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SINGULAR KLEIN SURFACES AND STABLE AND REAL VECTOR BUNDLES

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Abstract: Here we introduce the concept of singular bordered Klein surface using the normalization map and the complex double of a smooth compact Klein surface. As an example we define in this way bielliptic singular bordered Klein surfaces and produce stable vector bundles on their complex double which are defined over $\mathbf{R}$.

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1. Singular Klein Surfaces

Recall that a connected compact Klein surface (possibly with border) is a compact two dimensional real analytic manifold X with border $\partial(X)$ equipped with an open covering $\{U_i\}$ and charts $f_i : U_i \to \mathbf{C}^+$ (where $\mathbf{C}^+ := \{x + iy \in \mathbf{C} : y \geq 0\}$ such that for all $i \neq j$ and every connected component V of $U_i \cap U_j$ $f_i \circ f_j^{-1}|f_j(V)$ is either holomorphic or anti-holomorphic (a so-called dianalytic structure of X). ([1], [5], [4] or [2] for much more). For any connected compact Klein surface $(X, \partial(X))$ there are some geometrically important double coverings and here we will exploit the one called the complex double of X in [1], p. 40; it is a compact Riemann surface $X_{\mathbf{C}}$ (in the classical sense, i.e. equipped

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with a complex structure and without border) together with a morphism of Klein surfaces $f : X_{\mathbf{C}} \to X$ (i.e. a dianalytic map) which is an unramified double covering in the sense of [1], p. 37; notice that if $\partial(X) \neq \emptyset$, this notion of "unramified" does not imply that the map is everywhere two-to-one, because over $\partial(X)$, case (1.6.b) of [1], p. 37, occurs, i.e. over $\partial(X)$ f is essentialy a folding map and in particular $f|f^{-1}(\partial(X)$ induces a bijection between $f^{-1}(\partial(X)$ and $\partial(X)$. If $\partial(X) = \emptyset$ and X is orientable, the dianalytic structure induces a complex structure on X and f splits, i.e. $X_{\mathbf{C}}$ has two connected components. If either $\partial(X) \neq \emptyset$ or X is non-orientable, then $X_{\mathbf{C}}$ is connected, and conversely (see [1], Lemma 1.6.3). There is an anti-holomorphic involution $\sigma : X_{\mathbf{C}} \to X_{\mathbf{C}}$ such that $f \circ \sigma = f$; X is the quotient of $X_{\mathbf{C}}$ by σ. If $X_{\mathbf{C}}$ is connected (i.e. if X is not a classical compact Riemann surface without border), then the field of meromorphic function of X is the invariant field of the field of meromorphic functions on $X_{\mathbf{C}}$ by the action of σ. The existence of an anti-holomorphic involution on the complex projective curve $X_{\mathbf{C}}$ is equivalent to the fact that it may be defined over $\mathbf{R}$ and fixing an anti-holomorphic involution σ we fix one of its real structures, i.e. we fix $X_{\mathbf{R}} \to \mathrm{Spec}(\mathbf{R})$ such that $X_{\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C} \cong X_{\mathbf{C}}$. Since $X_{\mathbf{C}}$ is smooth, the set $X_{\mathbf{C}}(\mathbf{R}) := \{P \in X_{\mathbf{C}} : \sigma(P) = P\}$ is a union of finitely many disjoint circles; call $a(X)$ the number of these circles. The set $X_{\mathbf{C}}(\mathbf{R})$ is the set of real points of $X_{\mathbf{C}}$. We have $X_{\mathbf{C}}(\mathbf{R}) = f^{-1}(\partial(X))$ and in particular $X_{\mathbf{C}}(\mathbf{R}) = \emptyset$ if and only if X has no boundary. If $X_{\mathbf{C}}$ is not connected, then $X_{\mathbf{C}}(\mathbf{R}) = \emptyset$ and σ exchanges the two connected components of $X_{\mathbf{C}}$. Now assume $X_{\mathbf{C}}$ connected and call g the genus of $X_{\mathbf{C}}$. The topological type of the pair $(X_{\mathbf{C}}, \sigma)$, i.e. the topological type of X, is uniquely determined by the integer $a(X)$ and the connectedness or not of $X_{\mathbf{C}} \backslash X_{\mathbf{C}}(\mathbf{R})$, which has at most two connected components; if $X_{\mathbf{C}} \backslash X_{\mathbf{C}}(\mathbf{R})$ is connected, then $0 \leq a(X) \leq g$ and $a(X) \equiv g + 1 \pmod 2$; if $X_{\mathbf{C}} \backslash X_{\mathbf{C}}(\mathbf{R})$ has two connected components, then $1 \leq a(X) \leq g + 1$; all these admissible $a(X)$ arises for some compact Klein surface X ([6], Proposition 3.1). Now we will use the double covering f to define singular compact bordered Klein surfaces. Our main point is that the algebraic-geometrical concept of variety defined over $\mathrm{Spec}(\mathbf{R})$ is the right setting, even for somebody interested in the analytic or topological part of Klein surfaces. Fix a compact bordered Klein surface X; we will think of X as the normalization of our singular compact bordered Klein surface Y. First assume that $X_{\mathbf{C}}$ is not conneted (i.e. that X is orientable and without border) and call Z any of the two connected components of $X_{\mathbf{C}}$. Hence Z is a smooth and connected complex projective curve and hence it has a structural morphism $u : Z \to \mathrm{Spec}(\mathbf{C})$. The complex conjugation induces an order two map $\mathrm{Spec}(\mathbf{C}) \to \mathrm{Spec}(\mathbf{C})$ and composing it with u we obtain the conjugate variety

