International Journal of Pure and Applied Mathematics
Volume 11 No. 2 2004, 215-216

PROJECTIVE CURVES AND UNRAMIFIED PROJECTIONS

E. Ballico
Department of Mathematics
University of Trento
380 50 Povo (Trento) - Via Sommarive, 14, ITALY
e-mail: ballico@science.unitn.it

Abstract: Here we prove the following result. Let X be a smooth and connected projective curve, x a non-negative integer, $L \in \mathrm{Pic}(X)$ and $V \subseteq H^0(X, L)$ a linear subspace such that $\dim(V) \geq 3x + 4$, V spans L and the associated morphism $\phi_V : X \to \mathbf{P}^r$, $r := \dim(V) - 1$, is unramified and birational onto its image. Then for general $P_1, \dots, P_x$ the linear system $V(-2P_1 - \cdots - 2P_x)$ has no base points, $\dim(V(-2P_1 - \cdots - 2P_x)) = \dim(V) - 2x$ and the associated morphism $\phi_{V(-2P_1-\cdots-2P_x)}$ is unramified and birational onto its image.

AMS Subject Classification: 14N05, 14H50
Key Words: secant variety, tangent variety, fat point, projective curve

1. Unramified Morphism from Curves

For any projective variety Y, any $L \in \mathrm{Pic}(X)$ and any linear suspace $V \subset H^0(Y, L)$ spanning L, let $\phi_V : Y \to \mathbf{P}(V^*)$ be the associated morphism. For any closed subscheme Z of Y, set $V(-Z) := \{F \in V : f|Z \equiv 0\}$. We work over an algebraically closed field $\mathbf{K}$ with $\mathrm{char}(\mathbf{K}) = 0$ and prove the following result.

Theorem 1. *Let X be a smooth and connected projective curve, x a non-negative integer, $L \in \mathrm{Pic}(X)$ and $V \subseteq H^0(X, L)$ a linear subspace such that $\dim(V) \geq 3x + 4$, V spans Land the associated morphism $\phi_V : X \to$*

Received: December 23, 2003 © 2004, Academic Publications Ltd.

$\mathbf{P}^r$, $r := \dim(V) - 1$, is unramified and birational onto its image. Then for general $P_1, \dots, P_x$ the linear system $V(-2P_1 - \cdots - 2P_x)$ has no base points, $\dim(V(-2P_1 - \cdots - 2P_x)) = \dim(V) - 2x$ and the associated morphism $\phi_{V(-2P_1 - \cdots - 2P_x)}$ is unramified and birational onto its image.

Proof of Theorem 1. The equality $\dim(V(-2P_1 - \cdots - 2P_x)) = \dim(V) - 2x$ is a very particular case of [2]. □

Claim. *For general $P_1 \dots, P_x$ the linear system $V(-2P_1 - \cdots - 2P_x)$ has no base point.*

Proof of Claim. We use induction on x. Assume the Claim for the integer $x' := x - 1$, with the convention that for $x = 1$ instead of $V(-2P_1 - \cdots - 2P_{x'})$ we use V. The Claim for the integer x means that a general tangent line to $\phi_{V(-2P1 - \cdots - 2P_{x-1}}(X) \subset \mathbf{P}^{r-2x+2}$ does not intersect $\phi_{V(-2P1 - \cdots - 2P_{x-1})}(X)$. This is true by [3] because $r - 2(x-1) \geq 3$. By [1], Theorem 1.4, a general hyperplane of $\mathbf{P}^(V^*)$ tangent to $\phi_V(A)$ at $\phi_V(P_1), \dots, \phi_V(P_x)$ is not tangent to $\phi_V(X)$ at any other smooth point of $\phi_V(X)$ or to any branch of $\Phi_V(X)$ at one of its singular points (if any). The last part does not follow formally from the statement of [1], Theorem 1.4, but it is quite easy to prove it working on X instead of working on $\phi_V(X)$. By the Claim this is equivalent to the fact that $\phi_{V(-P_1 - \cdots - P_x)}$ is unramified. We obtain, using assumption $\dim(V) \geq 3x + 4$, that for general $P_1, \dots, P_{x+1}$ $V(-2P_1 - \cdots - 2P_{x+1})$ induces an unramified morphism. This implies that $\phi_{V(-2P_1 - \cdots - 2P_x)}$ is birational onto its image. □

Acknowledgements

This research was partially supported by MIUR and GNSAGA of INdAM (Italy).

References

[1] L. Chiantini, C. Ciliberto, Weakly defective varieties, *Trans. Amer. Math. Soc.*, **354**, No. 1 (2002), 151–178.

[2] C. Ciliberto, R. Miranda, Interpolations on curvilinear schemes, *J. Algebra*, **203**, No. 2 (1998), 677-678.

[3] H. Kaji, On tangentially degenerate curves, *J. London Math. Soc.*, **33**, No. 3 (1986), 430–440.