International Journal of Pure and Applied Mathematics

Volume 11 No. 2 2004, 217-227

HIGHER-ORDER DUALITY FOR A CLASS OF NONDIFFERENTIABLE MULTIOBJECTIVE PROGRAMMING PROBLEMS

S.K. Mishra[1], S.Y. Wang[2], K.K. Lai[3 §]

[1]Department of Mathematics, Statistics and Computer Science
College of Basic Sciences and Humanities
Govind Ballabh Pant University of Agriculture and Technology
Pantnagar, 263 145, INDIA

[2]Institute of Systems Sciences
Academy of Mathematics
Chinese Academy of Sciences
Beijing, 100 800, P.R. CHINA

[3] Department of Management Sciences
City University of Hong Kong
Tat Chee Avenue, Kowloon, HONG KONG
e-mail: mskklai@cityu.edu.hk

Abstract: In this paper we formulate a number of higher-order duals to a nondifferentiable multiobjective programming problem and establish higher-order duality results under the higher-order generalized invexity assumptions. A special case that appears repeatedly in the literature is that the support function is the square root of a positive semidefinite quadratic form. This and other special cases can be readily generated from our results.

AMS Subject Classification: 26A33
Key Words: higher order duality, invexity, multi-objective programming

1. Introduction

Mond and Schechter [13] studied nondifferentiable symmetric duality, in which

Received: January 6, 2004

§Correspondence author

the objective function contains a support function. Following Mond and Schechter [13], Yang et al [17] introduced a nondifferentiable multiobjective programming problem in which the objective function contains a support function of a compact convex set, for which the subdifferential may be simply expressed and duality theorems under (F, ρ)-convexity and generalized (F, ρ)-convexity assumptions.

Several approaches to duality for a multiobjective programming problem may be found in the literature. These include the use of the first order dual [1-3, 5-7, 9, 16] and the second order dual [4, 7, 8, 10, 11, 14, 15, 18] to establish duality theorems. Zhang and Mond [19] proposed a class of functions called second order (F, ρ)-convex as a generalization of (F, ρ)-convex functions and established duality theorems for second order Mangasarian, second order Mond-Weir and second order general Mond-Weir duals to the multiobjective programming problem.

A higher order dual to the scalar nonlinear programming problem:

(P)
$$\begin{array}{c} \text{Min } f(x) \, , \\ \text{subject to: } \quad g(x) \geq 0, \end{array}$$

was formulated by Mangasarian [8]. However, Mangasarian proved only a limited version of strong duality only. Later, Mond and Weir [14] gave the conditions for which duality holds between (P) and its higher order Mangasarian type dual.

Zhang [18] introduced higher order (F, ρ)-convexity and established higher order duality for multiobjective programming problems, thereby extended the results of Gulati [5], Mangasarian [8], Preda [9], Mishra and Rueda [11], Mond and Weir [14] and Zhang and Mond [19].

Mond [12] considered the class of nondifferentiable mathematical programming problems

(NDP)
$$\begin{array}{c} \text{Min } f(x) + \left(x^T B x\right)^{\frac{1}{2}} \, , \\ \text{subject to: } \quad g(x) \geq 0, \end{array}$$

where f and g are twice differentiable functions from $f_i(x) - f_i(u) \leq h_i(u,p) - p^T \nabla_p h_i(u,p) \Rightarrow \eta(x,u)^T \nabla_p h_i(u,p) \leq 0, \; \forall \, i \in \{1, 2, ..., l\}$ to R and R^m, respectively and B is an $n \times n$ positive semi-definite (symmetric) matrix.

Zhang [18] introduced Mangasarian type [8] and Mond-Weir type [14] higher order dual to (NDP) as follows:

(NDHMD)
$$\begin{array}{l} \text{Max } f(u) + h(u,p) + (u+p)^T Bw - y^T g(u) - y^T k(u,p) \\ \quad \text{subject to: } \quad \nabla_p h(u,p) + Bw = \nabla_p \left(y^T k(u,p)\right), \\ \qquad\qquad\qquad\qquad w^T Bw \leq 1, \; y \geq 0, \end{array}$$

where $u, w, p \in R^n$ and $y \in R^m$;

$$\text{(NDHD)} \quad \begin{array}{c} \text{Max } f(u) + h(u,p) + u^T Bw - p^T \nabla_p h(u,p) \\ \text{subject to: } \nabla_p h(u,p) + Bw = \nabla_p \left(y^T k(u,p) \right), \\ y^T g(u) + y^T k(u,p) - p^T \nabla_p \left(y^T k(u,p) \right) \leq 0, \\ w^T Bw \leq 1, \; y \geq 0, \end{array}$$

where $V\text{-Maximize}\begin{pmatrix} f_1(u) + h_1(u,p) + u^T w_1 - p^T \nabla_p h_1(u,p), ..., \\ f_l(u) + h_l(u,p) + u^T w_l - p^T \nabla_p h_l(u,p) \end{pmatrix}$

and $y \in R^m$;

