equations of the ocean, *Num. Func. Analy. Opti.*, **21**, No. 5, 6 (2000), 623-651.

[6] A. Belmiloudi, Existence and characterization of an optimal control for the problem of long waves in a shallow-water model, *SIAM J. Control and Optim.*, **39**, No. 5 (2001), 1558-1584; Published electronically (January 31, 2001).

[7] A. Belmiloudi, Robin-type boundary control problems for the nonlinear Boussinesq type equations, *J. Math. Anal. Appl.*, To appear.

[8] A. Belmiloudi, Mathematical analysis and optimal control problems for the perturbation of the primitive equations of the ocean with vertical viscosity, *J. Appl. Anal.*, To appear.

[9] A. Belmiloudi, F. Brossier, A control method for assimilation of surface data in a linearized Navier-Stokes type problem related to oceanography, *SIAM Journal on Control and Optimization*, **35**, No. 6 (1997), 2183-2197.

[10] M.W. Cantoni, K. Glover, Gap-metric robustness analysis of linear periodically time-varying feedback systems, *SIAM J. Control Optim.*, **38**, No. 6 (2000), 803-8222.

[11] W. Dale, M. C. Smith, Stabilizability and existence of system representations for discrete-time, time varying systems, *SIAM J. Control Optim.*, **31**, No. 6 (1993), 1538-1557.

[12] I. Ekeland, R. Temam, *Convex Analysis and Variational Problems,* North-Holland, Amsterdam (1976).

[13] C. Foias, H. Ozbay, A. Taunnenbaum, *Robust Control for Infinite-Dimensional Systems. Frequency Domain Methods,* Lecture Notes in Control and Information Sciences, **209**, Springer, Berlin (1996).

[14] T.T. Georgiou, Differential stabilisation and robust control of nonlinear systems, *Math. Control Signals Systems*, **6** (1993), 389-306.

[15] T.T. Georgiou, M.C. Smith, Robustness analysis of nonlinear feedback systems: an input-output approach, *IEEE Trans. Automat. Control*, **42** (1997), 1200-1221.

[16] M. Green, D.J.N. Limebeer, *Linear Robust Control,* Prentice-Hall, Englewood Cliffs, NJ (1995).

[17] C. Hu, R. Temam, Robust control of the Kuramoto-Sivashinsky equation, *Dyn. Cont. Impu. Syst.*, **8** (2001), 315-338.

[18] J.L. Lions, *Equations Differentielles Operationnelles*, Springer-Verlag, New York (1961).

[19] J.L. Lions, E.Magenes, *Problèmes aux Limites non Homogènes et Applications*, Tome **1** and **2**, Dunod, Paris (1968).

[20] G.I. Marchuk, *Mathematical Models in Environemental Problems,* North Holland, Amsterdam (1986).

[21] G.I. Marchuk, *Adjoint Equation and Analysis of Complex Systems*, Kluwer, Dordrecht, The Netherlands (1995).

[22] M.Z. Nashed, Differentiability and related properties of nonlinear operators: some aspects of the role of differentials in nonlinear functional analysis, *Nonlinear Functional Analysis and Application*, Publication No. 26 of the Mathematics Research Center, The University of Wisconsin (1971), 103-309.

[23] J. Reinschke, M.W. Cantoni, M.C. Smith, A robust control framework for linear, time invariant, spatially disturbuted systems, *SIAM J. Control Optim.*, **40**, No. 2 (2001), 610-627.

[24] T. Seidman, H.Z. Zhou, Existence and uniqueness of optimal controls for a quasilinear parabolic equation, *SIAM J. Control Optim.*, **20**, No. 6 (1982), 747-762.

[25] G. Zames, A.K. Elsakkary, Unstable sysems and feedback: the gap-metric, In: *Proceedings of the Allerton Conference*, University of Illinois, Monticello, IL (1980), 380-385.

[26] K. Zhou, J.C. Doyle, K. Glover, *Robust and Optimal Control*, Prentice-hall, Upper Saddle River, NJ (1996).

7. Appendix

Lemma of Gronwall. *Let $A \geq 0$ and $B \geq 0$ be constants and f be integrable and positive such that:*

$$f(t) \leq A + B\int_0^t f(s)ds, \quad \text{for} \;\; t \geq 0\,.$$

Then $\sup_{t\in[0,T]} f(t) \leq A\exp(BT)$.

Corollary. *Let $\alpha \geq 0$, $\beta \geq 0$ and $\gamma \geq -1$ be constants and f be integrable and positive such that:*

$$f(t) \leq \alpha^2 + 2\beta(\int_0^t f(s)ds)^{1/2} + \gamma\int_0^t f(s)ds, \quad \text{for} \;\; t \geq 0\,.$$

Then $\sup_{t\in[0,T]} f(t) \leq (\alpha^2+\beta^2)\exp((\gamma+1)T)$.

[1]This paper is a re-publication from *International Journal of Pure and Applied Mathematics*, **9**, No. 2 (2003), because of some technical problems, aroused in the printing process.