International Journal of Pure and Applied Mathematics
Volume 11 No. 2 2004, 165-176

A NEW REFINEMENT ON THE HARDY-HILBERT THEOREM FOR DOUBLE SERIES AND ITS APPLICATIONS

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Abstract: In this paper, it is shown that a new refinement on the Hardy-Hilbert inequality for double series can be established by introducing a proper weight function of the form $\frac{\pi}{\sin(\pi/p)} - \theta_r$, (with $\theta_r > 0$, $r = p, q$) and by means of the positive definiteness of Gram matrix and Euler-Maclaurin summation formula. In particular for case $p = 2$, a sharp result of Hubert Theorem for double series is obtained. As applications, a new extension and a new improvement on Hardy-Littlewood inequality are attained.

AMS Subject Classification: 26D15, 46C99, 15A03, 15A63
Key Words: Hardy-Hilbert inequality, double series, weight function, inner product, Gram matrix

Received: November 4, 2003

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1. Introduction

Let $\{a_n\}$ and $\{b_n\}$ and be two sequences of non-negative real numbers, $1/p + 1/q = 1$ and $p > 1$. If $\sum_{n=0}^{\infty} a_n^p < +\infty$ and $\sum_{n=0}^{\infty} b_n^q < +\infty$, then the Hardy-Hilbert inequality (see [1]) may be written in the form

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{a_m b_n}{m+n+1} \leq \frac{\pi}{\sin \pi/p} \left(\sum_{n=0}^{\infty} a_n^p\right)^{\frac{1}{p}} \left(\sum_{n=0}^{\infty} b_n^q\right)^{\frac{1}{q}}, \tag{1}$$

where the constant factor $\frac{\pi}{\sin \pi/p}$ in (1) is best possible. The equality contained in (1) holds if and only if $\{a_n\}$ and $\{b_n\}$ is a zero-sequence. Recently, the inequality (1) is generalized or improved in some papers (see [1], [2] and [8], etc.). The purpose of this paper is to give a new refinement of (1) which is different from the previous results obtained. At same time, its important applications are given. For convenience, we need to introduce the following notations:

$$(a^r, b^s) = \sum_{n=0}^{\infty} a_n^r b_n^s\,, \quad s_r(\alpha, x) = \left(\alpha^{r/2}, x\right) \|\alpha\|_r^{-r/2}\,,$$

$$\|x\|_r = \left(\sum_{n=0}^{\infty} x_n^r\right)^{1/r}, \quad \|x\|_2 = \|x\|\,.$$

2. Lemmas and Their Proofs

In order to verify our assertions, we need to build the following lemmas.

Lemma 1. *Let $a_n, b_n \geq 0$, $(n = 0, 1, 2, \dots)$, $1/p + 1/q = 1$ and $p > 1$. If $0 < \|a\|_p < +\infty$ and $0 < \|b\|_q < +\infty$, then*

$$(a, b) \leq \|a\|_p \|b\|_q (1 - r)^m\,, \tag{2}$$

where $r = (s_p(a, c) - s_q(b, c))^2$, $m = \min\{1/p, 1/q\}$, $\|c\| = 1$ and $(a^{p/2}, c)(b^{q/2}, c) > 0$. And:

(i) In the case $p \neq q$, the equality in (2) holds if $a^{p/2}$ and $b^{q/2}$ are linearly dependent.

(ii) In the case $p = q$, the equality in (2) holds if a and b are linearly dependent, or the vector c is a linear combination of a and b, and $(a, c)(b, c) = 0$ but the vector c is not simultaneously orthogonal to a and b.