Zhang [18] established higher order duality results under higher order invexity and generalized higher order invexity assumptions between (NDP) and (NDHMD) and (NDHD), respectively.

Mishra and Rueda [11] established higher order duality results between (NDP) and (NDHMD) and (NDHD) under higher order type I, higher order pseudo-type I and higher order quasi-type I functions.

In this paper, we formulate Mond-Weir type and general Mond-Weir type higher order duals to a nondifferentiable multiobjective programming problem in which every component of the objective function contains a term involving the support function of a compact convex set and establish higher order duality results under the higher order type I, higher order pseudo-type I and higher order quasi-type I assumptions.

2. Preliminaries

Throughout this paper, if x and $y \in R^n$, then by $x \leqq y$ we mean $x_i \leqq y_i$ for all i, $x \leq y$ means $x_i \leqq y_i$ for all i and $x_j < y_j$ for at least one j, $1 \leqq j \leqq n$. By $x < y$ we mean $x_i < y_i$ for all i and by $x \nleq y$ we mean the negation of $x \leq y$.

We consider the following nondifferentiable multiobjective programming problem:

$$\begin{array}{ll} (VP) & \text{Min } (f_1(x) + s(x : C_1), f_2(x) + s(x : C_2), ..., f_l(x) \\ & \qquad + s(x : C_l)) \\ & \text{subject to: } \; g(x) \geqq 0, x \in D, \end{array} \tag{2.1}$$

where f and g are twice differentiable functions from $f_i(x) - f_i(u) \leq h_i(u,p) - p^T \nabla_p h_i(u,p) \Rightarrow \eta(x,u)^T \nabla_p h_i(u,p) \leq 0, \; \forall\, i \in \{1, 2, ..., l\}$. to R^l and R^m, respectively; C_i for each $i \in L = \{1, 2, ..., l\}$ is a compact convex set of $f_i(x) - f_i(u) \leq h_i(u,p) - p^T \nabla_p h_i(u,p) \Rightarrow \eta(x,u)^T$

$\nabla_p h_i(u,p) \leq 0,\ \forall\ i \in \{1,2,...,l\}$. and D is an open subset of $f_i(x) - f_i(u) \leq h_i(u,p) - p^T \nabla_p h_i(u,p) \Rightarrow \eta(x,u)^T \nabla_p h_i(u,p) \leq 0,\ \forall\ i \in \{1,2,...,l\}$.

Let $f : R^n \to R^l$ be twice differentiable and $\eta : R^n \times R^n \to R^n$.

Definition 2.1 f is said to be *higher order invex* at u with respect to η and h if for all x,

$$f_i(x) - f_i(u) \geq \eta(x,u)^T \nabla_p h_i(u,p) + h_i(u,p) - p^T \nabla_p h_i(u,p), \quad \forall\ i \in \{1,2,...,l\}.$$

Definition 2.2 f is said to be *higher order pseudo invex* at u with respect to η and h if for all x,

$$\eta(x,u)^T \nabla_p h_i(u,p) \geq 0 \Rightarrow f_i(x) - f_i(u) - h_i(u,p) + p^T \nabla_p h_i(u,p) \geq 0, \quad \forall\ i \in \{1,2,...,l\}.$$

Definition 2.3 f is said to be *higher order quasi invex* at u with respect to η and h if for all x,

$$f_i(x) - f_i(u) \leq h_i(u,p) - p^T \nabla_p h_i(u,p) \Rightarrow \eta(x,u)^T \nabla_p h_i(u,p) \leq 0, \quad \forall\ i \in \{1,2,...,l\}.$$