$\bar{Z}$ of Z. $X_{\mathbf{C}}$ is isomorphic (as a curve over Spec($\mathbf{R}$)) to the disjoint union of Z and $\bar{Z}$. Let Y be an integral complex projective curve with Z as normalization. Since $f|Z : Z \to X$ is an isomorphism, we will call Y a singular compact bordered Klein surface with X as normalization. The composition of the complex conjugation Spec($\mathbf{C}$) $\to$ Spec($\mathbf{C}$) with the structural map $Y \to$ Spec($\mathbf{C}$) defines the conjugate variety $\bar{Y}$. We will say that $Y \cup \bar{Y}$ (disjoint union) is the complex double of Y. The projective curve $Y \cup \bar{Y}$ is defined over Spec($\mathbf{R}$); it is irreducible over Spec($\mathbf{R}$), but is has two irreducible component, Z and $\bar{Z}$, over Spec($\mathbf{C}$). Now assume that $X_{\mathbf{C}}$ is connected. We recall that the normalization of any reduced algebraic variety defined over a field is defined over the same field ([7], Example II.3.8) and in particular that for the normalization, T, of any projective curve defined over Spec($\mathbf{R}$) is defined over Spec($\mathbf{R}$) and the normalization map $h : T \to B$ is defined over Spec($\mathbf{R}$). Let C be any integral projective curve defined over Spec($\mathbf{R}$) and such that $X_{\mathbf{C}}$ (with the real structure determined by σ, i.e. by f) and $u : X_{\mathbf{C}} \to C$ the normalzitation map. Since C is defined over Spec($\mathbf{R}$), there is an anti-holomorphic involution σ' such that $C(\mathbf{R}) = \{P \in C : \sigma'(P) = P\}$. Since u is defined over $\mathbf{R}$, we have $u \circ \sigma = \sigma' \circ u$. Equip $X_{\mathbf{C}}$ and C with their Euclidean topology, not their Zariski topology, and call $\mathcal{A}_{X_{\mathbf{C}}}$ and $\mathcal{A}_C$ the corresponding sheaves of germs of holomorphic functions. Hence C is a singular compact Riemann surface. Let Y be the topological quotient of C by the equivalence relation induced by σ'. Set $\mathcal{A}_Y := \mathcal{A}_C^{\sigma'}$ (the sheaf of invariants). Thus $\mathcal{A}_Y$ is a sheaf of local rings on Y and we will say that the ringed space $(Y, \mathcal{A}_Y)$ is a singular compact bordered Klein surface or a singular Klein curve with X as normalization. The complex projective curve C will be called the complex double of Y and often denoted with $Y_{\mathbf{C}}$. Since $X = X_{\mathbf{C}}/\sigma$ (even as ringed spaces) the normalization map u induces a finite map of local ringed spaces $u : X \to Y$. From the point of view of holomorphic line bundles there is a fundamental difference between real schemes with empty real part and non-empty real part.