Proof. At first, we consider the case $p \neq q$. Without loss of generality, we suppose that $p > q > 1$. Since $1/p + 1/q = 1$, we have $p > 2$. Let $R = p/2$, $Q = p/(p-2)$. Then $1/R + 1/Q = 1$. By Hölder inequality we obtain

$$(a,b) = \sum_{k=0}^{\infty} a_k b_k = \sum_{k=0}^{\infty} \left(a_k b_k^{q/p}\right) b_k^{1-q/p} \leq \left(\sum_{k=0}^{\infty} \left(a_k b_k^{q/p}\right)^R\right)^{\frac{1}{R}} \times \left(\sum_{k=0}^{\infty} \left(b_k^{1-q/p}\right)^Q\right)^{\frac{1}{Q}} = \left(a^{p/2}, b^{q/2}\right)^{2/p} \|b\|_q^{q(1-2/p)}. \quad (3)$$

The equality in (3) holds if and only if $a^{p/2}$ and $b^{q/2}$ is linearly dependent. In fact, the equality in (3) holds if and only if to any k, there exists c_1 ($c_1 \neq 0$) such that $\left(a_k b_k^{q/p}\right)^R = c_1 \left(b_k^{1-q/p}\right)^Q$. It is easy to deduce that $a_k^{p/2} = c_1 b_k^{q/2}$.

In our previous papers (see [5], [7]), with the help of the positive definiteness of Gram matrix, it is established an important inequality of the form

$$(\alpha,\beta)^2 \leq \|\alpha\|^2\|\beta\|^2 - \left(\|\alpha\|x - \|\beta\|y\right)^2, \quad (4)$$

where $x = (\beta,\gamma)$, $y = (\alpha,\gamma)$ with $\|\gamma\| = 1$ and $xy \geq 0$ (by the way, this condition should be added in the paper [5]). And the equality in (4) holds if and only if α and β are linearly dependent, or the vector γ is a linear combination of α and β, and $xy = 0$ but x and y are not simultaneously equal to zero.

If α, β and γ in (4) are replaced by $a^{p/2}$, $b^{q/2}$ and c respectively, then we get

$$\left(a^{p/2}, b^{q/2}\right)^2 \leq \|a\|_p^p \|b\|_q^q (1-r), \quad (5)$$

where $r = \left(s_p(a,c) - s_q(b,c)\right)^2$. Substituting (5) into (3), we obtain after simplifications

$$(a,b) \leq \|a\|_p \|b\|_q (1-r)^{1/p}. \quad (6)$$

And the equality in (6) holds if and if $a^{p/2}$ and $b^{q/2}$ is linearly dependent.

Notice that the symmetry of p and q. The inequality (2) follows from (6).

Secondly, consider the case $p = 2$. Basing on (4), we obtain at once:

$$(a,b) \leq \|a\|\|b\|(1-\tilde{r})^{1/2}, \quad (7)$$

where $\tilde{r} = \left((a,c)\|a\|^{-1} - (b,c)\|b\|^{-1}\right)^2$, $\|c\| = 1$ and $(a,c)(b,c) > 0$. And the equality in (7) holds if and only if a and b are linearly dependent, or c is a linear combination of a and b, and $(a,c)(b,c) = 0$ but (a,c) and (b,c) are not simultaneously equal to zero. The proof of the lemma is completed. ◻

Lemma 2. *Let $r > 1$ and $n \in N_0$. Define a function H by*

$$H(r,n) = \int_0^{\frac{1}{2n+1}} \frac{1}{1+t}\left(\frac{1}{t}\right)^{1/r} dt\,.$$

Then

$$H(r,n) > \frac{(2n+1)^{1/r}}{2(n+1)}\left(\frac{1}{1-1/r} + \frac{1}{4(n+1)}\right). \tag{8}$$

Proof. Applying integration by parts to $H(r,n)$, we have

$$H(r,n) = \frac{(2n+1)^{1/r}}{2(1-1/r)(n+1)} + \frac{1}{1-1/r}\int_0^{\frac{1}{2n+1}} \frac{t^{1-1/r}}{(1+t)^2}\,dt\,. \tag{9}$$

Now, we have

$$\begin{aligned}\int_0^{\frac{1}{2n+1}} \frac{t^{1-1/r}}{(1+t)^2}\,dt &> \int_0^{\frac{1}{2n+1}} \left(1+\frac{1}{2n+1}\right)^{-2} t^{1-1/r}\,dt \\ &= \frac{(2n+1)^{1/r}}{4(1-1/r)(2-1/r)(n+1)^2} > \frac{(2n+1)^{1/r}}{8(n+1)^2}\,. \end{aligned}\tag{10}$$

It follows from (9) and (10) that the inequality (8) is valid. □

Lemma 3. *Let $\phi \downarrow 0$ $(t \to \infty)$. Then*

$$\int_0^{\infty} \rho(t)\phi(t)\,dt > -\frac{1}{8}\phi(0)\,, \tag{11}$$

where $\rho(t) = t - [t] - 1/2$.

