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AUXILIARY MINIMIZATION PROBLEM PRINCIPLE AND ITS APPLICATIONS TO NONLINEAR VARIATIONAL PROBLEMS

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Abstract: This paper deals with a crucial role of the partial relaxed monotonicity – a relatively new and less explored concept – in the approximation solvability of a class of nonlinear variational problems on Hilbert spaces, though the convergence analysis much depends on the general auxiliary minimization problem principle. The nonlinear variational inequality problem (NVIP) is stated as follows: find an element $x^* \in K$ such that $u^* \in T(x^*)$ and

$$\langle u^*, x-x^*\rangle + f(x)-f(x^*)\geq 0, \quad \forall x\in K,$$

where $T : K\rightarrow P(E^*)$ is a mapping from a nonempty closed convex subset K of a real reflexive Banach space E into the power set $P(E^*)$ of E^*, and $f : K\rightarrow R$ is a continuous convex functional on K. Here $\langle x^*, x\rangle$ denotes the duality pairing between elements $x^*\in E^*$ and elements $x\in E$.

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1. Introduction

Cohen [2] proposed the auxiliary problem principle in the context of analyzing some of the optimization problems, for instance, gradient / subgradient as well as decomposition / coordination algorithms. However, the auxiliary problem

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principle extends to handle the approximation solvability of a class of nonlinear variational inequalities as well. On the top of that, the convergence analysis seems to be better than that of projection / projection type methods, where the convergence analysis is achieved in a more restricted sense, especially the computation of solutions to variational inequalities. Iterative algorithms in which the elementary step amounts to solving an auxiliary problem, that is, equivalent to a minimization problem based on an auxiliary convex cost function chosen by the user. Unlike projection methods, the auxiliary problem principle and its generalizations can be applied to a more general class of problems arising from complementarity problems, mathematical programming, and other optimization problems. Verma [11-13] extended and applied the auxiliary problem principle and its modified versions to the approximation solvability of nonlinear variational inequality problems involving partially relaxed monotone mappings – a more general notion than cocoercivity and strong monotonicity – in different space settings, which generalizes investigations of Zhu and Marcotte [16] and El Farouq [5] on the approximation solvability of nonlinear variational inequality problems involving cocoercive mappings in different spaces, including in R^n. In this paper we first present a modified version of the auxiliary minimization problem principle to the case of multivalued mappings, and then apply it to approximate solutions of a general class of nonlinear variational inequality problems involving multivalued partially relaxed monotone mappings in a real reflexive Banach space setting. The exposition is divided into four sections as follows: introduction, special class of partially relaxed monotone mappings, general auxiliary minimization problem, and applications. Since the general class of partially relaxed monotone mappings is more general than the classes of cocoercive and strongly monotone mappings, general properties of partially relaxed monotone mappings and their connections to cocoercive and strongly monotone mappings are illustrated by examples of interest in Section 2. For more details on the auxiliary problem principle and the solvability of related variational inequalities, we refer to [1-17].

Let E be a real reflexive Banach space and let dual E^* of E be strictly convex. Let $\langle x^*, x\rangle$ denote the duality pairing between elements $x^* \in E^*$ and $x \in E$, and let the norm $\|x\|$ be defined by

$$||x|| = (\langle x^*, x\rangle)^{1/2} \quad \text{for} \quad \|x^*\| = \|x\|.$$

Let $T : K \to P(E^*)$ be a nonlinear mapping from a nonempty closed convex subset K of a real reflexive Banach space E into the power set $P(E^*)$ of E^*, and $f : K \to R$ be a continuous convex functional on K. We consider a class of nonlinear variational inequality problems (NVIP) based on the general auxiliary minimization problem principle (GAMPP) as follows: find an element $x^*\in K$ such that $u^*\in T(x^*)$ and

$$\langle u^*, x-x^*\rangle + f(x)-f(x^*) \geq 0\,, \quad \forall x\in K, \tag{1}$$

where $T : K\to P(E^*)$ is a mapping from a nonempty closed convex subset K of a real reflexive

Banach space E into the power set $P(E^*)$ of E^*, and $f : K\to R$ is a continuous convex functional on K.

Definition 1.1. A mapping $T : K\to P(E^*)$ is said to be $\gamma-\mu$-partially relaxed monotone if, for each $x, y, z\in K$, we have

$$\langle u-v, z-y\rangle \geq (-\gamma)||z-x||^2 + \mu||x-y||^2 \quad \text{for constants} \quad \gamma, \mu > 0,$$

where $u\in T(x)$ and $v\in T(y)$.

