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COMPLEX STEIN MANIFOLDS WITH AN ANTI-HOLOMORPHIC INVOLUTION

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Abstract: Here we study complex Stein manifolds equipped with an anti-holomorphic involution and prove some Theorem A type result for "real" holomorphic vector bundles on them.

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1. Introduction

Let $(X, \mathcal{O}_X)$ be an n-dimensional complex manifold and $\sigma : X \to X$ a C^∞ (or real analytic) involution. Let $T_X^{1,0}$ be the rank n complex vector bundle of holomorphic vector fields, T_X the rank r real vector bundle of real C^∞ vector fields on X and $T_X \otimes \mathbf{C}$ its complexification. Hence $T_X^{1,0}$ is a holomorphic vector rank n vector bundle, $T_X \otimes \mathbf{C}$ is a rank $2n$ differentiable complex vector bundle and $T_X^{1,0}$ is an integrable subbundle of $T_X \otimes \mathbf{C}$. We have $T_X \otimes \mathbf{C} \cong T_X^{1,0} \oplus T_X^{0,1}$ (as differentiable vector bundles), where $T_X^{0,1}$ is the complex conjugate of $T_X^{1,0}$. For any P in X the fiber $T_X \otimes \mathbf{C}|\{P\} \cong \mathbf{C}^{2n}$ of $T_X \otimes \mathbf{C}$ at P contains the following two n-dimensional complex linear subspaces $T_X^{1,0}|\{P\}$ and $\sigma^*(T_X^{1,0}|\{\sigma(P)\}$. Let

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$a(P)$ be the dimension of their intersection. We will say that σ has constant rank a if $a(P) = a$ for every $P \in X$. If this is the case, then $\mathcal{H}_{X,\sigma} := T_X^{1,0} \cap \sigma^*(T_X^{1,0})$ is a rank a C^∞ complex subbundle of $T_X \otimes \mathbf{C}$. Since $T_X^{1,0}$ and $\sigma^*(T_X^{1,0})$ are involutive subbundles of $T_X \otimes \mathbf{C}$, $\mathcal{H}_{X,\sigma}$ is an involutive subbundle of $T_X \otimes \mathbf{C}$. If σ has constant rank zero, then we will say that it is totally real. The involution is holomorphic if and only if $a = 0$. In the general case the integer $a(P)$ is an upper semicontinuous function of P. If $\sigma^*(T_X^{1,0} = T_X^{0,1}$, then we will see that σ is anti-holomorphic. Notice that an anti-holomorphic involution is totally real. Equip $Y := X/\sigma$ with the quotient

Definition 1. Let X be a complex manifold (even infinite-dimensional), $\sigma : X \to X$ an anti-holomorphic involution and E a holomorphic vector bundle with finite rank on X. We will say that F is strongly σ-invariant or strongly σ-real if there is an open trivializing cover $\{U_i\}_{i\in I}$ and a defining cocycle $\{g_{ij}\}_{i,j\in I}$ for F, $g_{ij} : U_i \cap U_j \to \mathbf{C}$ such that σ induces a permutation of I (denoted again with σ) such that $\sigma(U_i) = U_{\sigma(i)}$ for every i and $g_{\sigma(i)\sigma(j)} = \bar{g}_{ij}|\sigma$. Any $s \in H^0(X, F)$ will be said to be σ-real if $\bar{s} \circ \sigma = s$. Let $H^0_\sigma(X, F)$ be the $\mathbf{R}$-vector space of all σ-real sections of F. We can make the same definitions for topological or differentiable vector bundles on X.

Remark 1. Let X be a complex manifold, $\sigma : X \to X$ an anti-holomorphic involution and E a strongly σ-real holomorphic vector bundle X. For every open $U \subseteq X$ and every $s \in H^0(U, F)$ we have $\bar{s} \circ \sigma \in H^0(\sigma(U), F)$. Hence if $F|U$ is trivial, then $F|\sigma(U)$ is trivial.

Theorem 1. *Let X be a Stein complex manifold, σ an anti-holomorphic involution and F a strongly σ-real holomorphic vector bundle on X. Then F is spanned by $H^0_\sigma(X, F) \otimes_{\mathbf{R}} \mathbf{C}$.*

Proof. Fix $P \in X$ and let r be the rank of F at P. First assume $P \neq \sigma(P)$. By Theorem A of Cartan-Serre there are $f_i \in H^0(X, F)$, $1 \le i \le r$, such that $f_i(\sigma(P)) = 0$ for every i and $f_1(P), \dots, f_r(P)$ is a basis of the fiber $F|\{P\} \cong \mathbf{C}^r$. Set $h_i := f_i + \bar{f}_i \circ \sigma$. By Remark 1 $h_i \in H^0(X, F)$ for every i. Since σ is an involution, we have $h_i \in H^0_\sigma(X, F)$. By construction $h_i(P) = f_i(P)$ for every i and hence $h_1(P), \dots, h_r(P)$ span the $\mathbf{C}$-vector space $F|\{P\}$. Now assume $P = \sigma(P)$. Since F is strongly σ-real, $F|\{P\}$ gas a real structure for which σ induces the complex conjugations. By Theorem A of Cartan-Serre we may find $f_1, \dots f_r \in H^0(X, F)$ such that $f_1(P), \dots, f_r(P)$ are a $\mathbf{C}$-basis of $F|\{P\}$ and $f_i(P)$ are real for all i. Set $g_i := f_i + \bar{f}_i \circ \sigma$. Since $h_i(P) = 2f_i(P)$ for every i, we are done. □

The same proof gives the following result.

Proposition 1. *Let X be a complex manifold and σ an anti-holomorphic involution on X. If $H^0(X, \mathcal{O}_X)$ gives local coordinates at each point of X, then $H^0_\sigma(X, \mathcal{O}_X)$ gives local coordinates at each point of X. If $H^0(X, \mathcal{O}_X)$ separates points, then for all $P, Q \in X$ such that $Q \neq P$ and $Q \neq \sigma(P)$ there is $f \in H^0_\sigma(X, \mathcal{O}_X)$ such that $f(P) = 1$ and $f(Q) = 0$. If X is a Stein manifold and $\{x_n\}_{n \geq 1}$ is a sequence of points on X without any accumulating point, then there is $f \in H^0_\sigma(X, \mathcal{O}_X)$ such that $\limsup |f(x_n)| = +\infty$; if we also assume $x_i \in \{x_j, \sigma(x_j)\}$ for every $j \neq i$, then there is $u_i \in H^0_\sigma(X, \mathcal{O}_X)$ such that $u_i(x_i) = 1$ and $u_i(x_j) = 0$ for every $j \neq i$.*

For several examples and the general theory when X is one-dimensional and compact, see [1], [2], [3], [4], [5] and references therein.

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