The proof has been given by Zhao (see [7]) and we omit it.

Lemma 4. *Let $r > 0$, $n \in N_0$. Define the function θ by*

$$\theta(r,n) = H(r,n) - \frac{1}{2}f(0) - \int_0^{\infty} \rho(t)f'(t)\,dt\,, \tag{12}$$

where the function f is defined by

$$f(t) = \frac{1}{t+n+1}\left(\frac{2n+1}{2t+1}\right)^{1/r}, \quad t \in [0,+\infty)\,. \tag{13}$$

Then $\theta(r,n) > \frac{(r+1)(2n+1)^{1/r}}{4r(r-1)(n+1)} > 0$.

Proof. Let us denote $\phi(t) = -f'(t)$, where $f(t)$ is defined by (13). It is easy to verify that $\phi(t) \downarrow 0$ $(t \to +\infty)$ and

$$\phi(0) = \frac{(2n+1)^{1/r}}{8(n+1)} \left(\frac{2}{r} - \frac{1}{n+1} \right) .$$

By Lemma 2 and Lemma 3, we have

$$\begin{aligned} \theta(r,n) &= H(r,n) - \frac{(2n+1)^{1/r}}{2(n+1)} + \int_0^\infty \rho(t)\phi(t)\,dt \\ &> \frac{(2n+1)^{1/r}}{2(n+1)} \left\{ \frac{1}{1-1/r} + \frac{1}{4(n+1)} \right\} - \frac{(2n+1)^{1/r}}{2(n+1)} - \frac{1}{8}\phi(0) \\ &= \frac{(2n+1)^{1/r}}{2(n+1)} \left(\frac{1}{r-1} - \frac{1}{2r} \right) = \frac{(r+1)(2n+1)^{1/r}}{4r(r-1)(n+1)} > 0 . \quad \square \end{aligned}$$

3. Theorem and its Corollaries

For sake of convenience, we introduce the following notations:

$$A = \sum_{n=0}^\infty \left(\frac{\pi}{\sin \pi/p} - \theta(q,n) \right) a_n^p , \quad B = \sum_{n=0}^\infty \left(\frac{\pi}{\sin \pi/p} - \theta(p,n) \right) b_n^p ,$$

$$C = \sum_{n=0}^\infty \left(\frac{\pi}{\sin \pi/p} - \theta(p,n) \right) a_n^q ,$$

$$R = \left(\frac{a_0^{p/2}}{\sqrt{A}} - \frac{b_0^{q/2}}{\sqrt{B}} \right)^2 , \quad \theta_r = \frac{(r+1)(2n+1)^{1/r}}{4r(r-1)(n+1)} , \quad r = p, q , \tag{14}$$

where $\theta(r,n)$ is the function defined by (12).

Theorem 1. *Let* $a_0, b_0 > 0$ *and* $a_n, b_n \geq 0$, $n = 1, 2, \ldots$. *If* $0 < \|a\|_p < +\infty$, $0 < \|b\|_q < +\infty$, $1/p + 1/q = 1$ *and* $p > q > 1$, *then*

$$\begin{aligned} \sum_{m=0}^\infty \sum_{n=0}^\infty \frac{a_m b_n}{m+n+1} &< \left(\sum_{n=0}^\infty \left(\frac{\pi}{\sin \pi/p} - \theta_q \right) a_n^p \right)^{1/p} \\ &\times \left(\sum_{n=0}^\infty \left(\frac{\pi}{\sin \pi/p} - \theta_p \right) b_n^q \right)^{1/q} (1 - R/p) , \end{aligned} \tag{15}$$

where θ_r $(r = p, q)$ and R are defined by (14).