We note that the $\gamma-\mu$-partial relaxed monotonicity implies the γ-partial relaxed monotonicity, that is,

$$\langle u-v, z-y\rangle \geq (-\gamma)||z-x||^2\,, \quad \forall x, y, z\in K,$$

where $u\in T(x)$, $v\in T(y)$ and γ is a positive constant.

2. Special Class of Partially Relaxed Monotone Mappings

In this section, we present a special class of partially relaxed monotone mappings and their connections to other classes of related mappings in R^n. On the top of that, this section deals with some examples for partially relaxed monotone, cocoercive and strongly monotone mappings in R^n. Let $x = (x_1, \ldots, x_n)\in R^n$ and $y = (y_1, \ldots, y_n)\in R^n$. Then $[x, y]$ and $\|x\|$, respectively, shall denote the Euclidean inner product and the standard Euclidean norm defined by

$$[x, y] = \Sigma_{i=1}^n(x_i y_i) \quad \text{and } \|x\| = [x, x]^{1/2}_.$$

For a given matrix A, the corresponding induced matrix norm $\|A\|$ is given by

$$\|A\| = \max\|Ax\| \quad \text{for} \quad \|x\| = 1.$$

Let K be a nonempty closed convex subset of R^n. A mapping $T : K\to R^n$ is called r-strongly monotone if there exists a positive constant r such that

$$[T(x) - T(y), x - y] \geq r\|x-y\|^2\,, \quad \forall x, y\in K.$$

Clearly, the strong monotonicity implies the monotonicity, that is, T is monotone if

$$[T(x)-T(y), x-y] \geq 0\,, \quad \forall x, y{\in}K.$$

The mapping T is called β-Lipschitz continuous if for each $x, y{\in}K$, we have

$$\|T(x)-T(y)\| \leq \beta\|x{-}y\| \quad \text{for} \quad \beta \geq 0.$$

A mapping $T : K{\rightarrow}R^n$ is called α-cocoercive [3, 17] if there exists a positive constant α such that

$$[T(x)-T(y), x-y] \geq \alpha\|T(x)-T(y)\|^2\,, \quad \forall x, y{\in}K.$$

T is called α-strictly cocoercive if there exists an $\alpha > 0$ such that

$$[T(x)-T(y), x-y] > \alpha\|T(x)-T(y)\|^2\,, \quad \forall x, y{\in}K.$$

We note that if T is α-strongly monotone and β-Lipschitz continuous, then T is (α/β^2)-cocoercive for $\beta > 0$, while the converse is not true in general. The α-cocoercivity of T implies the $(1/\alpha)$-Lipschitz continuity of T. Any cocoercive mapping is clearly monotone, but converse is not true.

Example 2.1. (see [17]) A mapping T, defined by

$$T(x) = x^3 \quad \text{for} \quad x{\in}[-1, 1],$$

is strictly (1/3)-cocoercive, while it is not strongly monotone. Since x, $y{\in}[-1, 1]$ for $x{\neq}y$, we have

$$[T(x)-T(y), x-y] = |x^3{-}y^3||x{-}y|,$$

$$0 < |x^2+xy+y^2| < 3,$$

and

$$(x^3-y^3)(x-y) = |x^3{-}y^2||x-y| > 0,$$

it implies that

$$\begin{aligned} \langle T(x)-T(y), x-y\rangle &= |x^3-y^3||x{-}y| \\ &> (1/3)|x^3-y^3||x^2+xy+y^2||x{-}y| \\ &= (1/3)|x^3-y^3|^2 = (1/3)\|T(x)-T(y)\|^2, \end{aligned}$$

which means, T is strictly (1/3)-cocoercive.

A mapping $T : K\to R^n$ is said to be α-partially relaxed monotone [11] if there is a positive constant α such that

$$[T(x) - T(y), z - y] \geq (-\alpha)\|z-x\|^2, \quad \forall x, y, z\in K.$$

Proposition 2.1. *Let $T : K\to R^n$ be $(1/\alpha)$-strongly monotone and $(1/\alpha)$-Lipschitz continuous for $\alpha>0$. Then we have the following:*

(i) T is α-cocoercive.

(ii) T is $(1/4\alpha)$-partially relaxed monotone.

