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# A CYLINDER THEOREM FOR A (1,1)-GEODESIC AFFINE IMMERSION

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**Abstract:** We prove a cylinder theorem for a $(1,1)$-geodesic affine immersion from a complete complex manifold with complex affine connection to an affine space under a certain assumption relating to the relative nullity space.

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**Key Words:** cylinder theorem, relative nullity distribution, $(1,1)$-geodesic affine immersion

## 1. Introduction

Some sufficient conditions for an isometric immersion from a Riemannian manifold to a Euclidean space to be a cylinder are given in [9] and [12] for example. In the case of a holomorphic isometric immersion from a Kähler manifold to a complex Euclidean space, a cylinder theorem is given in [16]. An isometric immersion from a Kähler manifold to a Riemannian manifold is said to be $(1,1)$-geodesic if the $(1,1)$-part of the complexified second fundamental form vanishes, equivalently, the immersion is pluriharmonic as a map. A pluriharmonic map can be regarded as a generalization of a holomorphic map. Cylinder theorems for a $(1,1)$-geodesic isometric immersion from a complete Kähler manifold to a Euclidean space are given in [2] and [6]. In the case, where the ambient space is

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a pseudo-Euclidean space, it is proved in [8] that if the index of relative nullity of such an immersion from a $2m$-dimensional complete Kähler manifold is not less than $2m-2$, then the immersion is $(2m-2)$-cylindrical.

In this paper, we prove a generalization of the last result, that is, for a $(1,1)$-geodesic affine immersion from a $2m$-dimensional complex manifold with complex affine connection to an affine space, if the manifold is complete with respect to the connection and the index of relative nullity is not less than $2m-2$, then the immersion is $(2m-2)$-cylindrical. Here we say that an affine immersion from a complex manifold with complex affine connection to a real manifold with affine connection is $(1,1)$-geodesic if the $(1,1)$-part of the complexified affine fundamental form vanishes. A $(1,1)$-geodesic isometric immersion can be considered as a $(1,1)$-geodesic affine immersion when we take the normal bundle as a transversal bundle and use Levi-Civita connections. A cylinder theorem for an affine hypersurface in an affine space is shown in [10] under a certain assumption concerning the relative nullity distribution. The result is generalized in [1] to the case of general codimension. To prove our main theorem, we consider a smooth distribution which is contained in the relative nullity distribution and show a cylinder theorem for an affine immersion from a manifold with affine connection to an affine space. This cylinder theorem does not depend on the choice of transversal bundle and can be considered as a generalization of those in [1] and [10].

The paper is organized as follows. Section 2 provides basic definitions and tools for the preceding sections. In Section 3, we prepare fundamental results on an affine immersion between manifolds with affine connections. Especially, we prove the prescribed cylinder theorem for an affine immersion. In Section 4, we prove our cylinder theorem for a $(1,1)$-geodesic affine immersion.

## 2. Splitting of the Tangent Bundle.

In this paper, all manifolds are assumed to be connected and all manifolds and morphisms are assumed to be smooth. Let $M$ be a manifold and $S$, $\widetilde{S}$ real vector bundles over $M$. We denote by $TM$ the tangent bundle and $T^*M$ its cotangent bundle. The fibre of a vector bundle $S$ at $x \in M$ is denoted by $S_x$. We denote by $\mathcal{C}(S)$ the set of connections on $S$ and $\Gamma(S)$ the space of cross sections of $S$. We denote by $A^p = \Gamma(\wedge^p T^*M)$ and $A^p(S) = \Gamma(\wedge^p T^*M \otimes S)$ the space of $p$-forms on $M$ and the space of $p$-forms on $M$ with values in $S$. Let $\mathrm{Hom}(S, \widetilde{S})$ be the vector bundle of which fibre $\mathrm{Hom}(S, \widetilde{S})_x$ at $x \in M$ is the vector space $\mathrm{Hom}(S_x, \widetilde{S}_x)$ of linear maps from $S_x$ to $\widetilde{S}_x$, $\mathrm{HOM}(S, \widetilde{S})$ the space of vector bundle homomorphisms from $S$ to $\widetilde{S}$. Note that $\mathrm{HOM}(S,\widetilde{S})$ can be identified with $\Gamma(\mathrm{Hom}(S,\widetilde{S}))$. For $F \in \mathrm{HOM}(S,\widetilde{S})$ and $x \in M$, we put $F_x := F|_{S_x}$. We will generally use the same symbol to denote a vector bundle homomorphism and the induced linear map on the space of cross sections.

In this paper, the subset $T := \cup_{x\in M} T(x)$ of $TM$, where $T(x)$ is a linear subspace of $T_xM$, is called a distribution on $M$. For $r \in \mathbb{N}$, if $\dim T(x) = r$ for each $x \in M$ and $T$ is a smooth subbundle of $TM$, we say that $T$ is of rank $r$ and smooth. For an open set $U \subset M$, we define $T|_U$ by $T|_U := \cup_{x\in U} T(x) \subset TM|_U$.

Let $T_1$ and $T_2$ be smooth distributions such that $TM = T_1 \oplus T_2$, $\iota_i : T_i \to TM$ the inclusion and $\pi_i : TM \to T_i$ the projection homomorphism, $i = 1, 2$. Hereafter in this section, we always assume that $i, j = 1, 2$ and $i \neq j$. Let $\mathcal{C}_0(TM)$ be the set of torsion free affine connections on $M$.

**Definition 2.1.** For $\nabla \in \mathcal{C}_0(TM)$, we define splitting tensors $C^{T_i}$ and $\hat{C}^{T_i}$ by

$$C^{T_i}_{X_j} X_i := -\pi_i \nabla_{X_i} X_j, \quad \hat{C}^{T_i}_{X_i} Y_i := \pi_j \nabla_{Y_i} X_i\,,$$

for any $X_i, Y_i \in \Gamma(T_i)$.

For a vector bundle $S$ over $M$, a subbundle $S'$ of $S$ and $\nabla \in \mathcal{C}(S)$, we say that $S'$ is *parallel with respect to* $\nabla$ if

$$(\nabla_X \xi)_x \in S'_x\,,$$

for any $\xi \in \Gamma(S')$ and $X \in T_xM$, $x \in M$.

Note that $T_i$ is integrable if and only if $\hat{C}^{T_i}$ is symmetric, that is, $\hat{C}^{T_i}_{X_i} Y_i = \hat{C}^{T_i}_{Y_i} X_i$ for any $X_i, Y_i \in T_i(x)$, $x \in M$ and $T_i$ is parallel with respect to $\nabla$ if and only if both $\hat{C}^{T_i}$ and $C^{T_j}$ vanish.

For $\nabla \in \mathcal{C}_0(TM)$, we define $\nabla^i \in \mathcal{C}(T_i)$ by

$$\nabla^i_Z := (\pi_i \nabla \iota_i)_Z := \pi_i \nabla_Z \iota_i\,,$$

for $Z \in T_xM$, $x \in M$. Then the covariant derivatives of $C^{T_i}$ and $\hat{C}^{T_i}$ are given by

$$\begin{aligned}
(\hat{\nabla}_Z C^{T_i})_{X_j} X_i &:= \nabla^i_Z (C^{T_i}_{X_j} X_i) - C^{T_i}_{\nabla^j_Z X_j} X_i - C^{T_i}_{X_j} \nabla^i_Z X_i, \\
(\hat{\nabla}_Z \hat{C}^{T_i})_{X_i} Y_i &:= \nabla^j_Z (\hat{C}^{T_i}_{X_i} Y_i) - \hat{C}^{T_i}_{\nabla^i_Z X_i} Y_i - \hat{C}^{T_i}_{X_i} \nabla^i_Z Y_i\,,
\end{aligned}$$

for $X_i, Y_i \in \Gamma(T_i)$ and $Z \in \Gamma(TM)$. Next we will show some basic facts concerning splitting tensors, which can be regarded as a generalization of those in [3], [8] and [15] for the Levi-Civita connection and the orthogonal decomposition.