Proof. Let us define two functions by

$$\begin{aligned} \alpha &= \frac{a_m}{(m+n+1)^{1/p}} \left(\frac{2m+1}{2n+1}\right)^{1/pq}, \\ \beta &= \frac{b_m}{(m+n+1)^{1/q}} \left(\frac{2n+1}{2m+1}\right)^{1/pq}. \end{aligned} \tag{16}$$

Applying the inequality (2) to estimate the right hand side of (15) as follows:

$$\begin{aligned} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{a_m b_n}{m+n+1} &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \alpha\beta \\ &\le \left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \alpha^p\right)^{1/p} \left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \beta^q\right)^{1/q} (1-\tilde{R})^{1/p} \\ &= \left(\sum_{n=0}^{\infty} \omega(q,n) a_n^p\right)^{1/p} \left(\sum_{n=0}^{\infty} \omega(p,n) b_n^q\right)^{1/q} (1-\tilde{R})^{1/p}, \end{aligned} \tag{17}$$

where the weight function ω is defined by

$$\omega(r,n) = \sum_{m=0}^{\infty} \frac{1}{m+n+1} \left(\frac{2n+1}{2m+1}\right)^{1/r}, \quad r = p, q,$$

and $\tilde{R}$ is defined by

$$\begin{aligned} \tilde{R} = \Bigg\{ &\left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \alpha^{p/2}\gamma\right) \left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \alpha^p\right)^{-1/2} \\ &- \left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \beta^{q/2}\gamma\right) \left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \beta^q\right)^{-1/2} \Bigg\}^2, \end{aligned} \tag{18}$$

where γ is a variable unit-vector.

Consider the function $f(t)$ defined by (13). According to the paper [8], we have

$$\sum_{k=n+1}^{m} f(k) = \int_n^m f(t)\,dt + \frac{1}{2}\left(f(m) - f(n)\right) + \int_n^m \rho(t) f'(t)\,dt,$$

where $\rho(t) = t - [t] - 1/2$.

Let $m \to +\infty$ and $n = 0$. Then we obtain the Euler-Maclaurin summation formula of the form:

$$\sum_{k=0}^{\infty} f(k) = \int_0^{\infty} f(t)\,dt + \frac{1}{2}f(0) + \int_0^{\infty} \rho(t) f'(t)\,dt\,, \tag{19}$$

We may apply the formula (19) to compute the weight function ω in (17).

$$\omega(r,n) = \int_0^{\infty} \frac{1}{t+n+1}\left(\frac{2n+1}{2t+1}\right)^{1/r} dt + \frac{1}{2}f(0) + \int_0^{\infty} \rho(t) f'(t)\,dt =$$

$$\int_0^{\infty} \frac{1}{1+t}\left(\frac{1}{t}\right)^{1/r} dt - \left(\int_0^{\frac{1}{2n+1}} \frac{1}{1+t}\left(\frac{1}{t}\right)^{1/r} dx - \frac{1}{2}f(0) - \int_0^{\infty} \rho(t) f'(t)\,dt \right)$$

$$= \frac{\pi}{\sin \pi/p} - \theta(r,n)\,, \quad r = p, q\,, \tag{20}$$

where the function $\theta(r,n)$ is defined by (12). Hence we get

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \alpha^p = \sum_{n=0}^{\infty} \omega(q,n) a_n^p = A\,, \tag{21}$$

and similarly, we have

$$\sum_{n=0}^{\infty}\sum_{n=0}^{\infty} \beta^q = B\,, \tag{22}$$

where A and B are defined by (14).

Let us choose a unit-vector γ such that

$$\gamma = \begin{cases} 1\,, & m = n = 0\,; \\ 0\,, & m, n \in N_0 \text{ but } m, n \text{ are not simultaneously equal to zero.} \end{cases}$$

Obviously, $\|\gamma\|^2 = \sum\limits_{m=0}^{\infty}\sum\limits_{n=0}^{\infty} \gamma^2 = 1$. It is easy to deduce that

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \alpha^{p/2}\gamma = a_0^{p/2}\,, \text{ and } \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \beta^{q/2}\gamma = b_0^{p/2}\,. \tag{23}$$

Substituting (21), (22) and (23) into (18), $\tilde{R}$ is reduced to R, where R is defined by (14). The inequality (17) therefore may be written in form