Proof. Clearly T is α-cocoercive, and so it implies for $x, y, z\in H$ that

$$[T(x)-T(y), z-y] = [T(x)-T(y), x-y] + [T(x)-T(y), z-x]$$
$$\geq \alpha\|T(x)-T(y)\|^2 + [T(x)-T(y), z-x]$$
$$= \alpha\{\|T(x)-T(y)\|^2 + (1/\alpha)[T(x)-T(y), z-x]\}$$
$$\geq (-1/4\alpha)\|z-x\|^2,$$

that is, T is $(1/4\alpha)$-partially relaxed monotone. □

Proposition 2.2. *The cocoercivity implies the partial relaxed monotonicity, while the converse is not true in general.*

Let $T : K\to R^n$ be $(1/\alpha)$-cocoercive for $\alpha>0$. Then we can express

$$[T(x) - T(y), z - y] = [T(x) - T(y), x - y + z - x]$$
$$= [T(x) - T(y), x - y] + [T(x) - T(y), z - x]$$
$$\geq (1/\alpha)\|T(x) - T(y)\|^2 + [T(x) - T(y), z - x]$$
$$= (1/\alpha)\{\|T(x) - T(y)\|^2 + \alpha[T(x) - T(y), z - x]\}.$$

Since

$$\|T(x) - T(y)\|^2 + [T(x) - T(y), \alpha(z - x)] \geq (-\alpha^2/4)\|z-x\|^2,$$

it follows that

$$[T(x) - T(y), z - y] \geq (-\alpha/4)\|z-x\|^2,$$

that is, T is $(\alpha/4)$-partially relaxed monotone.

Note that Proposition 2.1 and Proposition 2.2 do hold for general Hilbert spaces and beyond.

The mapping T is called $\alpha - r$-partially relaxed monotone if, for each $x, y, z \in K$, there exist positive constants α and r such that

$$[T(x) - T(y), z - y] \geq (-\alpha)\|z - x\|^2 + r\|x - y\|^2.$$

Clearly, the $\alpha - r$-partial relaxed monotonicity implies the α-partial relaxed monotonicity.

The mapping T is called strictly α-partially relaxed monotone if there exists a positive constant α such that

$$[T(x) - T(y), z - y] > (-\alpha)\|z - x\|^2, \quad \forall x, y, z \in K.$$

Example 2.2. Consider a function T defined by

$$T(x) = x^3 \quad \text{for} \quad x \in [-1, 1].$$

Then T is strictly (3/4)-partially relaxed monotone, but not strongly monotone. For $x, y, z \in K$, we have

$$[T(x) - T(y), z - y] = [T(x) - T(y), x - y + z - x]$$

$$= [T(x) - T(y), x - y] + [T(x) - T(y), z - x]. \tag{2}$$

Since $[T(x) - T(y), x - y] > (1/3)\|T(x) - T(y)\|^2$ from Example 2.1, we have

$$[T(x) - T(y), z - y] > (1/3)\|T(x) - T(y)\|^2 + [T(x) - T(y), z - x]$$

$$= (1/3)(\|T(x) - T(y)\|^2 + 3[T(x) - T(y), z - x]). \tag{3}$$

Since

$$\|T(x) - T(y)\|^2 + 3[T(x) - T(y), z - x] \geq (-9/4)\|z - x\|^2, \tag{4}$$

it follows from (3) that

$$[T(x) - T(y), z - y] > (1/3)(-9/4)\|z - x\|^2,$$

that is, T is strictly (3/4)-partially relaxed monotone.

Example 2.3. Let $T : R^n \to R^n$ be defined by

$$T(x) = cI(x) + v,$$

where $c > 0$, $x, v \in R^n$ with v fixed, and I is the $n \times n$ identity matrix. Then T is an $\alpha - \alpha r$-partially relaxed monotone mapping for $c = \alpha > 0$ and $0 \leq r < 1$. For $x, y, z \in R^n$, we have