**Lemma 2.2.** *For $\nabla \in \mathcal{C}_0(TM)$, we have*

$$
\begin{aligned}
\pi_i R_{X_i,Y_i} Z_i &= R^i_{X_i,Y_i} Z_i - C^{T_i}_{\hat{C}^{T_i}_{Z_i} Y_i} X_i + C^{T_i}_{\hat{C}^{T_i}_{Z_i} X_i} Y_i, && (2.1)\\
\pi_i R_{X_i,Y_j} Z_i &= R^i_{X_i,Y_j} Z_i + C^{T_i}_{C^{T_j}_{Z_i} Y_j} X_i - \hat{C}^{T_j}_{\hat{C}^{T_i}_{Z_i} X_i} Y_j, && (2.2)\\
\pi_i R_{X_j,Y_j} Z_i &= R^i_{X_j,Y_j} Z_i - \hat{C}^{T_j}_{C^{T_j}_{Z_i} Y_j} X_j + \hat{C}^{T_j}_{C^{T_j}_{Z_i} X_j} Y_j, && (2.3)\\
\pi_j R_{X_i,Y_i} Z_i &= (\hat{\nabla}_{X_i} \hat{C}^{T_i})_{Z_i} Y_i - (\hat{\nabla}_{Y_i} \hat{C}^{T_i})_{Z_i} X_i && (2.4)\\
&\quad + C^{T_j}_{Z_i} (\hat{C}^{T_i}_{Y_i} X_i - \hat{C}^{T_i}_{X_i} Y_i), &&\\
\pi_j R_{X_i,Y_j} Z_i &= -(\hat{\nabla}_{X_i} C^{T_j})_{Z_i} Y_j - (\hat{\nabla}_{Y_j} \hat{C}^{T_i})_{Z_i} X_i && (2.5)\\
&\quad + \hat{C}^{T_i}_{Z_i} C^{T_i}_{Y_j} X_i + C^{T_j}_{Z_i} C^{T_j}_{X_i} Y_j, &&\\
\pi_j R_{X_j,Y_j} Z_i &= (\hat{\nabla}_{Y_j} C^{T_j})_{Z_i} X_j - (\hat{\nabla}_{X_j} C^{T_j})_{Z_i} Y_j && (2.6)\\
&\quad + \hat{C}^{T_i}_{Z_i} (\hat{C}^{T_i}_{X_j} Y_j - \hat{C}^{T_i}_{Y_j} X_j) &&
\end{aligned}
$$

*for any $X_i, Y_i, Z_i \in \Gamma(T_i)$.*

**Definition 2.3.** For $\nabla \in \mathcal{C}_0(TM)$, if $T_i$ is integrable and the induced connection of the maximal integral manifold of $T_i$ is complete, we say that $T_i$ is *complete with respect to* $\nabla$.

We obtain the following, which is given for the Levi-Civita connection and the orthogonal decomposition in [8] and [15].

**Lemma 2.4.** *For $\nabla \in \mathcal{C}_0(TM)$, assume that $T_1$ is complete, $\hat{C}^{T_1} = 0$ and $\pi_2 R_{X_2,Y_1} Z_1 = 0$ for any $Y_1, Z_1 \in T_1(x)$ and $X_2 \in T_2(x)$, $x \in M$. Then the only possible eigenvalue of $C^{T_2}_{X_1} : T_2(x) \to T_2(x)$ is zero for any $X_1 \in T_1(x)$, $x \in M$.*

*Proof.* For $x \in M$, let $L$ be the leaf of $T_1$ through $x$ and $\gamma$ the geodesic in $L$ such that $\gamma(0) = x$ and $\dot{\gamma}(0) = X_1 \in T_1(x)$. Since $T_1$ is complete, the domain of $\gamma$ is $\mathbb{R}$. From (2.5), assumptions that $\hat{C}^{T_1} = 0$ and $\pi_2 R_{X_2,Y_1} Z_1 = 0$ for any $Y_1, Z_1 \in T_1(x)$ and $X_2 \in T_2(x)$, $x \in M$, we get

$$
\begin{cases} (C^{T_2}_{\dot{\gamma}(t)})' = (C^{T_2}_{\dot{\gamma}(t)})^2, \\ C^{T_2}_{\dot{\gamma}(0)} = C^{T_2}_{X_1}. \end{cases} \tag{2.7}
$$

Let $P^{\nabla^2}_t : T_2(x) \to T_2(\gamma(t))$ be the parallel translation with respect to $\nabla^2$ along $\gamma$ and define $C_t$ by

$$
C_t \quad := \quad (P^{\nabla^2}_t)^{-1} \circ C^{T_2}_{\dot{\gamma}(t)} \circ P^{\nabla^2}_t.
$$

Then the domain of $C_t$ is $\mathbb{R}$ since $T_1$ is complete and $C_t$ satisfies (2.7).

Define $\widetilde{C}(t) \in \mathrm{End}(T_2(x))$ by

$$\widetilde{C}(t) \;\; := \;\; C^{T_2}_{X_1}(id_{T_2(x)} - tC^{T_2}_{X_1})^{-1}.$$

Let $\lambda_1, \ldots, \lambda_k$ be all distinct real eigenvalues of $C^{T_2}_{X_1}$ $(k \geq 1)$. Then the domain of $\widetilde{C}(t)$ is $\mathbb{R} \setminus \{\frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_k}\}$. Note that $\frac{1}{0} := \infty \notin \mathbb{R}$. For each $t \in \mathbb{R} \setminus \{\frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_k}\}$, we get

$$\widetilde{C}'(t) = (\widetilde{C}(t))^2, \qquad \widetilde{C}(0) = C^{T_2}_{X_1},$$

that is, $\widetilde{C}(t)$ satisfies the same differential equation as $C_t$. Since $C_t$ is defined for each $t \in \mathbb{R}$, the domain of $\widetilde{C}(t)$ is also $\mathbb{R}$. Hence $\mathbb{R} = \mathbb{R} \setminus \{\frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_k}\}$, that is, $k = 1$ and $\lambda_1 = 0$. □

## 3. Affine Immersions and a Cylinder Theorem

We prepare a notion of an affine immersion and introduce the relative nullity space. Let $M$ and $\widetilde{M}$ be manifolds, $f : M \to \widetilde{M}$ a smooth map, $f^{\#}T\widetilde{M}$ and $f_{\#} : f^{\#}T\widetilde{M} \to T\widetilde{M}$ the induced bundle and its bundle map. We define $i^f : TM \to f^{\#}T\widetilde{M}$ by $i^f_x := (f_{\#x})^{-1} f_{*x}$ for each $x \in M$. Hereafter we consider the case, where $f$ is an immersion in this paper. For a subbundle $N$ of $f^{\#}T\widetilde{M}$, if

$$f^{\#}T\widetilde{M} \;\; = \;\; i^f(TM) \oplus N, \tag{3.1}$$

then we call such an immersion an *immersion with transversal bundle* $N$. Let $\iota_{i^f(TM)} : i^f(TM) \to f^{\#}T\widetilde{M}$, $\iota_N : N \to f^{\#}T\widetilde{M}$ be the inclusions and $\pi_{i^f(TM)} : f^{\#}T\widetilde{M} \to i^f(TM)$, $\pi_N : f^{\#}T\widetilde{M} \to N$ the projection homomorphisms. Put $\hat{i}^f := \pi_{i^f(TM)} i^f \in \mathrm{ISO}(TM, i^f(TM))$. Let $(M, \nabla)$ and $(\widetilde{M}, \widetilde{\nabla})$ be manifolds with torsion free affine connections $\nabla$ and $\widetilde{\nabla}$. We denote by $f^{\#}\widetilde{\nabla}$ the pull-back of $\widetilde{\nabla}$. For an immersion $f : M \to \widetilde{M}$ with transversal bundle $N$, if the induced connection $\pi_{i^f(TM)}(f^{\#}\widetilde{\nabla})\iota_{i^f(TM)}$ on $i^f(TM)$ for $f^{\#}\widetilde{\nabla}$ coincides with $\hat{i}^f \nabla (\hat{i}^f)^{-1}$, we call such a morphism $(f, N) : (M, \nabla) \to (\widetilde{M}, \widetilde{\nabla})$ an *affine immersion with transversal bundle* $N$ and denote it by $f : (M, \nabla) \to (\widetilde{M}, \widetilde{\nabla})$ for simplicity.

In this case, we define the affine fundamental form $B \in A^1(\mathrm{Hom}(TM, N))$, the shape tensor $A \in A^1(\mathrm{Hom}(N, TM))$ and the transversal connection $\nabla^N \in \mathcal{C}(N)$ by

$$B := \pi_N (f^\# \widetilde{\nabla}) \iota_{i^f(TM)} \hat{i}^f, \; A := -(\hat{i}^f)^{-1} \pi_{i^f(TM)} (f^\# \widetilde{\nabla}) \iota_N,$$
$$\nabla^N := \pi_N (f^\# \widetilde{\nabla}) \iota_N.$$

Since $\widetilde{\nabla}$ is torsion free, $B$ is symmetric, that is, $B_X Y = B_Y X$ for any $X, Y \in T_x M$, $x \in M$. Note that $B_X Y$ (resp. $A_X \xi$) is usually denoted by $\alpha(X, Y)$ (resp. $A_\xi X$) for any $X, Y \in T_x M$ and $\xi \in N_x$, $x \in M$. Then we can write Gauss and Weingarten formulas as

$$\begin{aligned} (f^\# \widetilde{\nabla})_X i^f Y &= i^f \nabla_X Y + B_X Y, \\ (f^\# \widetilde{\nabla})_X \xi &= -i^f A_X \xi + \nabla^N_X \xi \,, \end{aligned}$$

for $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(N)$. Then Gauss, Codazzi and Ricci equations are given by

$$\begin{aligned} (\hat{i}^f)^{-1} \pi_{i^f(TM)} \widetilde{R}_{X,Y} i^f Z &= R_{X,Y} Z - A_X B_Y Z + A_Y B_X Z, \\ \pi_N \widetilde{R}_{X,Y} i^f Z &= (\hat{\nabla}_X B)_Y Z - (\hat{\nabla}_Y B)_X Z, \\ (\hat{i}^f)^{-1} \pi_{i^f(TM)} \widetilde{R}_{X,Y} \xi &= -(\hat{\nabla}_X A)_Y \xi + (\hat{\nabla}_Y A)_X \xi, \\ \pi_N \widetilde{R}_{X,Y} \xi &= R^N_{X,Y} \xi - B_X A_Y \xi + B_Y A_X \xi, \end{aligned}$$

where $\widetilde{R}$, $R$, $R^N$ are the curvatures of $(f^\# \widetilde{\nabla})$, $\nabla$, $\nabla^N$, respectively, $(\hat{\nabla}_X A)$ and $(\hat{\nabla}_X B)$ are given by

$$\begin{aligned} (\hat{\nabla}_X B)_Y Z &= \nabla^N_X (B_Y Z) - B_{\nabla_X Y} Z - B_Y \nabla_X Z, \\ (\hat{\nabla}_X A)_Y \xi &= \nabla_X (A_Y \xi) - A_{\nabla_X Y} \xi - A_Y \nabla^N_X \xi \,, \end{aligned}$$

for $X, Y, Z \in \Gamma(TM)$ and $\xi \in \Gamma(N)$.