Definition 1. Let Y be a singular connected and compact Klein surface. We will say that Y is bi-elliptic if there is a dianalytic degree two morphism $h : Y \to C$, where C is a smooth (possibly bordered) compact Klein surface such that $C_{\mathbf{C}}$ is a smooth elliptic curve.

Definition 2. Let T be an integral complex projective curve. For any rank r torsion-free sheaf F on T, set $\deg(F) := \chi(F) - r(\chi(\mathcal{O}_X))$ and $\mu(F) := \deg(F)/r$ (the slope of F). F is said to be stable (resp. semistable) if and only if for every subsheaf M of F with $1 \leq \mathrm{rank}(M) < r$ we have $\mu(M) < \mu(F)$ (resp. $\mu(M) \leq \mu(F)$). If T and F are defined over Spec($\mathbf{R}$) we have the same definition (called real stability and real semistability) taking as M only the subsheaves of F defined over Spec($\mathbf{R}$).

Remark 1. Let T be a connected algebraic scheme defined over Spec($\mathbf{R}$) and $T_{\mathbf{C}}$ the associated complex algebraic scheme. $T_{\mathbf{C}}$ is connected. The complex conjugation induces an order two map Spec($\mathbf{C}$) $\to$ Spec($\mathbf{C}$) and composing with the structural map $T_{\mathbf{C}} \to$ Spec($\mathbf{C}$) we obtain a morphism of schemes (not of Spec($\mathbf{C}$)-schemes) $\sigma : T_{\mathbf{C}} \to T_{\mathbf{C}}$. For any coherent sheaf $\mathcal{F}$ on T, let $\mathcal{F}_{\mathbf{C}}$ denote the associated sheaf on $T_{\mathbf{C}}$. Then $\sigma^*(\mathcal{F}_{\mathbf{C}}) \cong \mathcal{F}_{\mathbf{C}}$. Conversely if $T(\mathbf{R}) \neq \emptyset$ and M is a line bundle on $T_{\mathbf{C}}$ such that $\sigma^*(M) \cong M$, then there is a line bundle L on T such that $M \cong L_{\mathbf{C}}$ (see [8], Example 1.17). In [6], Section 2 and Section 3, the reader may find several examples (with T a smooth projective curve) for which this is not true when $T(\mathbf{R}) = \emptyset$. By Serre GAGA Theorems, T is projective, there is a natural equivalence betwen the category of vector bundles (or coherent sheaves or line bundles) on T (as algebraic objects) and the corresponding categories of holomorphic objects if we equip T with the euclidean topology instead of the Zariski topology.

Remark 2. Let A be a smooth and geometrically connected curve defined over a field K. Let L be the algebraic closure of K. Let X_L denote the extension of X to L. For any vector bundle E on X let E_L denote the corresponding vector bundle on X_L. E is semistable if and only if E_L is semistable ([9], Proposition 3). Obviously, if E_L is stable, then E is stable. If $K = \mathbf{R}$, $L = \mathbf{C}$ and E is stable, then either E_L is stable or there is a stable bundle F on X_L such that $E_L \cong F \oplus \sigma^*(F)$, where σ denotes the anti-holomorphic involution on A induced by the complex conjugation.