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{a_m b_n}{m+n+1} \le A^{1/p}B^{1/q}(1-R)^{1/p}\,. \tag{24}$$

Since α, β and γ are linearly independent, it is impossible to take equality in (24). Notice that $0 < R < 1$ and $0 < 1/p < 1$. Hence we have

$$(1-R)^{1/p} < 1 - R/p\,. \tag{25}$$

By Lemma 4, $\theta(r,n) > \theta$, so $A < \sum_{n=0}^{\infty}\left(\frac{\pi}{\sin\pi/p} - \theta_q\right)a_n^p$, and $B < \sum_{n=0}^{\infty}\left(\frac{\pi}{\sin\pi/p} - \theta_p\right)b_n^q$.

Consequently, it follows from (24) and (25) that the inequality (15) keeps valid, and the proof of the theorem is completed. □

Corollary 1. *Let $a_0 > 0$, $a_n > 0$, $n = 1, 2, \ldots$. If $0 < \|a\|_p < +\infty$, $0 < \|b\|_q < +\infty$, $1/p + 1/q = 1$ and $p \ge q > 1$, then*

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{a_m b_n}{m+n+1} < \left\{\sum_{n=0}^{\infty}\left(\frac{\pi}{\sin\pi/p} - \theta_q\right)a_n^p\right\}^{1/p} \times \left\{\sum_{n=0}^{\infty}\left(\frac{\pi}{\sin\pi/p} - \theta_p\right)b_n^q\right\}^{1/q}(1-\tilde{r}/p)\,, \tag{26}$$

where θ_r $(r = p, q)$ is defined by (14), and $\tilde{r}$ is defined by

$$\tilde{r} = \left\{(a_0^p/A)^{1/2} - (b_0^q/B)^{1/2}\right\}^2, \tag{27}$$

and A, C are defined by (14).

Corollary 2. *If $0 < \|a\| < +\infty$ and $0 < \|b\| < +\infty$, then*

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{a_m b_n}{m+n+1} < \left(\sum_{n=0}^{\infty}(\pi-\theta)a_n^2\right)^{1/2}\left(\sum_{n=0}^{\infty}(\pi-\theta)b_n^2\right)^{1/2}(1-s/2)\,, \tag{28}$$

where $\theta = 2\arctan\frac{1}{2n+1} - \frac{(2n+1)^{1/2}}{2(n+1)} - \frac{(n+2)(2n+1)^{1/2}\xi}{12(n+1)^2} > 0$ $(0 < \xi < 1)$ and s is defined by

$$s = \left\{a_0\left(\sum_{n=0}^{\infty}(\pi-\theta)a_n^2\right)^{-1/2} - b_0\left(\sum_{n=0}^{\infty}(\pi-\theta)b_n^2\right)^{-1/2}\right\}^2.$$

Proof. We need only to give the expression of θ.

Let us consider the function f defined by

$$f(t) = \frac{1}{t+n+1}\left(\frac{2n+1}{2t+1}\right)^{1/2} .$$

Let ω is a weight function defined by

$$\omega(n) = \sum_{n=0}^{\infty} \frac{1}{m+n+1}\left(\frac{2n+1}{2m+1}\right)^{1/2} .$$

We may apply the Euler-Maclaurin summation formula to ω as

$$\begin{aligned}\omega(n) &= \int_0^\infty f(t)\,dt + \frac{1}{2}f(0) - \frac{\xi}{12}f'(0)\\ &= \pi - \left\{2\arctan\frac{1}{2n+1} - \frac{(2n+1)^{1/2}}{2(n+1)} - \frac{(n+2)(2n+1)^{1/2}\xi}{12(n+1)^2}\right\},\end{aligned}$$

where $0 < \xi < 1$ and

$$\theta = \left\{2\arctan\frac{1}{2n+1} - \frac{(2n+1)^{1/2}}{2(n+1)} - \frac{(n+2)(2n+1)^{1/2}\xi}{12(n+1)^2}\right\} = \theta(2,n).$$

It follows from Lemma 4 that

$$\theta(2,n) > \frac{3(2n+1)^{1/2}}{8(n+1)} > 0 .$$

Hence the inequality (28) keeps valid. □

Remark. The vector γ is a variable unit-vector. It may be chosen properly in accordance with our requirements (cf. [5], [7], [6], etc.). Hence the inequality (17) has generality.