$$\|y-z\|^2 + \|y-x\|^2 + \|x-z\|^2 \geq 0\,,$$

or

$$\|y-z\|^2 + \|y-x\|^2 + \|x-z\|^2 \geq r\|x-y\|^2\,,$$

or

$$-[x, y] - [y, z] + [y, y] + [z, z] - [z, x] + [x, x] \geq r\|x-y\|^2\,,$$

or

$$\alpha([x-y, z-y] + [z-x, z-x]) \geq \alpha r\|x-y\|^2\,,$$

or

$$[\alpha x - \alpha y, z-y] + \alpha\|z-x\|^2 \geq \alpha r\|x-y\|^2\,,$$

or

$$[T(x) - T(y), z-y] \geq (-\alpha)\|z-x\|^2 + \alpha r\|x-y\|^2\,,$$

that is, T is an $\alpha - \alpha r$-partially relaxed monotone mapping. Clearly, one can derive from this that T is α-partially relaxed monotone (for $c = \alpha$) as well. On the top of that, T is an α-cocoercive mapping [3, 17] for $0 < \alpha < c$, and it is also an r-strongly monotone mapping for $0 < r < c$. For more details on general classes of partial relaxed monotone and cocoercive mappings, we recommend [1, 3, 11, 17].

Example 2.4. Consider a function T defined by

$$T(x) = 0 \;\text{ for }\; x \leq 0 \;\text{ and }\; T(x) = x \;\text{ for }\; x > 0.$$

Then T is (1/4)-partially relaxed monotone. For distinct element $x, y, z > 0$, we have

$$[T(x) - T(y), z - y] = [T(x) - T(y), x - y] + [T(x) - T(y), z - x]$$
$$= [x - y, x - y] + [T(x) - T(y), z - x]$$
$$= \|x-y\|^2 + [T(x) - T(y), z - x]$$
$$= \|T(x) - T(y)\|^2 + [T(x) - T(y), z - x] \geq (-1/4)\|z-x\|^2,$$

that is, T is (1/4)-partially relaxed monotone. We note that T is also $(3/4) - (2/3)$-partially relaxed monotone.

3. General Auxiliary Minimization Problem

Before we discuss the approximation solvability of the NVIP (1.1), we introduce a general multivalued version of the existing auxiliary minimization problem principle (AMPP) initiated by Cohen [2] and later studied by others, which can also be characterized by the variational inequality known as the general auxiliary minimization problem principle (GAMPP). The general auxiliary minimization problem (GAMP) is described as follows: for some $x^*\in K$ and $u^*\in T(x^*)$,

$$\underset{x\in K}{\text{minimize}} \quad h(x) + \langle \rho u^* - h'(x^*), x\rangle + \rho f(x), \tag{5}$$

where $\rho > 0, T : K\to P(E^*)$ is a nonlinear mapping from a nonempty closed convex subset K of a real reflexive Banach space E into power set $P(E^*)$ of E^*, and $f : K\to R$ is a continuous convex functional on K. If y^* denotes the solution of this problem, then this is equivalent to the solution of a class of general nonlinear variational inequality problems (GNVIP):

$$\langle h'(y^*) + \rho u^* - h'(x^*), x - x^*\rangle + \rho\{f(x) - f(y^*)\} \geq 0\,, \quad \forall x\in K. \tag{6}$$

Note that if $y^* = x^*$ in the GNVIP (6), then x^* is a solution to the NVIP (1).

Based on the GNVIP (6), we can state the general auxiliary minimization problem principle (GAMPP) in the following manner.

Algorithm 3.1. For an arbitrarily chosen initial point $x^0\in K$ and corresponding $u^0\in T(x^0)$, determine an iterate x^{k+1} such that (for $k \geq 0$)

$$\langle \rho u^k + h'(x^{k+1}) - h'(x^k), x - x^{k+1}\rangle + \rho\{f(x) - f(x^{k+1})\} \geq 0\,, \quad \forall x\in K, \tag{7}$$

where $h : K\to R$ is continuously Fréchet-differentiable on K, a convex subset of E, and $\rho > 0$ (the large step).

Next, we recall an auxiliary result crucial to the approximation solvability of the NVIP (1).

Lemma 3.1. *Let E be a Banach space and K be a nonempty convex subset of E. Let $h : K\to R$ be continuously Fréchet-differentiable on K and h' (the derivative of h) be b-strongly monotone.*

Then we have

$$h(x) - h(x^*) - \langle h'(x^*), x - x^*\rangle \geq (b/2)\|x - x^*\|^2\,, \quad \forall x, x^*\in K,$$

where b is a positive constant.