Theorem 1. *Fix positive integers r, d and k such that $k < d/2$. Let Y be a singular Klein bi-elliptic curve. If $(Y_{\mathbf{C}_{reg}})(\mathbf{R}) = \emptyset$, assume d even and k even. Then there is a real rank r stable vector bundle F on $Y_{\mathbf{C}}$ such that $\deg(F) = d$ and $h^0(Y_{\mathbf{C}}, F) = k$.*

Proof. First assume $Y_{\mathbf{C}}$ connected. Let $h : Y \to C$ be the degree two dianalitic map defining a bi-elliptic structure on Y and $m : Y_{\mathbf{C}} \to C_{\mathbf{C}}$ the associated double covering. Hence $C_{\mathbf{C}}$ is an elliptic curve defined over $\mathbf{R}$ and m is defined over $\mathbf{R}$. There is a real semistable vector bundle M on $C_{\mathbf{C}}$ with rank r and degree k; if $C_{\mathbf{C}}$ has no real point, we use that k is even in this case. By Riemann-Roch and the semistability of M we have $h^0(C_{\mathbf{C}}, M) = k$. Set $E := m^*(F)$. Hence E is a rank r and degree $2k$ semistable real vector bundle on $Y_{\mathbf{C}}$ with $h^0(Y_{\mathbf{C}}, E) \geq k$. Since $h^0(Y_{\mathbf{C}}, E) = h^0(C_{\mathbf{C}}, F \otimes m_*(\mathcal{O}_{Y_{\mathbf{C}}}))$ and $m_*(\mathcal{O}_{Y_{\mathbf{C}}})$ is an extension of $\mathcal{O}_{Y_{\mathbf{C}}}$ by a very negative line bundle, we obtain $h^0(Y_{\mathbf{C}}, E) =$

k. Take as F a bundle obtained from E making $d-2k$ positive elementary transformations supported either by points of $(Y_{\mathbf{C}_{reg}})(\mathbf{R})$ or by conjugate pairs of points of $Y_{\mathbf{C}_{reg}}$; here we use that if $(Y_{\mathbf{C}_{reg}})(\mathbf{R}) = \emptyset$, then $d-2k$ is even. For the proof of the stability of F, see the proof of the corresponding result for smooth complex curves done in [3], Theorem 5.3, with the additional checking of the non-existence of destabilizing proper non-locally free subsheaves of F. □

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References

[1] N.L. Allings, N. Greenleaf, *Foundations of the Theory of Klein Surfaces*, Lect. Notes in Math., **219**, Springer, Berlin-Heidelberg-New York (1971).

[2] P. Arés-Gatesi, I. Biswas, The Jacobian of a nonorientable Klein surface, *Proc. Indian Acad. Sci., Math. Sci.*, **113**, No. 2 (2003), 139–152.

[3] E. Ballico, Brill-Noether theory for vector bundles on projective curves, *Math. Proc. Camb. Phil. Soc.*, **124** (1998), 483–499.

[4] E. Bujalance, F.-J. Cirre, J.-M. Gamboa, G. Gromadzki, Symmetry types of hyperelliptic Riemann surfaces, *Mém. Soc. Math. France*, **86** (2001).

[5] E. Bujalance, J.J. Etayo, J.M. Gamboa, G. Gromadzki, *Automorphism Groups of Compact Bordered Klein Surfaces*, Lect. Notes in Math., **1439**, Springer, Berlin-Heidelberg-New York (1990).

[6] B. Gross, J. Harris, Real algebraic curves, *Ann. Scient. Éc. Norm. Sup.*, **14** (1981), 157–182.

[7] R. Hartshorne, *Algbraic Geometry*, Springer, Berlin-Heidelberg-New York (1977).

[8] J. Kollár, Real algebraic surfaces, arXiV: mathAG/971203.

[9] S.G. Langton, Valuative criteria for families of vector bundles on algebraic varieties, *Ann. of Math.*, **101**, No. 1 (1985), 88–110.