When there is no danger of confusion, we simply write $f : (M, \nabla) \to (\widetilde{M}, \widetilde{\nabla})$ to denote an affine immersion with transversal bundle.

**Definition 3.1.** For an affine immersion $f : (M, \nabla) \to (\widetilde{M}, \widetilde{\nabla})$, the subspace of $T_x M$ defined by

$$\Delta(x) := \{ X \in T_x M \mid B_X = 0 \}$$

is called the *relative nullity space* of $f$ at $x \in M$, the distribution $\Delta := \cup_{x \in M} \Delta(x)$ is called the *relative nullity distribution* and its dimension $\nu(x)$ is called the *index of relative nullity* at $x \in M$.

Note that the relative nullity distribution does not depend on the choice of transversal bundle. Put $\nu_0 := \min\{\nu(x)|x \in M\}$ and $G := \{x \in M|\nu(x) = \nu_0\}$. Then $G$ is an open set in $M$.

We say that an affine immersion is *totally geodesic* if its affine fundamental form vanishes identically. The property that an affine immersion is totally geodesic does not depend on the choice of transversal bundle, which is shown in [11]. Hence we do not specify the transversal bundle if an affine immersion is totally geodesic. The following is shown in [3] when an immersion is an isometric immersion from a pseudo-Riemannian manifold to a pseudo-Riemannian space form.

**Lemma 3.2.** *For an affine immersion* $f : (M, \nabla) \to (\widetilde{M}, \widetilde{\nabla})$, *we have the following:*

(1) $\Delta|_G$ *is integrable if and only if* $\pi_N \widetilde{R}_{S,T} i^f = 0$ *for any* $S, T \in \Delta(x)$, $x \in G$.

(2) $\Delta|_G$ *is integrable and their leaves are totally geodesic in* $M$ *and* $\widetilde{M}$ *if and only if* $\pi_N \widetilde{R}_{S,Z} i^f T = 0$ *for any* $S, T \in \Delta(x)$ *and* $Z \in T_x M$, $x \in G$.

*Proof.* First we prove (1). From Codazzi equation for $B$, we get

$$\pi_N \widetilde{R}_{S,T} i^f Z = (\hat{\nabla}_S B)_T Z - (\hat{\nabla}_T B)_S Z = B_{[T,S]} Z ,$$

for any $S, T \in \Gamma(\Delta|_G)$ and $Z \in \Gamma(TM|_G)$. Since $Z \in \Gamma(TM|_G)$ is taken arbitrary, $\Delta|_G$ integrable if and only if $\pi_N \widetilde{R}_{S,T} i^f = 0$ for any $S, T \in \Delta(x)$, $x \in G$.

Next we prove (2). From the definition of $\Delta|_G$, we have

$$f^{\#} \widetilde{\nabla}_T i^f S = i^f \nabla_T S + B_T S = i^f \nabla_T S ,$$

for any $S, T \in \Gamma(\Delta|_G)$. Hence if $\Delta|_G$ is integrable, then their leaves are totally geodesic in $\widetilde{M}$. From Codazzi equation for $B$, it follows that

$$\pi_N \widetilde{R}_{S,Z} i^f T = (\hat{\nabla}_S B)_Z T - (\hat{\nabla}_Z B)_S T = -B_{\nabla_S T} Z ,$$

for any $S, T \in \Gamma(\Delta|_G)$ and $Z \in \Gamma(TM|_G)$. Therefore $\Delta|_G$ is integrable and their leaves are totally geodesic in $M$ and $\widetilde{M}$ if and only if $\pi_N \widetilde{R}_{S,Z} i^f T = 0$, for any $S, T \in \Delta(x)$ and $Z \in T_x M$, $x \in G$ . □

We mention that the conditions $\pi_N \widetilde{R}_{S,T} i^f = 0$ and $\pi_N \widetilde{R}_{S,Z} i^f T = 0$, for any $S, T \in \Delta(x)$ and $Z \in T_x M$, $x \in G$ in Lemma 3.2 do not depend on the choice of transversal bundle. Hereafter in this paper, we denote by $(\mathbb{R}^{n+p}, D)$ an $(n + p)$-dimensional affine space with the standard affine connection $D$.

For a manifold $M_i$ and $\nabla^i \in \mathcal{C}_0(TM_i)$, $i = 1, 2$, we always consider the connection $\nabla^{M_1 \times M_2} \in \mathcal{C}_0(T(M_1 \times M_2))$ on $M_1 \times M_2$ such that

$$(\nabla^1_{X_1} Y_1 + \nabla^2_{X_2} Y_2) = \nabla^{M_1 \times M_2}_{(X_1 + X_2)} (Y_1 + Y_2)\,,$$

for any $X_i \in T_{x_i} M_i$, $x_i \in M_i$ and $Y_i \in \Gamma(TM_i)$, $i = 1, 2$, where we use the same symbol to denote $X_i \in T_{x_i} M_i$ and the lift of $X_i \in T_{x_i} M_i$ to $T_{(x_1, x_2)}(M_1 \times M_2)$, $x_i \in M_i$. By $(M_1, \nabla^1) \times (M_2, \nabla^2)$, we mean the product manifold $M_1 \times M_2$ with $\nabla^{M_1 \times M_2} \in \mathcal{C}_0(T(M_1 \times M_2))$.

For manifolds $M$, $M'$ and $\nabla \in \mathcal{C}_0(TM)$, $\nabla' \in \mathcal{C}_0(TM')$, we say that $(M, \nabla)$ and $(M', \nabla')$ are *affine isomorphic* if there exists a local diffeomorphism $\Phi : M \to M'$ such that

$$\Phi_* \nabla_X Y = \nabla'_{\Phi_* X} \Phi_* Y\,,$$

for any $X \in T_x M$, $x \in M$ and $Y \in \Gamma(TM)$. Next we define a cylindrical affine immersion.

**Definition 3.3.** For an $n$-dimensional manifold $M$, we say that an affine immersion $f : (M, \nabla) \to (\mathbb{R}^{n+p}, D)$ is $r$-*cylindrical* if there exist an $(n-r)$-dimensional manifold $M'$, $\nabla' \in \mathcal{C}_0(TM')$ such that $(M, \nabla)$ is affine isomorphic to $(M', \nabla') \times (\mathbb{R}^r, D)$ with respect to $\Phi : M \to M' \times \mathbb{R}^r$ and an affine immersion $f' : (M', \nabla') \to (\mathbb{R}^{n-r+p}, D)$ such that

$$f = (f' \times id_{\mathbb{R}^r}) \circ \Phi.$$

For an $n$-dimensional affine space $\mathbb{R}^n$, an $r$-dimensional $(n > r)$ linear subspace $V$ and each $p \in \mathbb{R}^n$, we denote by $p + V$ the affine subspace of $\mathbb{R}^n$ through $p$ which is parallel to $V$, that is,

$$p + V := \{p + v \mid v \in V\}.$$

To verify the smoothness of the decomposition, our proof is longer and more complicated.