Proof. For $x, x^*\in K$ and $\theta_1\in[0,1]$, we have

$$h(x)-h(x^*)-\langle h'(x^*), x-x^*\rangle$$

$$=\int_0^1 h'[x^*+\theta_1(x-x^*)](x-x^*)d\theta_1-\langle h'(x^*), x-x^*\rangle$$

$$=\int_0^1 \langle h'[x^*+\theta_1(x-x^*)], (x-x^*)\rangle d\theta_1-\langle h'(x^*), x-x^*\rangle$$

$$=\int_0^1 \langle h'[x^*+\theta_1(x-x^*)]-h'(x^*), x-x^*\rangle d\theta_1$$

$$\geq\int_0^1 b\|x-x^*\|^2\theta_1 d\theta_1=(b/2)\|x-x^*\|^2.$$

Now we are just about ready to present, based on Algorithm 3.1, the approximation solvability of the NVIP (1).

Theorem 3.1. *Let E be a reflexive Banach space and $T: K\to P(E^*)$ be the $\gamma-\mu$-partially relaxed monotone mapping from a nonempty closed convex subset K of E into $P(E^*)$, the power set of E^*.*

Let $f: K\to R$ be proper, convex and lower semicontinuous (lsc) on K and $h: K\to R$ be continuously Fréchet-differentiable on K. Suppose that h', the derivative of h, is α-strongly monotone on K. Then there exists a unique solution x^{k+1} to (7). If, in addition, $x^*\in K$ is a solution to the NVIP (1) and

$$0<\rho<\alpha/2\gamma,$$

then the sequence $\{x^k\}$ converges to x^*.

Proof. Since h' is α-strongly monotone on K, it implies x^{k+1} is a unique solution to (7). Assume that y^{k+1} is another solution to (7), that means, we have

$$\langle\rho u^k+h'(x^{k+1})+h'(x^k), x-x^{k+1}\rangle+\rho[f(x)-f(x^{k+1})]\geq 0, \quad \forall x\in K, \quad (8)$$

$$\langle \rho u^k + h'(y^{k+1}) + h'(x^k), x - y^{k+1}\rangle + \rho[f(x) - f(y^{k+1})] \geq 0\,, \qquad \forall x\in K\,. \quad (9)$$

If we replace x by y^{k+1} in (8) and replace x by x^{k+1} in (9), and if we combine, we get

$$-\langle h'(x^{k+1}) - h'(y^{k+1}), x^{k+1} - y^{k+1}\rangle \geq 0\,,$$

or

$$\langle h'(x^{k+1}) - h'(y^{k+1}), x^{k+1} - y^{k+1}\rangle \leq 0. \quad (10)$$

Since h' is α-strongly monotone, it implies that

$$\alpha\|x^{k+1} - y^{k+1}\|^2 \leq \langle h'(x^{k+1}) - h'(y^{k+1}), x^{k+1} - y^{k+1}\rangle \leq 0\,,$$

or

$$\alpha\|x^{k+1} - y^{k+1}\|^2 \leq 0,$$

that is, $x^{k+1} = y^{k+1}$, which shows the uniqueness. □

To show the sequence $\{x^k\}$ converges to x^*, a solution of the NVIP (1), we compute the estimates. Let us first define a function Λ^* by

$$\Lambda^*(x) := h(x^*) - h(x) - \langle h'(x), x^* - x\rangle.$$

Then, by Lemma 3.1, we have

$$\Lambda^*(x) = h(x^*) - h(x) - \langle h'(x), x^* - x\rangle \geq (\alpha/2)||x - x^*||^2,$$

where x^* is any fixed solution of the NVIP (1). It further follows that

$$\Lambda^*(x^{k+1}) = h(x^*) - h(x^{k+1}) - \langle h'(x^{k+1}), x^* - x^{k+1}\rangle.$$

Now we find that

$$\begin{aligned} \Lambda^*(x^k) - \Lambda^*(x^{k+1}) &= h(x^{k+1}) - h(x^k) \\ &- \langle h'(x^k), x^{k+1} - x^k\rangle + \langle h'(x^{k+1}) - h'(x^k), x^* - x^{k+1}\rangle \\ &\geq (\alpha/2)||x^{k+1} - x^k||^2 + \langle h'(x^{k+1}) - h'(x^k), x^* - x^{k+1}\rangle \\ &\geq (\alpha/2)||x^{k+1} - x^k||^2 + \rho\langle u^k, x^{k+1} - x^*\rangle + \rho\{f(x^{k+1}) - f(x^*)\}, \end{aligned} \quad (11)$$

for $x = x^*$ in (7).