3. Mond-Weir Type Duality

In relation to (VP) consider the problem

(NDVHD)

$$V - \text{Maximize} \begin{pmatrix} f_1(u) + h_1(u,p) + u^T w_1 - p^T \nabla_p h_1(u,p), ..., \\ f_l(u) + h_l(u,p) + u^T w_l - p^T \nabla_p h_l(u,p) \end{pmatrix}$$

subject to

$$\sum_{i=1}^{l} \lambda_i \left[\nabla_p h_i(u,p) + w_i\right] - \sum_{j=1}^{m} y_j \nabla_p k_j(u,p) = 0, \tag{1}$$

$$\sum_{j=1}^{m} y_j \left[g_j(u) + k_j(u,p) - p^T \nabla_p k_j(u,p)\right] \leqq 0, \tag{2}$$

$$y \geqq 0, \quad \lambda \in \Lambda = \left\{\lambda \in R^l : \lambda \geq 0, \sum_{i=1}^{l} \lambda_i = 1\right\}, \tag{3}$$

$$w_i \in C_i,\ i = 1, 2, ..., l,$$

where $h : R^n \times R^n \to R^l$ and $k : R^n \times R^n \to R^m$ are differentiable functions; $\nabla_p h_i(u,p)$ denotes the $n \times 1$ gradient of h_i with respect to p, and $\nabla_p \left(y^T k_i(u,p)\right)$ denotes the $n \times 1$ gradient of $y^T k_i$ with respect to p.

The problem (NDVHD) may be regarded as a multiple objective higher order nondifferentiable Mond-Weir type [14] vector dual to (VP).

Remark 3.1 If $h_i(u,p) = p^T \nabla f_i(u)$, $\forall i \in \{1, 2, ..., l\}$ and $k_j(u,p) = p^T \nabla g_j(u)$ $\forall\, j \in \{1, 2, ..., m\}$, (NDVHD) then becomes the Mond-Weir type vector dual [17] for (VP). If $h_i(u,p) = p^T \nabla f_i(u) + \frac{1}{2} p^T \nabla^2 f_i(u)\, p$, $\forall\, i \in \{1, 2, ..., l\}$ and $k_j(u,p) = p^T \nabla g_j(u) + \frac{1}{2} p^T \nabla^2 g_j(u)\, p$, $\forall j \in \{1, 2, ..., m\}$, then (NDVHD) becomes the second order nondiffereniabl version of Mond-Weir type vector dual [19] for (VP).

Theorem 3.1. (Weak Duality) *Let x be feasible for (VP), $(u, \lambda, w,\ y, p)$ feasible for (NDVHD) and $\lambda > 0$. Assume that $f_i(\cdot) + \cdot^T w_i$ is higher order $\eta-$invex with respect to $h_i(\cdot, p)$ and $-g_j(\cdot)$ is higher order η-invex with respect to $-k_j(\cdot, p)$, then the following cannot hold:*

$$f_i(x) + s(x : C_i) \leqq f_i(u) + u^T w_i + h_i(u,p) - p^T \nabla_p h_i(u,p)\ , \quad \forall i \in \{1, 2, ..., l\}$$

$$f_{i_0}(x) + s(x : C_{i_0}) \le f_{i_0}(u) + u^T w_{i_0} + h_{i_0}(u,p) - p^T \nabla_p h_{i_0}(u,p) \quad \textit{for } i_0 \in \{1, 2, ..., l\}\,.$$

Proof. Since x is feasible for (VP) and (u, λ, w, y, p) is feasible for (NDVHD), by higher order η-invexity of $f_i(\cdot) + \cdot^T w_i$ with respect to $h_i(\cdot, p)$, we get

$$\begin{aligned} & f_i(x) + x^T w_i - f_i(u) - u^T w_i - h_i(u,p) + p^T \nabla_p h_i(u,p) \\ & \quad \geq \eta(x,u)^T \left[\nabla_p h_i(u,p) + w_i\right]\ ,\ \forall i \in \{1, 2, ..., l\}\,. \end{aligned}$$