**Proposition 3.4.** *Let $f : (M, \nabla) \to (\mathbb{R}^{n+p}, D)$ be an affine immersion, $\overline{\Delta}$ a smooth distribution of rank $r$ such that $\overline{\Delta} \subset \Delta$. If $\overline{\Delta}$ is parallel and complete with respect to $\nabla$, then $f$ is $r$-cylindrical.*

*Proof.* For each $x \in M$, we denote by $L(x)$ the leaf of $\overline{\Delta}$ through $x$. Since $\overline{\Delta}$ is of rank $r$, parallel and complete with respect to $\nabla$ and it holds that $\overline{\Delta} \subset \Delta$, there is an $r$-dimensional linear subspace $V$ of $\mathbb{R}^{n+p}$ such that

$$f(L(x)) \;=\; f(x) + V, \tag{3.2}$$

that is, $f(L(x))$ is an $r$-dimensional affine subspace of $\mathbb{R}^{n+p}$ through $f(x)$ with associated vector space $V$. Let $W$ be an $(n-r+p)$-dimensional linear subspace of $\mathbb{R}^{n+p}$ such that

$$\mathbb{R}^{n+p} = V \oplus W. \tag{3.3}$$

Fix a point $x_0 \in M$ and set an $(n-r+p)$-dimensional affine subspace of $\mathbb{R}^{n+p}$ through $f(x_0)$ which is transversal to $f(L(x))$ for each $x \in M$ by

$$A^{n-r+p}(f(x_0)) \quad := \quad f(x_0) + W. \tag{3.4}$$

Then from (3.2), (3.3) and (3.4), we have

$$\mathbb{R}^{n+p} = A^{n-r+p}(f(x_0)) \oplus f(L(x_0))\,, \tag{3.5}$$

as affine spaces. According to (3.5), let $\widetilde{p}_1 : \mathbb{R}^{n+p} \to A^{n-r+p}(f(x_0))$ and $\widetilde{p}_2 : \mathbb{R}^{n+p} \to f(L(x_0))$ be the projections. Then

$$M' := (\widetilde{p}_2 \circ f)^{-1}(f(x_0))$$

is an $(n-r)$-dimensional submanifold of $M$ since $M'$ is the inverse image of a single point. Put $f' := f|_{M'}$. Then from the definition of a pull-back bundle, we obtain

$$f'^{\#}T_{f(x_0)}f(L(x_0)) \quad = \quad M' \times T_{f(x_0)}f(L(x_0)). \tag{3.6}$$

Let $f'_{\#} : f'^{\#}T_{f(x_0)}f(L(x_0)) \to T_{f(x_0)}f(L(x_0))$ be the bundle map and define $F : \overline{\Delta}|_{M'} \to T_{f(x_0)}f(L(x_0))$ by

$$F \quad := \quad (\widetilde{p}_2 \circ f)_*|_{\overline{\Delta}|_{M'}}.$$

Then $\widetilde{F} : \overline{\Delta}|_{M'} \to f'^{\#}T_{f(x_0)}f(L(x_0))$ given by

$$\widetilde{F}_y := (f'_{\# y})^{-1}F_y : \overline{\Delta}|_{M'}(y) \to (f'^{\#}T_{f(x_0)}Tf(L(x_0)))_y\,,$$

for $y \in M'$ is a diffeomorphism. Define $\alpha : M' \times T_{f(x_0)}Tf(L(x_0)) \to M' \times \overline{\Delta}(x_0)$ by

$$\alpha(y,v) \quad := \quad (y, (f_{*x_0})^{-1}v)\,,$$

for $(y,v) \in M' \times T_{f(x_0)}Tf(L(x_0))$. Then $\alpha$ is a diffeomorphism. Next define $\beta : M' \times \overline{\Delta}(x_0) \to M' \times L(x_0)$ by

$$\beta(y,u) \quad := \quad (y, \exp_{x_0} u)\,,$$

for $(y,u) \in M' \times \overline{\Delta}(x_0)$. Then $\beta$ is a diffeomorphism. Hence $\Psi : M' \times L(x_0) \to M$ given by

$$\Psi := \exp^{\overline{\Delta}|_{M'}} \circ (\widetilde{F})^{-1} \circ \alpha^{-1} \circ \beta^{-1}$$

is a diffeomorphism. Then

$$\begin{aligned} f(\Psi(y,z)) &= f((\exp^{\overline{\Delta}|_{M'}} \circ (\widetilde{F})^{-1} \circ \alpha^{-1} \circ \beta^{-1})(y,z)) \\ &= f(y) + (\widetilde{p}_2 \circ f)_{*x_0} \overline{f_{*x_0} \exp_{x_0}^{-1} z}_{f(y)} \\ &= f(x_0) + (\overrightarrow{f(x_0)f(y)} + \overrightarrow{(\widetilde{p}_2 \circ f)_{*x_0} f_{*x_0} \exp_{x_0}^{-1} z}) \end{aligned}$$

holds for each $(y,z) \in M' \times L(x_0)$. Put $f'' := f|_{L(x_0)}$ and let $\widetilde{\exp} : T\mathbb{R}^{n+p} \to \mathbb{R}^{n+p}$ be an exponential map. Then we get

$$\begin{aligned} (f' \times f'')(y,z) &= \widetilde{\exp}_{f(y)}(F(f_{*x_0} \exp_{x_0}^{-1} z)_{f(y)}) \\ &= f(x_0) + (\overrightarrow{f(x_0)f(y)} + \overrightarrow{(\widetilde{p}_2 \circ f)_{*x_0} f_{*x_0} \exp_{x_0}^{-1} z}) \,, \end{aligned}$$

for each $(y,z) \in M' \times L(x_0)$. By putting $\Phi := \Psi^{-1} : M \to M' \times L(x_0)$, we have $f = (f' \times f'') \circ \Phi$.

For any $U_1 \in (f'^{\#} T A^{n-r+p}(f(x_0)))_y$, $y \in M'$, we define $\overline{U_1}^1 \in (f^{\#} T(A^{n-r+p}(f(x_0)) \oplus f(L(x_0)))_{(y,z)}$ by

$$\overline{U_1}^1 := (f_{\#\Psi(y,z)})^{-1}(f'_{\#y} U_1) \,,$$

for $z \in L(x_0)$, where we always use the same symbol to denote $f'_{\#y} U_1 \in T_{f(y)} A^{n-r+p}(f(x_0))$ and its lift to $T_{(f(y),f(z))}(A^{n-r+p}(f(x_0)) \oplus f(L(x_0)))$. For any $U_2 \in f''^{\#} T(f(L(x_0)))_z$, $z \in L(x_0)$, we define

$$\overline{U_2}^2 \in (f^{\#} T(A^{n-r+p}(f(x_0)) \oplus f(L(x_0)))_{(y,z)}, \quad y \in M' \,,$$

by a similar way. For any $X_1 \in T_y M'$ and $X_2 \in T_z L(x_0)$, $y \in M'$, $z \in L(x_0)$, we get

$$i^f_{\Psi(y,z)}(\Psi_{*(y,z)}(\widetilde{X_1}^1 + \widetilde{X_2}^2)) \quad = \quad \overline{i^{f'}_y X_1}^1 + \overline{i^{f''}_z X_2}^2 .$$

Hence we have

$$\begin{aligned} i^f_{\Psi(y,z)}(T_{\Psi(y,z)} M) &= i^f((\Psi_{*(y,z)})(T_{(y,z)}(M' \times L(x_0)))) \\ &= \overline{i^{f'}_y (T_y M')}^1 \oplus \overline{i^{f''}_z (T_z L(x_0))}^2 \,, \end{aligned}$$

for each $(y,z) \in M' \times L(x_0)$. Since $f : (M,\nabla) \to (\mathbb{R}^{n+p}, D)$ is an affine immersion, we have

$$\begin{aligned} f^{\#}T\mathbb{R}^{n+p} &= i^f(TM) \oplus N \\ &= \overline{i^{f'}(TM')}^1 \oplus \overline{i^{f''}(T(L(x_0)))}^2 \oplus N, \end{aligned}$$

where $N$ is the transversal bundle of the affine immersion $f$. By the above equation, we obtain

$$\begin{aligned} T_{f(y)}A^{n-r+p}(f(x_0)) &= \widetilde{p}_{1*f(y)}\mathbb{R}^{n+p} = \widetilde{p}_{1*f(y)}f_{\#y}(i^f_y(T_yM) \oplus N_y) \\ &= f'_{*y}T_yM' \oplus \widetilde{p}_{1*f(y)}f_{\#y}N_y\,, \end{aligned}$$

for each $y \in M'$. Then we have

$$f'^{\#}TA^{n-r+p}(f(x_0))_y = i^{f'}_y(T_yM') \oplus (f'_{\#y})^{-1}\widetilde{p}_{1*\Psi(y,z)}f_{\#\Psi(y,z)}N_{\Psi(y,z)}\,,$$

for each $y \in M'$ and $z \in L(x_0)$. Hence $f$ is an immersion with transversal bundle $N'$ given by

$$N'_y \quad := \quad (f'_{\#y})^{-1}\widetilde{p}_{1*\Psi(y,z)}f_{\#\Psi(y,z)}N_{\Psi(y,z)}\,,$$

for each $y \in M'$ and $z \in L(x_0)$.

