International Journal of Pure and Applied Mathematics

Volume 11 No. 2 2004, 211-213

REAL FLEXES OF REAL PLANE CURVES

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Abstract: Fix an integer $s \geq 2$, $s \neq 3$, s distinct real lines $L_1, \dots, L_s \subset \mathbb{P}^2(\mathbb{R})$ and points $P_i \in L_i(\mathbb{R})$, $1 \leq i \leq s$, such that $P_i \notin L_j$ if $i \neq j$. Fix an integer $d \geq 2s + \epsilon$ with $\epsilon = 0$ for s even, $\epsilon = 3$ for s odd and such that $d \equiv s \pmod 2$. Here we prove the existence of a smooth degree d curve $C \subset \mathbb{P}^2(\mathbb{R})$ defined over $\mathbb{R}$, with only ordinary flexes, with exactly s real flexes such that $P_1, \dots, P_s$ are the real flexes of C and for all i L_i is the tangent line to C at P_i.

AMS Subject Classification: 14P99, 14H50
Key Words: real flex, real plane curve

1. Introduction

Here we stress how easy to construct real objects with prescribed inflexional behaviour on the set of their real points if "the amount of inflexional behaviour is small". As an example in this note we prove the following result concerning real flexes of smooth real plane curves.

Theorem 1. *Fix an integer $s \geq 2$, $s \neq 3$, s distinct real lines $L_1, \dots, L_s \subset \mathbb{P}^2(\mathbb{R})$ and points $P_i \in L_i(\mathbb{R})$, $1 \leq i \leq s$, such that $P_i \notin L_j$ if $i \neq j$. Fix an integer $d \geq 2s + \epsilon$ with $\epsilon = 0$ for s even, $\epsilon = 3$ for s odd and such that $d \equiv s$ (mod 2). Then there is a smooth degree d curve $C \subset \mathbb{P}^2(\mathbb{R})$ defined over $\mathbb{R}$, with only ordinary flexes, with exactly s real flexes such that $P_1, \dots, P_s$ are the real flexes of C and for all i L_i is the tangent line to C at P_i.*

Received: December 12, 2003

Since a smooth plane curve of degree d has exactle $3d(d-2)$ complex flexes (counting multiplicities) (see [3], III.6.3), the assumption "$d \equiv s \pmod 2$" in the statement of Theorem 1 is obviously necessary. We excluded the case $s = 3$ because a smooth real plane cubic C with $C(\mathbb{R}) \neq \emptyset$ has exactly 3 real flexes (see citest, p. 42, or [3]) and hence the configuration of real flexes in this case has a very strong restriction (indeed, this is the only restriction for $d = 3$).

Definition 1. Fix an integer $x \geq 3$, a plane curve C, $P \in C$ and a line $D \subset \mathbb{P}^2$ such that $P \in D$. We will say that D is a flex with length x for C at P if $P \in C_{reg}$ and the connected component supported by P of the scheme-theoretic intersection $C \cap D$ is the effective degree x divisor xP of D.

Proposition 1. *Fix an integer $s \geq 2$, $s \neq 3$, s distinct real lines $L_1, \dots, L_s \subset \mathbb{P}^2(\mathbb{R})$ and points $P_i \in L_i(\mathbb{R})$, $1 \leq i \leq s$, such that $P_i \notin L_j$ if $i \neq j$. Fix an integer $d \geq 2s + \epsilon$ with $\epsilon = 2$ for s even, $\epsilon = 5$ for s odd such that $d \equiv s \pmod 2$. Then there is a smooth degree d curve $C \subset \mathbb{P}^2(\mathbb{R})$ defined over $\mathbb{R}$, with only ordinary flexes, exept one, P_1, with exactly $s + 1$ real flexes such that $P_1, \dots, P_s$ are the real flexes of C, P_1 is a flex of length 4, and for all i L_i is the tangent line to C at P_i.*

Many variations of Proposition 1 concerning real flexes with length x (in particular the case of one flex with length d) and of both Theorem 1 and Proposition 1 for real singular plane curves are obviously possible, but left to the interested reader.

Proof of Theorem 1. We divide the proof into three parts.

(a) Here we assume $s = 2$. By [1], p. 174, there is a smooth degree 4 real plane curve D such that $D(\mathbb{R})$ has exactly one connected component. By [1], Proposition 7.3, D has exactly two real flexes, say Q_1 and Q_2 with real tangent lines T_1 and T_2. We may find D such that $Q_1 \notin T_2$, $Q_2 \notin T_1$ and all flexes of D are ordinary. The configurations of line and points $(P_1, L_1; P_2, L_2)$ and $(Q_1, T_1; Q_2, T_2)$ are projectively equivalent. Hence there is $g \in PGL(2, \mathbb{R})$ such that $g(D)$ is real, with only ordinary flexes, with P_1 and P_2 as only real flex points and with L_1 and L_2 as the corresponding tangent lines. Set $m := (d-s)/2$ and choose m sufficiently general smooth real lane conics M_x, $1 \leq x \leq m$ without real points. Taking the se conics sufficiently general, we may assume that $A := D \cup M_1 \cup \cdot \cup M_m$ has only ordinary nodes as singularieties. By construction A has exactly the same real flexes as D. Since no point $M_j \cap D$ is real, for every small real smoothing, C, of A, C has exactly two real flexes. For general small smoothing we may even assume that C has only ordinary flexes. Taking again a real projective transformation we may assume that C has P_1 and P_2 as flex points and L_1, L_2 as corresponding flex tangent lines.

(b) Here we assume $s \geq 4$ and s even. Let $Z \subset \mathbb{P}^2(\mathbb{R})$ the degree $3s$ zero-dimensional scheme union of the degree 3 divisors $3P_i$ of L_i, $1 \leq i \leq s$. For every j with $1 \leq j \leq s/2$ call D_j, A_j and C_j the curve constructed in part (a) for the lines L_j and $L_{j+s/2}$. We may find D_j, $1 \leq j \leq s/2$, such that $D_1 \cup \cdots \cup D_{s/2}$ is nodal and none of its singular point is nodal. Hence we may find A_j as in part (a) such that no singular point of $A_1 \cup \cdots \cup A_{s/2}$ is real. A small real smooting of $A_1 \cup \cdots \cup A_{s/2}$ has only ordinary flexes and exactly s real flexes. Since $d \gg s$ we may find such real smooting among the plane curves containing Z. Such a smoothing gives a solution.

(c) Here we assume $s \geq 5$ and s odd. We start with a real smooth plane curve D of degree 5 whose real locus $D(\mathbb{R})$ has exactly one connected component (see [1], p. 177) and with one or three or five real flexes. Then we add real plane conics without real points and we make a smoothing, as we did in parts (a) and (b). □

Proof of Proposition 1. Let B, B' be distint real smooth plane conics with $P_1 \in B \cap B'$ and such L_1 is the tangent line to both B and B' at P_1. Fix a general real line R. To obtain a smooth real plane D quartic with L_1 as flex tangent to D at P_1 use a small real smoothing of $B \cup B'$ among the plane curves containing the divisor $4P_1$ of L_1. To obtain a smooth real plane D quintic with L_1 as flex tangent to D at P_1 use a small real smoothing of $B \cup B'$ among the plane curves containing the divisor $4P_1$ of L_1. Then copy the proof of Theorem 1. □

Acknowledgements

The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

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