Since $\lambda \in \Lambda$, we get

$$\begin{aligned} & \sum_{i=1}^{l} \lambda_i \left[f_i(x) + x^T w_i - f_i(u) - u^T w_i - h_i(u,p) + p^T \nabla_p h_i(u,p)\right] \\ & \geq \eta(x,u)^T \sum_{i=1}^{l} \lambda_i \left[\nabla_p h_i(u,p) + w_i\right] \\ & = \eta(x,u)^T \sum_{j=1}^{m} y_j \left[\nabla_p k_j(u,p)\right] \\ & \geqq \sum_{j=1}^{m} y_j \left[g_j(x) - g_j(u) - k_j(u,p) + p^T \nabla_p k_j(u,p)\right] \\ & \geqq 0, \qquad (\text{since } \ g_j(x) \ \geqq 0,\ y_j \geq 0), \end{aligned}$$

that is,

$$\sum_{i=1}^{l} \lambda_i \left[f_i(x) + x^T w_i - f_i(u) - u^T w_i - h_i(u,p) + p^T \nabla_p h_i(u,p) \right] \geqq 0.$$

Since $x^T w_i \leq s(x : C_i)$, $\forall i = 1, 2, ..., l,$ we get

$$\sum_{i=1}^{l} \lambda_i \left[f_i(x) + s(x : C_i) \right] \geqq \sum_{i=1}^{l} \lambda_i \left[f_i(u) + u^T w_i + h_i(u,p) - p^T \nabla_p h_i(u,p) \right].$$

Therefore, the following implications are not true:

$$f_i(x) + s(x : C_i) \leqq f_i(u) + u^T w_i + h_i(u,p) - p^T \nabla_p h_i(u,p) \quad \forall i \in \{1, 2, ..., l\},$$

$$f_{i_0}(x) + s(x : C_{i_0}) < f_{i_0}(u) + u^T w_{i_0} + h_{i_0}(u,p) - p^T \nabla_p h_{i_0}(u,p) \quad \text{for } i_0 \in \{1, 2, ..., l\}. \quad \square$$

Theorem 3.2. (Weak Duality) *Let x be feasible for (VP), (u, λ, w, y, p) feasible for (NDVHD). Assume that $f_i(\cdot) + \cdot^T w_i$ is higher order η-pseudo-invex with respect to $h_i(\cdot, p)$ for every $i = 1, 2, ..., l$ and $-g_j(\cdot)$ is higher order η-quasi-invex with respect to $-k_j(\cdot, p)$ for every $j = 1, 2, ..., m$, then the following cannot hold:*

$$f_i(x) + s(x : C_i) \leqq f_i(u) + u^T w_i + h_i(u,p) - p^T \nabla_p h_i(u,p), \quad \forall i \in \{1, 2, ..., l\}$$

$$f_{i_0}(x) + s(x : C_{i_0}) < f_{i_0}(u) + u^T w_{i_0} + h_{i_0}(u,p) - p^T \nabla_p h_{i_0}(u,p) \quad \text{for } i_0 \in \{1, 2, ..., l\}.$$

Proof. Since x is feasible for (VP) and (u, λ, w, y, p) is feasible for (NDVHD), by (2.1) and (2), we get

$$\sum_{j=1}^{m} y_j \left[g_j(x) - g_j(u) - k_j(u,p) + p^T \nabla_p k_j(u,p) \right] \geqq 0.$$

By higher order η-quasi-invex of $-g_j(\cdot)$ with respect to $-k_j(\cdot,p)$ for every $j=1,2,...,m$, and the above inequality, we get

$$\eta(x,u)^T \sum_{j=1}^{m} y_j \left[\nabla_p k_j(u,p)\right] \geqq 0. \tag{4}$$

From (4) and (1), we get

$$\eta(x,u)^T \sum_{i=1}^{l} \lambda_i \left[\nabla_p h_i(u,p) + w_i\right] \geqq 0. \tag{5}$$

From higher order η-pseudo-invex of $f_i(\cdot) + \cdot^T w_i$ with respect to $h_i(\cdot,p)$ for every $i=1,2,...,l$and (5), we have

$$\sum_{i=1}^{l} \lambda_i \left[f_i(x) + x^T w_i - f_i(u) - u^T w_i - h_i(u,p) + p^T \nabla_p h_i(u,p)\right] \geqq 0. \tag{6}$$

Since $x^T w_i \leq s(x:C_i)$, $\forall i=1,2,...,l$, from (6), we get

$$\sum_{i=1}^{l} \lambda_i \left[f_i(x) + s(x:C_i)\right]$$
$$\geqq \sum_{i=1}^{l} \lambda_i \left[f_i(u) + u^T w_i + h_i(u,p) - p^T \nabla_p h_i(u,p)\right],$$