Next we will show that $f$ is an affine immersion. Let $\nabla' \in \mathcal{C}(TM')$, $\nabla'' \in \mathcal{C}(TL(x_0))$, $D' \in \mathcal{C}(TA^{n-r+p}(f(x_0)))$ and $D'' \in \mathcal{C}(Tf(L(x_0)))$ be the connections defined by

$$\begin{aligned} \Psi_*(\nabla'_{X_1}Y_1 + \nabla''_{X_2}Y_2) &= \nabla_{\Psi_*(X_1+X_2)}\Psi_*(Y_1+Y_2), \\ D'_{U_1}V_1 + D''_{U_2}V_2 &= D_{U_1+U_2}V_1 + V_2\,, \end{aligned}$$

for $X_1, Y_1 \in \Gamma(TM')$, $X_2, Y_2 \in \Gamma(TL(x_0))$, $U_1, V_1 \in \Gamma(TA^{n-r+p}(f(x_0)))$ and $U_2, V_2 \in \Gamma(Tf(L(x_0)))$. Then it follows that $\nabla' \in \mathcal{C}_0(TM')$, $\nabla'' \in \mathcal{C}_0(TL(x_0))$, $D' \in \mathcal{C}_0(TA^{n-r+p}(f(x_0)))$, $D'' \in \mathcal{C}_0(Tf(L(x_0)))$. We also see that $(M,\nabla)$ and $(\mathbb{R}^{n+p}, D)$ are affine isomorphic to $(M',\nabla')\times (L(x_0),\nabla'')$ and $(A^{n-r+p}(f(x_0)), D') \times (f(L(x_0)), D'')$ respectively. Since we have

$$f'_{\#y}\widetilde{p}_{1*f(y)}(f_{\#y})^{-1}B_{X_1}Y_1 \in N'_y\,,$$

for any $X_1, Y_1 \in T_yM'$, $y \in M'$, we see that

$$f' : (M',\nabla') \to (A^{n-r+p}(f(x_0)), D')$$

is an affine immersion with transversal bundle $N'$.

On the other hand, it holds that

$$f''_{\#z}\widetilde{p}_{2*}(f_{\#z})^{-1}B_{X_2}Y_2 = 0\,,$$

for any $X_2, Y_2 \in T_zL(x_0)$, $z \in L(x_0)$. Since $TL(x_0) = \overline{\Delta}|_{L(x_0)}$, $f''$ is totally geodesic and connection preserving. $\square$

When $\overline{\Delta} = \Delta$, Proposition 3.4 is shown in [10] for $p = 1$ and in [1] for general $p$. Our cylinder theorem does not depend on the choice of transversal bundle and can be applied to a lightlike submanifold in a pseudo-Riemannian manifold.

Note that a cylinder theorem for an affine immersion to an affine space in other formulation is presented in [11], [13] and [14] for $p = 1$ and in [4] for general $p$.

Let $\overline{\Delta}$ be a smooth distribution such that $\overline{\Delta} \subset \Delta$ and $\overline{E}$ a smooth distribution such that $TM = \overline{\Delta} \oplus \overline{E}$. According to this decomposition, let $pr_{\overline{\Delta}} : TM \to \overline{\Delta}$ and $pr_{\overline{E}} : TM \to \overline{E}$ be the projection homomorphisms. Note that for any $Z, W \in T_xM$, $x \in M$

$$B_ZW = B_{pr_{\overline{E}}Z}W = B_Z pr_{\overline{E}}W = B_{pr_{\overline{E}}Z} pr_{\overline{E}}W. \tag{3.7}$$

We mention that if $\overline{\Delta} = \Delta$, for any $S \in \Gamma(\Delta)$ and $Z \in \Gamma(TM)$, we get

$$pr_{\overline{E}}(\nabla_S Z) = pr_{\overline{E}}(\nabla_S pr_{\overline{E}}Z), \tag{3.8}$$
$$pr_{\overline{E}}(\nabla_Z S) = pr_{\overline{E}}(\nabla_{pr_{\overline{E}}Z} S). \tag{3.9}$$

Next we define splitting tensors for an affine immersion as a special case of those defined in the previous section.

**Definition 3.5.** For an affine immersion $f : (M, \nabla) \to (\widetilde{M}, \widetilde{\nabla})$, we define the splitting tensors $C^{\overline{\Delta}}$, $\hat{C}^{\overline{\Delta}}$, $C^{\overline{E}}$ and $\hat{C}^{\overline{E}}$ by

$$C^{\overline{\Delta}}_X S := -pr_{\overline{\Delta}}\nabla_S X, \quad \hat{C}^{\overline{\Delta}}_S T := pr_{\overline{E}}\nabla_T S,$$
$$C^{\overline{E}}_S X := -pr_{\overline{E}}\nabla_X S, \quad \hat{C}^{\overline{E}}_X Y := pr_{\overline{\Delta}}\nabla_Y X\,,$$

for any $S, T \in \Gamma(\overline{\Delta})$ and $X, Y \in \Gamma(\overline{E})$.

When $f$ is an isometric immersion from a Riemannian manifold, $\overline{\Delta} = \Delta$ and $\overline{E}$ is the orthogonal complement of $\Delta$ in $TM$, the splitting tensor $C^{\overline{E}}$ is studied in [3], [7], [8] and [16].

Hereafter in this section, we assume that $(\widetilde{M}, \widetilde{\nabla}) = (\mathbb{R}^{n+p}, D)$. Note that for an affine immersion, we have

$$B_{C^{\overline{E}}_T X} Y = B_X C^{\overline{E}}_T Y\,, \tag{3.10}$$

for any $T \in \overline{\Delta}(x)$ and $X, Y \in \overline{E}(x)$, $x \in M$, from Codazzi equation for $B$. From Proposition 3.4 and the definitions of splitting tensors, we obtain the following lemma.

**Lemma 3.6.** *If $\overline{\Delta}$ is of rank $r$ and complete with respect to $\nabla$, then the splitting tensors $C^{\overline{E}}$ and $\hat{C}^{\overline{\Delta}}$ vanish identically if and only if $f$ is $r$-cylindrical.*

Note that the conditions $C^{\overline{E}} = 0$ and $\hat{C}^{\overline{\Delta}} = 0$ in Lemma 3.6 do not depend on the choice of transversal bundle and that of $\overline{E}$ such that $TM = \overline{\Delta} \oplus \overline{E}$.

We will show an example of an affine immersion which is cylindrical with respect to a distribution which is properly contained in the relative nullity distribution which is not parallel.

**Example 3.7.** Let $(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)$ be a coordinate of $\mathbb{R}^8$. Define $f : \mathbb{R}^8 \to \mathbb{R}^9$ by

$$\begin{aligned}
&f(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)\\
&:= (x_1, x_2 + x_1(x_3 + x_4 + x_5), x_3 + x_4 + x_5, x_3 + x_4, x_3 + x_5,\\
&\quad x_6 + x_1 x_7, x_7, x_8, x_1(x_3 + x_4 + x_5)x_7 + x_2 x_7 + (x_3 + x_4 + x_5)x_6).
\end{aligned}$$

We define $\xi$ by $\xi := (0, 0, 0, 0, 0, 0, 0, 0, 1)$ and a transversal bundle $N$ by

$$N_p := \mathrm{Span}\{(f_{\#p})^{-1}\xi\}\,,$$

for each $p \in \mathbb{R}^8$. We regard $f$ as an affine immersion with transversal bundle $N$ as follows. Let $D$ be the standard connection on $\mathbb{R}^9$ and we denote by $\nabla$ the induced connection on $\mathbb{R}^8$ determined by the immersion and the transversal bundle. Then $f : (\mathbb{R}^8, \nabla) \to (\mathbb{R}^9, D)$ is an affine immersion with transversal bundle $N$ by a direct calculation. We mention that in this case, $\mathbb{R}^8$ is complete with respect to $\nabla$. For simplicity, we put $\partial_1 = \frac{\partial}{\partial x_1}$, etc.. Then we have

$$\Delta(p) = \mathrm{Span}\{\partial_1|_p, \partial_8|_p, \partial_3|_p - \partial_4|_p, \partial_3|_p - \partial_5|_p\}.$$

Since $\nabla_{\partial_7}\partial_1 = \partial_6$, $\Delta$ is not parallel with respect to $\nabla$. On the other hand, the distributions $\overline{\Delta}$ and $\overline{E}$ given by

$$\begin{aligned}
\overline{\Delta}(p) &:= \mathrm{Span}\{\partial_8|_p, \partial_3|_p - \partial_4|_p, \partial_3|_p - \partial_5|_p\},\\
\overline{E}(p) &:= \mathrm{Span}\{\partial_1|_p, \partial_2|_p, \partial_6|_p, \partial_7|_p, \partial_3|_p + \partial_5|_p\}\,,
\end{aligned}$$

for each $p \in \mathbb{R}^8$, we obtain $TM = \overline{\Delta} \oplus \overline{E}$, $\hat{C}^{\overline{\Delta}} = 0$ and $C^{\overline{E}} = 0$ by direct calculations. Hence from Lemma 3.6, $f$ is 3-cylindrical. In fact, for $\rho : \mathbb{R}^8 \to \mathbb{R}^5 \oplus \mathbb{R}^3$ given by

$$\begin{aligned}\rho(&x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)\\ &:= (x_1, x_2, x_3 + x_4 + x_5, x_3 + x_4, x_3 + x_5, x_6, x_7, x_8)\\ &=: (y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8)\,,\end{aligned}$$

and $\Psi : \mathbb{R}^8 \to \mathbb{R}^5 \oplus \mathbb{R}^3$ by

$$\Psi(y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8) := ((y_1, y_2, y_3, y_6, y_7), (y_4, y_5, y_8)),$$