4. Applications

Let $f(x) \in L^2(0,1)$ and $f(x) \neq 0$ for all x. Define a sequence $\{a_n\}$ by

$$a_n = \int_0^1 x^n f(x)\,dx\,, \quad n = 0,1,2,\dots .$$

Hardy-Littlewood (see [1]) proved that

$$\sum_{n=0}^{\infty} a_n^2 < \pi \int_0^1 f^2(x)\, dx\,, \tag{29}$$

where π is the best constant that the inequality (29) keeps valid. The following extension and improvement of (29) will be obtained by means of Corollary 1.

Theorem 2. *Under the assumptions just describe, if $1/p + 1/q = 1$ and $p \geq q > 1$, then*

$$\left(\sum_{n=0}^{\infty} a_n^2\right)^2 < \left\{\sum_{n=0}^{\infty}\left(\frac{\pi}{\sin \pi/p} - \theta_q\right) a_n^p\right\}^{1/p} \times \left\{\sum_{n=0}^{\infty}\left(\frac{\pi}{\sin \pi/p} - \theta_q\right) b_n^q\right\}^{1/q} (1 - \tilde{r}/p) \int_0^1 f^2(t)\, dt\,, \tag{30}$$

where θ_r $(r = p, q)$ and $\tilde{r}$ are defined by (14) and (27) respectively.

Proof. By our assumptions, we may write a_n^2 in the form:

$$a_n^2 = \int_0^1 a_n x^n f(x)\, dx\,.$$

Applying Cauchy-Schwarz inequality and Corollary 1 to estimate the right-hand side of (30) as follows:

$$\left(\sum_{n=0}^{\infty} a_n^2\right)^2 = \left(\sum_{n=0}^{\infty}\int_0^1 a_n x^n f(x)\, dx\right)^2 = \left\{\int_0^1 \left(\sum_{n=0}^{\infty} a_n x^n\right) f(x)\, dx\right\}^2 \leq \int_0^1 \left(\sum_{n=0}^{\infty} a_n x^n\right)^2 dx \int_0^1 f^2(x)\, dx$$

$$= \int_0^1 \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_m a_n x^{m+n}\,dx \int_0^1 f^2(x)\,dx$$

$$= \left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{a_m a_n}{m+n+1}\right) \int_0^1 f^2(x)\,dx$$

$$\le \left\{\sum_{n=0}^{\infty}\left(\frac{\pi}{\sin \pi/p} - \theta_q\right) a_n^p\right\}^{1/p} \left\{\sum_{n=0}^{\infty}\left(\frac{\pi}{\sin \pi/p} - \theta_p\right) a_n^q\right\}^{1/q}$$

$$\times (1 - \tilde{r}/p) \int_0^1 f^2(x)\,dx\,, \quad (31)$$

where θ_r $(r = p, q)$ and $\tilde{r}$ is defined by (14) and (27) respectively. Since $f(x) \ne 0$, for all x, $a_n \ne 0$ for all $n \ge 0$. Consequently, it is impossible to take equality in (31). It follows that the inequality (30) is valid and the theorem is therefore proved. □

In particular, for case $p = 2$, we have the following result.

Corollary 3. *Let the assumptions in Theorem 2 hold. Then*

$$\left(\sum_{n=0}^{\infty} a_n^2\right)^2 < \left\{\sum_{n=0}^{\infty} (\pi - \theta) a_n^2\right\} \int_0^1 f^2(x)\,dx\,, \quad (32)$$

where $\theta = 2\arctan \dfrac{1}{2n+1} - \dfrac{(2n+1)^{1/2}}{2(n+1)} - \dfrac{(n+2)(2n+1)^{1/2}\xi}{12(n+1)^2} > 0 \ (0 < \xi < 1)$.

Proof. By Corollary 2, $\theta = \theta(2, n) > 0$.

Obviously, the inequality (30) is an extension of (29), and the inequality (32) is an improvement of (29). □

5. Acknowledgements

Authors would like to express their thanks to Professor L. C. Hsu for his encouragement.

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