If we replace x by x^{k+1} in (1) and if we combine with (11), we obtain

$$\Lambda^*(x^k) - \Lambda^*(x^{k+1}) \geq (\alpha/2)||x^{k+1} - x^k||^2$$
$$+\rho\langle u^k, x^{k+1} - x^*\rangle - \rho\langle u^*, x^{k+1} - x^*\rangle$$
$$= (\alpha/2)||x^{k+1} - x^k||^2 + \rho\langle u^k - u^*, x^{k+1} - x^*\rangle.$$

Since T is $\gamma - \mu$-partially relaxed monotone, it implies that

$$\Lambda^*(x^k) - \Lambda^*(x^{k+1}) \geq (\alpha/2)||x^{k+1} - x^k||^2 - \rho\gamma||x^{k+1} - x^k||^2 + \rho\mu||x^k - x^*||^2$$
$$= (1/2)[\alpha - 2\rho\gamma]||x^{k+1} - x^k||^2 + \rho\mu||x^k - x^*||^2$$
$$\geq \rho\mu||x^k - x^*||^2, \quad \text{for} \quad \alpha - 2\rho\gamma\rangle 0,$$

that is,

$$\Lambda^*(x^k) - \Lambda^*(x^{k+1}) \geq \rho\mu||x^k - x^*||^2, \quad \text{for} \quad \alpha - 2\rho\gamma > 0. \tag{12}$$

It follows from (12) that (except for $x^k = x^*$) the sequence $\{\Lambda^*(x^k)\}$ is strictly decreasing for

$$0 < \rho < \alpha/2\gamma,$$

and nonnegative by its definition. As a result, the difference of two consecutive terms tends to zero, which implies in turn that $x^k \to x^*$ (strongly) as $k \to \infty$. Clearly, when $x^k = x^*$, it is a solution to (1).

4. Applications

In this section, we discuss some applications of Theorem 3.1 in R^n setting, so that these results can unify some of the results of Zhu and Marcotte [16] and Verma [11]. Let K be a nonempty closed convex subset of R^n and let $T : K \to R^n$ be a nonlinear mapping on K. Let $f : K \to R$ be a continuous convex function and $h : K \to R$ be a continuously Fréchet-differentiable function on K. We consider a class of nonlinear variational inequality problems (NVIP) as follows: determine $x^* \in K$ such that

$$[T(x^*), x - x^*] + f(x) - f(x^*) \geq 0\,, \quad \forall x \in K. \tag{13}$$

Next we need the general auxiliary problem principle (GAPP), which is stated as:

Algorithm 4.1. For an arbitrarily chosen initial point $x^0 \in K$, compute an iterate x^{k+1} such that

$$[\sigma T(x^k) + h'(x^{k+1}) - h'(x^k), x - x^{k+1}] + \sigma\{f(x) - f(x^{k+1})\} \geq 0\,, \quad \forall x \in K, \tag{14}$$

where $\sigma > 0$.

Lemma 4.1. *Let K be a nonempty closed convex subset of R^n and $h : K \to R$ be a continuously Fréchet-differentiable function on K. If h', the derivative of h, is α-strongly monotone, then*

$$h(x) - h(x^*) - [h'(x^*), x - x^*] \geq (\alpha/2)\|x - x^*\|^2\,, \quad \forall x, y \in K. \tag{15}$$

Now we can present the solvability of the NVIP (13) based on Algorithm 4.1 and by an application of Lemma 4.1.

Theorem 4.1. *Let $T : K \to R^n$ be $\gamma - \mu$-partially relaxed monotone on a nonempty closed convex subset K of R^n. Let $h : K \to R$ be Fréchet-differentiable on K and $f : K \to R$ be continuous convex on K. If h', the derivative of h, is α-strongly monotone on K, then there exists a unique solution x^{k+1} to (14).*

In addition, if $x^* \in K$ is any fixed solution to the NVIP (13) and

$$0 < \sigma < \alpha/2\gamma,$$

then the sequence $\{x^k\}$ converges strongly to x^*.

Proof. The proof is similar to that of Theorem 3.1. □

Zhu and Marcotte [16] studied the convergence analysis for the auxiliary problem principle in the context of the approximation solvability of the NVIP (13) involving cocoercive mapping – a more general class of mappings than the class of strongly monotone mappings in R^n.

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