$\Phi := \Psi \circ \rho$ is a diffeomorphism. For maps defined by

$$\begin{aligned}f' : \mathbb{R}^5 \ni (y_1, y_2, y_3, y_6, y_7) \mapsto (&y_1, y_2 + y_1 y_3, y_3, y_6 + y_1 y_7, y_7,\\ &y_1 y_3 y_7 + y_2 y_7 + y_3 y_6) \in \mathbb{R}^6,\\ id_{\mathbb{R}^3} : \mathbb{R}^3 \ni (y_4, y_5, y_8) \mapsto (&y_4, y_5, y_8) \in \mathbb{R}^3,\end{aligned}$$

we get

$$f = (f' \times id_{\mathbb{R}^3}) \circ \Phi.$$

Moreover, for $\eta := (0, 0, 0, 0, 0, 1)$ when we define $N'$ defined by

$$N'_q := \mathrm{Span}\{(f'_{\#q})^{-1}\eta\}\,,$$

for each $q \in \mathbb{R}^5$ and $\nabla' \in \mathcal{C}_0(T\mathbb{R}^5)$by

$$\nabla'_X Y := p_{1*}\Phi_* \nabla_{(\Phi^{-1})_* X}(\Phi^{-1})_* Y,$$

for any $X, Y \in \Gamma(T\mathbb{R}^5)$, where $p_1 : \mathbb{R}^5 \oplus \mathbb{R}^3 \to \mathbb{R}^5$ is the projection. Then $f' : (\mathbb{R}^5, \nabla) \to (\mathbb{R}^6, D)$ is an affine immersion with transversal bundle $N'$.

Next we prepare the notion of a real analytic affine immersion and derive fundamental results to apply them to a $(1,1)$-geodesic affine immersion in the next section.

**Definition 3.8.** Let $M$ and $\widetilde{M}$ be real analytic manifolds, $\nabla \in \mathcal{C}_0(TM)$, $\widetilde{\nabla} \in \mathcal{C}_0(T\widetilde{M})$ real analytic connections and $f : (M, \nabla) \to (\widetilde{M}, \widetilde{\nabla})$ an affine immersion with transversal bundle $N$. We say that $f$ is a *real analytic affine immersion* if $f$ is a real analytic map and $N$ is a real analytic vector subbundle of $f^{\#}T\widetilde{M}$, that is, $N$ is a real analytic vector bundle and the inclusion $\iota_N : N \to f^{\#}T\widetilde{M}$ is real analytic.

For a real analytic manifold $M$ and real analytic vector bundles $S$, $\widetilde{S}$ over $M$, we say that $\Psi \in \mathrm{HOM}(S, \widetilde{S})$ is *real analytic* if for any open set $V \subset M$ and real analytic local section $\xi \in \Gamma(S|_V)$, $\Psi\xi$ is a real analytic local section of $\widetilde{S}|_V$.

**Lemma 3.9.** *Let $M$ be a real analytic manifold, $S$, $\widetilde{S}$ real analytic vector bundles over $M$, $\nabla \in \mathcal{C}(S)$, $\widetilde{\nabla} \in \mathcal{C}(\widetilde{S})$ real analytic connections and $\Psi \in \mathrm{HOM}(S, \widetilde{S})$ a real analytic vector bundle homomorphism. If there is a non-empty open set $U \subset M$, such that $\mathrm{Ker}\Psi|_U$ is of rank $q$ and parallel with respect to $\nabla$, then for each $y \in M$ and any real analytic curve $c$ such that $c(0) = x \in U$ and $c(1) = y$, it holds that*

$$P_1^{\nabla}(\mathrm{Ker}\Psi_x) \subset \mathrm{Ker}\Psi_y,$$

*where we denote by $P_t^{\nabla} : T_xM \to T_{c(t)}M$ the parallel translation along $c$. Moreover, it holds that $\dim \mathrm{Ker}\Psi_y \geq q$ for each $y \in M$.*

*Proof.* We fix a point $x \in U$. For each $y \in M$, let $c$ be a real analytic curve such that $c(0) = x$ and $c(1) = y$. For any $\xi \in \mathrm{Ker}\Psi_x$, we denote by $P_t^{\nabla}\xi$ the parallel translation of $\xi$ along $c$. For a basis $\eta_1, \ldots, \eta_r$ of $\widetilde{S}_x$, we denote by $\widetilde{\eta}_l := P_t^{\widetilde{\nabla}}\eta_l$ the parallel translation of $\eta_l$ along $c$, $l = 1, \ldots, r$, where $r$ is the rank of $\widetilde{S}$. According to this frame field, we have

$$\Psi_{c(t)} P_t^{\nabla}\xi = \sum_{l=1}^{r} k^l(t)(\widetilde{\eta}_l)_{c(t)},$$

where $k^l : \mathrm{dom}c \to \mathbb{R}$ is a real analytic function. Let $\delta > 0$ be a constant such that $c((0, \delta)) \subset U$. Then $P_s^{\nabla}\xi \in \mathrm{Ker}\Psi_{c(s)}$ for each $s \in (0, \delta)$. Hence $k^l(s) = 0$ for each $s \in (0, \delta)$. Since $k^l$ is real analytic, we see that $k^l(t) = 0$ for each $t \in \mathrm{dom}c$. Especially, $k^l(1) = 0$. Then we get $P_1^{\nabla}\xi \in \mathrm{Ker}\Psi_y$ and $P_1^{\nabla}\mathrm{Ker}\Psi_x \subset \mathrm{Ker}\Psi_y$. Moreover, $\dim \mathrm{Ker}\Psi_y \geq \dim\mathrm{Ker}\Psi_x = q$. □

In the case where $S = TM$, $\widetilde{S} = \mathrm{Hom}(TM, N)$ and $B \in \mathrm{HOM}(TM, \mathrm{Hom}(TM, N))$, Lemma 3.9 reduces to the following corollary.

**Corollary 3.10.** *Let $f : (M, \nabla) \to (\mathbb{R}^{n+p}, D)$ be a real analytic affine immersion. If there is a non-empty open set $U$ such that $\Delta|_U$ is of rank $q$ and parallel with respect to $\nabla$, then the index of relative nullity is not less than $q$ for each point of $M$. Especially, if the set of all geodesic points has an interior point, then $f$ is totally geodesic.*

When an affine immersion to an affine space is real analytic, we have the following cylinder theorem.

**Lemma 3.11.** *Let $M$ be complete with respect to $\nabla$ and $f : (M, \nabla) \to (\mathbb{R}^{n+p}, D)$ a real analytic affine immersion. Assume that for $q \in \mathbb{N}$, the set of all points $x \in M$ such that $\dim \Delta(x) \neq q$ has no interior point. If there is a non-empty open set $U \subset M$ such that $\Delta|_U$ is of rank $q$ and parallel with respect to $\nabla$, then $f$ is $q$-cylindrical.*

*Proof.* Put $G := \{y \in M \mid \dim \Delta(y) = q\}$. Thus $G \supset U$. Fix a point $x \in U$. Then there is a $q$-dimensional vector subspace $V_x$ of $\mathbb{R}^{n+p}$ such that $f_{*x}\Delta(x) = (f(x) + V_x)_{f(x)}$. For each $y \in M$, let $c$ be a curve such that $c(0) = x$ and $c(1) = y$. Let $P_t^\nabla : T_xM \to T_{c(t)}M$ be the parallel translation along $c$ and for $Y_0 \in \Delta(x)$, we denote by $P_t^\nabla Y_0$ the vector field along $c$ given by the parallel translation of $Y_0$. Then $P_t^\nabla Y_0 \in \Delta(c(t))$ for each $t \in [0, 1]$. Thus, we get

$$(f^\# D)_{\dot{c}} i^f P_t^\nabla Y_0 = i^f \nabla_{\dot{c}} P_t^\nabla Y_0 + B_{\dot{c}} P_t^\nabla Y_0 = 0,$$

that is, $f_{*c(t)} P_t^\nabla Y_0$ is a parallel vector field along $f \circ c$. Let $P_t^D : T_{f(x)}\mathbb{R}^{n+p} \to T_{f(c(t))}\mathbb{R}^{n+p}$ be the parallel translation in $\mathbb{R}^{n+p}$ along $f \circ c$. Then we get

$$f_{*c(t)} P_t^\nabla Y_0 = P_t^D f_{*x} Y_0 \,, \tag{3.11}$$

for each $t \in [0, 1]$. From (3.11), we have

$$f_{*y}\Delta(y) \supset f_{*y} P_1^\nabla \Delta(x) = P_1^D f_{*x}\Delta(x) = P_1^D (f(x) + V_x)_{f(x)}. \tag{3.12}$$

If $y \in G$, there is a $q$-dimensional vector subspace $V_y$ of $\mathbb{R}^{n+p}$ such that $f_{*y}\Delta(y) = (f(y) + V_y)_{f(y)}$. From (3.12), we obtain

$$\begin{aligned}(f(y) + V_y)_{f(y)} &= f_{*y}\Delta(y) = f_{*y} P_1^\nabla \Delta(x) \\ &= P_1^D f_{*x}\Delta(x) = P_1^D (f(x) + V_x)_{f(x)}\end{aligned}$$

and $V_x = V_y$. Put $V_x = V_y = V$. By (3.12), we have

$$P_1^D (f(x) + V)_{f(x)} \subset f_{*y}\Delta(y) \subset f_{*y} T_y M \,,$$

for each $y \in M$. Therefore a distribution $\overline{\Delta}$ defined by

$$\overline{\Delta} := \bigcup_{y \in M} (f_{*y})^{-1} (f(y) + V)_{f(y)} \,,$$

is of rank $q$, parallel with respect to $\nabla$ and satisfies $\overline{\Delta} \subset \Delta$.

Next we prove that $\overline{\Delta}$ is complete with respect to $\nabla$. For each $y \in M$, let $L(y)$ be a leaf of $\overline{\Delta}$ through $y$ and $\gamma$ a geodesic in $L(y)$. Since $\overline{\Delta}$ is parallel with respect to $\nabla$, $L(y)$ is totally geodesic in $M$ and $\gamma$ is a geodesic in $M$. Since $M$ is complete with respect to $\nabla$, $\gamma$ can be extended as the geodesic $\widetilde{\gamma}$ in $M$ such that dom$\widetilde{\gamma} = \mathbb{R}$. Since $\overline{\Delta}$ is parallel with respect to $\nabla$, we obtain $\dot{\widetilde{\gamma}}(t) \in \overline{\Delta}(\widetilde{\gamma}(t))$ for each $t \in \mathbb{R}$. Then the image of $\widetilde{\gamma}$ is contained in $L(y)$ since $\gamma$ is a geodesic in $L(y)$. Hence $L(y)$ is complete with respect to the connection induced from $\nabla$. Therefore $f$ is $q$-cylindrical from Proposition 3.4. □

## 4. (1,1)-Geodesic Affine Immersions

In this section, we study a $(1,1)$-geodesic affine immersion from a complex manifold with complex affine connection to a manifold with affine connection and prove our cylinder theorem for such an immersion to an affine space. For a complex manifold $(M, J)$ with complex structure $J$, we denote by $\mathcal{C}_0(TM, J)$ the set of torsion free affine connections $\nabla$ which satisfies $\nabla_X J = J\nabla_X$ for any $X \in T_xM$, $x \in M$.

**Definition 4.1.** For a complex manifold $(M, J)$, $\nabla \in \mathcal{C}_0(TM, J)$ and an affine immersion $f : (M, \nabla) \to (\widetilde{M}, \widetilde{\nabla})$, we call $f$ $(1,1)$-*geodesic* if

$$B_{JX}Y \;=\; B_X JY\,,$$

for any $X, Y \in T_xM$, $x \in M$.

Note that the equation above is equivalent to the condition that $(1,1)$-part of the complexified affine fundamental form vanishes. The property that an affine immersion is $(1,1)$-geodesic does not depend on the choice of transversal bundle. A $(1,1)$-geodesic isometric immersion is often called a pluriharmonic isometric immersion and such an immersion can be considered as a $(1,1)$-geodesic affine immersion when we take the normal bundle as a transversal bundle and use Levi-Civita connections. We mention that any complex affine immersion between complex manifolds with complex affine connections is a $(1,1)$-geodesic affine immersion.

Let $(M, J)$ be a complex manifold, $\nabla \in \mathcal{C}_0(TM, J)$ and $f : (M, \nabla) \to (\widetilde{M}, \widetilde{\nabla})$ a $(1,1)$-geodesic affine immersion. Hereafter in this section, we assume that the distributions $\overline{\Delta}$ and $\overline{E}$ are $J$-invariant. We obtain fundamental properties of splitting tensors for a $(1,1)$-geodesic affine immersion.

**Lemma 4.2.** *For a* $(1,1)$*-geodesic affine immersion, we have*

$$\begin{aligned} C^{\overline{\Delta}}_{JX}S &= JC^{\overline{\Delta}}_{X}S, & (4.1)\\ \hat{C}^{\overline{E}}_{JX}Y &= J\hat{C}^{\overline{E}}_{X}Y, & (4.2)\\ B_Y C^{\overline{E}}_{JS}X &= B_Y JC^{\overline{E}}_{S}X, & (4.3)\\ B_{C^{\overline{E}}_{T}JX}Y &= B_{C^{\overline{E}}_{JT}X}Y\,, & (4.4) \end{aligned}$$

*for any* $S,T\in\overline{\Delta}(x)$ *and* $X,Y\in\overline{E}(x)$*,* $x\in M$*.*

*Proof.* Since $\overline{\Delta}$ is $J$-invariant, it holds that

$$C^{\overline{\Delta}}_{JX}S = -pr_{\overline{\Delta}}J\nabla_S X = -Jpr_{\overline{\Delta}}\nabla_S X = JC^{\overline{\Delta}}_{X}S,$$
$$\hat{C}^{\overline{E}}_{JX}Y = pr_{\overline{\Delta}}J\nabla_Y X = Jpr_{\overline{\Delta}}\hat{C}^{\overline{E}}_{X}Y = J\hat{C}^{\overline{E}}_{X}Y\,,$$

for any $S\in\Gamma(\overline{\Delta})$ and $X,Y\in\Gamma(\overline{E})$. Since the immersion is $(1,1)$-geodesic, we get

$$\begin{aligned} B_Y C^{\overline{E}}_{JS}X &= -B_Y pr_{\overline{E}}\nabla_X JS = -B_Y J\nabla_X S = -B_{JY}\nabla_X S\\ &= -B_{JY}pr_{\overline{E}}\nabla_X S = B_{JY}C^{\overline{E}}_{S}X = B_Y JC^{\overline{E}}_{S}X\,, \end{aligned}$$

for any $S\in\Gamma(\overline{\Delta})$ and $X,Y\in\Gamma(\overline{E})$. From (3.10) and (4.3), we obtain

$$B_{C^{\overline{E}}_{T}JX}Y \;=\; B_{JX}C^{\overline{E}}_{T}Y = B_X JC^{\overline{E}}_{T}Y = B_X C^{\overline{E}}_{JT}Y = B_{C^{\overline{E}}_{JT}X}Y$$

for any $T\in\overline{\Delta}(x)$ and $X,Y\in\overline{E}(x)$, $x\in M$. □

As a corollary, we get the following corollary.

**Corollary 4.3.** *For a* $(1,1)$*-geodesic affine immersion, assume that* $pr_{\Delta(x)}C^{\overline{E}}=0$ *for each* $x\in M$*. Then we have*

$$\begin{aligned} C^{\overline{E}}_{JS}X &= JC^{\overline{E}}_{S}X, & (4.5)\\ C^{\overline{E}}_{T}JX &= C^{\overline{E}}_{JT}X\,, & (4.6) \end{aligned}$$

*for any* $S,T\in\overline{\Delta}(x)$ *and* $X,Y\in\overline{E}(x)$*,* $x\in M$*.*

*Proof.* For each $x\in M$, consider a subspace $F(x)$ of $T_xM$ such that $T_xM=\Delta(x)\oplus F(x)$. From the assumption that $pr_{\Delta(x)}C^{\overline{E}}=0$ and (4.3), we get

$$C^{\overline{E}}_{JS}X - JC^{\overline{E}}_{S}X \in \Delta(x)\cap F(x)\,,$$

for any $S\in\overline{\Delta}(x)$ and $X\in\overline{E}(x)$, $x\in M$. Hence we have (4.5). By the same way as above, we obtain (4.6). □

The equations (4.5) and (4.6) are already given in [8] when an immersion is a $(1,1)$-geodesic isometric immersion from a Kähler manifold to a pseudo-Euclidean space, $\overline{\Delta} = \Delta$ and $\overline{E}$ is the orthogonal complement of $\overline{\Delta}$.

**Lemma 4.4.** *For a $(1,1)$-geodesic affine immersion, we assume that $\overline{\Delta}$ is complete, $\hat{C}^{\overline{\Delta}} = 0$ and $pr_{\Delta(x)} C^{\overline{E}} = 0$ for each $x \in M$. Then the eigenvalues of $C^{\overline{E}}_T$ are zero.*

*Proof.* We consider $\overline{E}(x)$ as a complex vector space $\overline{E}_{\mathbb{C}}(x)$, $x \in M$ as follows. For $p, q \in \mathbb{R}$ and $X \in \overline{E}(x)$, put

$$(p + q\sqrt{-1})X := pX + qJX.$$

We fix $x \in M$ and take $T \in \overline{\Delta}(x)$ arbitrary. Since $C^{\overline{E}}_T$ satisfies (4.6), we can consider a complexification $C^{\overline{E}}_{\mathbb{C}T}$ of $C^{\overline{E}}_T$. Let $a + b\sqrt{-1}$ be an eigenvalue of $C^{\overline{E}}_{\mathbb{C}T}$, $a, b \in \mathbb{R}$ and $Y \in \overline{E}_{\mathbb{C}}(x)$ an eigenvector of $C^{\overline{E}}_{\mathbb{C}T}$ such that

$$C^{\overline{E}}_{\mathbb{C}T} Y = (a + b\sqrt{-1})Y = aY + bJY.$$

If we put $S := aT - bJT \in \overline{\Delta}(x)$, $x \in M$, it holds from (4.5) that

$$C^{\overline{E}}_{\mathbb{C}S} Y = C^{\overline{E}}_{\mathbb{C}aT - bJT} Y = aC^{\overline{E}}_{\mathbb{C}T} Y - bJC^{\overline{E}}_{\mathbb{C}T} Y = (a^2 + b^2)Y$$

and hence $C^{\overline{E}}_{\mathbb{C}S}$ has a real eigenvalue $a^2 + b^2$. In this case, $C^{\overline{E}}_S$ also has a real eigenvalue $a^2 + b^2$ with respect to $Y$. Then it follows from Lemma 2.4 that $a^2 + b^2 = 0$, which implies the eigenvalues of $C^{\overline{E}}_T$ are zero. □

Note that a similar result is given in [8] for a $(1,1)$-geodesic isometric immersion from a complete Kähler manifold to a pseudo-Euclidean space.

From Lemmas 3.2, 3.11, 4.4 and Corollary 3.10, we have the following cylinder theorem for a $(1,1)$-geodesic affine immersion.

**Theorem 4.5.** *Let $f : (M, \nabla) \to (\mathbb{R}^{2m+p}, D)$ be a $(1,1)$-geodesic affine immersion. If $M$ is complete with respect to $\nabla$ and the index of relative nullity is not less than $2m - 2$ everywhere, then $f$ is $(2m-2)$-cylindrical.*

*Proof.* For a $(1,1)$-geodesic affine immersion $f : (M, \nabla) \to (\mathbb{R}^{2m+p}, D)$, we can show that the map $f$ is real analytic by a similar method for a $(1,1)$-geodesic isometric immersion from a Kähler manifold to a Euclidean space in [5]. Since the property that an affine immersion is $(1,1)$-geodesic does not depend on the choice of transversal bundle and we can choose a transversal bundle for the affine immersion to be real analytic, we assume that $f : (M, \nabla) \to (\mathbb{R}^{2m+p}, D)$ is a real analytic affine immersion.

If the set of geodesic points has an interior point, then from Corollary 3.10, $f$ is totally geodesic.

Hereafter, we consider the only case, where the set of geodesic points has no interior point. Then $G = \{x \in M \mid \dim \Delta(x) = 2m - 2\}$ is an open set in $M$ and $\Delta|_G$ is smooth. Let $E$ be a $J$-invariant smooth distribution on $G$ such that $TM|_G = \Delta|_G \oplus E$. From (2) of Lemma 3.2, we have $\hat{C}^{\Delta|_G} = 0$. Since $E$ is of rank 2, Lemma 4.4 yields $C^E_T = 0$ for any $T \in \Delta(y)$, $y \in G$. Since $T$ is taken arbitrary, we get $C^E = 0$. Thus $\Delta|_G$ is parallel with respect to $\nabla$. Hence from Lemma 3.11, $f$ is $(2m-2)$-cylindrical. □

For a $(1,1)$-geodesic isometric immersion from a Kähler manifold to a pseudo-Euclidean space, Theorem 4.5 is already shown in [8].

We will show two examples of $(1,1)$-geodesic affine immersions which are cylinders.

**Example 4.6.** We denote by $(\mathbb{R}^4, \widetilde{J})$ a 4-dimensional real affine space with the complex structure $\widetilde{J}$ which is induced from the standard complex structure $J_0$ on $\mathbb{R}^4$. Let $(x_1, x_2, x_3, x_4)$ be a coordinate of $\mathbb{R}^4$ such that $\widetilde{J}\frac{\partial}{\partial x_k} = \frac{\partial}{\partial x_{k+2}}$ and $\widetilde{J}\frac{\partial}{\partial x_{k+2}} = -\frac{\partial}{\partial x_k}$, $k = 1, 2$. Define $f : \mathbb{R}^4 \to \mathbb{R}^5$ by

$$f(x_1, x_2, x_3, x_4) := (x_1 + \frac{x_2^2}{2} - \frac{x_4^2}{2}, x_2, x_3 + x_2 x_4, x_4, x_2 x_4 + \frac{x_2^2}{2} - \frac{x_4^2}{2}).$$

For $\xi := (0, 0, 0, 0, 1)$, we define $N$ by

$$N_p := \mathrm{Span}\{(f_{\#p})^{-1}\xi\},$$

for each $p \in \mathbb{R}^4$. We regard $f$ as an affine immersion with transversal bundle $N$ as follows. Let $D$ be the standard affine connection on $\mathbb{R}^5$ and $\nabla$ the induced connection on $\mathbb{R}^4$ determined by the immersion $f$ and the transversal bundle $N$. Then $f : (\mathbb{R}^4, \nabla) \to (\mathbb{R}^5, D)$ is a $(1,1)$-geodesic affine immersion with transversal bundle $N$ and $\mathbb{R}^4$ is complete with respect to $\nabla$ by direct calculations. Then we have

$$\Delta(p) = \mathrm{Span}\{\partial_1|_p, \partial_3|_p\},$$

for each $p \in \mathbb{R}^4$ and the index of relative nullity is two everywhere. We see that $\Delta$ is parallel with respect to $\nabla$ by a straightforward computation and $f$ is 2-cylindrical.

The following cylinder theorem is given in [2] (resp. [8]) for a $(1,1)$-geodesic isometric immersion from a complete Kähler manifold to a Euclidean space (resp. pseudo-Euclidean space) of codimension one.

**Theorem.** ([2], [8]) *For a complete Kähler manifold $M$ and a $(1,1)$-geodesic isometric immersion $f : M \to \mathbb{R}^{2m+1}_N$ to a pseudo-Euclidean space of index $N$, the index of relative nullity is not less than $2m-2$ everywhere and $f$ is $(2m-2)$-cylindrical.*

The next example shows that this theorem does not hold in the case of a $(1,1)$-geodesic affine immersion of codimension one.

**Example 4.7.** Denote by $(\mathbb{R}^8, \widetilde{J})$ an 8-dimensional real affine space with the complex structure $\widetilde{J}$ which is induced from the standard complex structure $J_0$ on $\mathbb{R}^8$. Let $(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)$ be a coordinate of $\mathbb{R}^8$ such that $\widetilde{J}\frac{\partial}{\partial x_k} = \frac{\partial}{\partial x_{k+4}}$ and $\widetilde{J}\frac{\partial}{\partial x_{k+4}} = -\frac{\partial}{\partial x_k}$, $k = 1,2,3,4$. Define $f : \mathbb{R}^8 \to \mathbb{R}^9$ by

$$\begin{aligned} &f(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \\ &\quad := (x_1, x_2 + x_1x_3 - x_5x_7, x_3, x_4, x_5, x_6 + x_1x_7 + x_3x_5, x_7, x_8, \\ &\qquad\qquad x_1x_3x_7 + x_2x_7 + x_3x_6 + \frac{x_5(x_3^2 - x_7^2)}{2}). \end{aligned}$$

For $\xi := (0,0,0,0,0,0,0,0,1)$, we define $N$ by

$$N_p := \mathrm{Span}\{(f_{\#p})^{-1}\xi\},$$

for each $p \in \mathbb{R}^8$. We regard $f$ as an affine immersion with transversal bundle $N$ as follows. Let $D$ be the standard affine connection $D$ and $\nabla$ the induced connection on $\mathbb{R}^8$ determined by the immersion $f$ and the transversal bundle $N$. Then $f : (\mathbb{R}^8, \nabla) \to (\mathbb{R}^9, D)$ is a $(1,1)$-geodesic affine immersion with transversal bundle $N$ by a direct calculation. We mention that $\mathbb{R}^8$ is complete with respect to $\nabla$. We have

$$\Delta(p) = \mathrm{Span}\{\partial_1|_p, \partial_4|_p, \partial_5|_p, \partial_8|_p\},$$

for each $p \in \mathbb{R}^8$ and the index of relative nullity is four everywhere. In this case, $\Delta$ is not parallel with respect to $\nabla$ since $\nabla_{\partial_7}\partial_5 = -\partial_2$. However for distributions $\overline{\Delta}$ and $\overline{E}$ given by

$$\begin{aligned} \overline{\Delta}(p) &:= \mathrm{Span}\{\partial_4|_p, \partial_8|_p\}, \\ \overline{E}(p) &:= \mathrm{Span}\{\partial_1|_p, \partial_2|_p, \partial_3|_p, \partial_5|_p, \partial_6|_p, \partial_7|_p\}, \end{aligned}$$

for each $p \in \mathbb{R}^8$, the splitting tensors $C^{\overline{E}}$ and $\hat{C}^{\overline{\Delta}}$ vanish identically by straightforward computations. Hence from Lemma 3.6, $f$ is 2-cylindrical.

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