Therefore, the following implications are not true:

$$f_i(x) + s(x:C_i) \leqq f_i(u) + u^T w_i + h_i(u,p) - p^T \nabla_p h_i(u,p), \quad \forall i \in \{1,2,...,l\}$$

$$f_{i_0}(x) + s(x:C_{i_0}) < f_{i_0}(u) + u^T w_{i_0} + h_{i_0}(u,p) - p^T \nabla_p h_{i_0}(u,p) \quad \text{for } i_0 \in \{1,2,...,l\}. \quad \square$$

Theorem 3.3. (Strong Duality) *Let x_0 be an efficient solution for (VP) at which a Kuhn-Tucker constraint qualification is satisfied and let*

$$h(x_0,0)=0,\ k(x_0,0)=0,\ \nabla_p h(x_0,0) = \nabla f(x_0) \ \textit{and} \ \nabla_p k(x_0,0) = \nabla g(x_0).$$

Then there exist $y \in R^m$ and $\lambda \in R^l$ such that $(x_0, \lambda, p = 0)$ is feasible for (VP) and $(x_0, y, w, \lambda, p = 0)$ is feasible for (NDVHD) and the corresponding values of (VP) and (NDVHD) are equal. If for all feasible (x_0, y, w, λ, p) the assumptions of Theorem 3.1 or Theorem 3.2 are satisfied, then $(x_0, y, w, \lambda, p = 0)$ is efficient for (NDVHD).

Proof. Since x_0 is an efficient solution and hence also a weak minimum for (VP) at which a Kuhn-Tucker constraint qualification is satisfied, then by Theorem 3.1 or 3.2, $(x_0, y, w, \lambda, p = 0)$ must be an efficient solution for (NDVHD). □

4. General Mond-Weir Type Higher Order Nondifferentiable Duality

In this section, we consider the following general higher order nondifferentiable dual to (VP):

(NDHGVD)

$$V\text{-Maximize}\begin{pmatrix} f_1(u) + h_1(u,p) + u^T w_1 - \sum_{i\in I_0} y_i g_i(u) - \sum_{i\in I_0} y_i k_i(u,p) \\ -p^T\left[\nabla_p h_1(u,p) - \nabla_p\left(\sum_{i\in I_0} y_i k_i(u,p)\right)\right], ..., \\ f_l(u) + h_l(u,p) + u^T w_l - \sum_{i\in I_0} y_i g_i(u) - \sum_{i\in I_0} y_i k_i(u,p) \\ -p^T\left[\nabla_p h_l(u,p) - \nabla_p\left(\sum_{i\in I_0} y_i k_i(u,p)\right)\right] \end{pmatrix},$$

subject to:

$$\sum_{i=1}^{l} \lambda_i\left[\nabla_p h_i(u,p) + w_i\right] - \sum_{j=1}^{m} y_j \nabla_p k_j(u,p) = 0\,,$$

$$\sum_{j\in I_\alpha} y_j\left[g_j(u) + k_j(u,p) - p^T\nabla_p k_j(u,p)\right] \leqq 0, \alpha = 1, 2, ..., r\,,$$

$$y \geqq 0, \quad \lambda \in \Lambda = \left\{\lambda \in R^l : \lambda \geq 0, \sum_{i=1}^{l}\lambda_i = 1\right\}, \quad w_i \in C_i,\; i = 1, 2, ..., l,$$

where $I_\alpha \subseteq M = \{1, 2, ..., m\}\,, \quad \alpha = 0, 1, 2, ..., r$ with $I_\alpha \cap I_\beta = \phi$ if $\alpha \neq \beta$ and $\bigcup_{\alpha=0}^{r} I_\alpha = M$.

Remark 4.1 If $h_i(u,p) = p^T\nabla f_i(u)$, $\forall\, i \in \{1, 2, ..., l\}$ and $k_j(u,p) = p^T\nabla g_j(u) \quad \forall\, j \in \{1, 2, ..., m\}$, (NDHGVD) then becomes the Mond-Weir type vector dual [17] for (VP). If $h_i(u,p) = p^T\nabla f_i(u) + \frac{1}{2}p^T\nabla^2 f_i(u)$

$p, \quad \forall\ i \in \{1, 2, ..., l\}$ and $k_j(u, p) = p^T \nabla g_j(u) + \frac{1}{2} p^T \nabla^2 g_j(u) p, \forall\ j \in \{1, 2, ..., m\}$, then (NDHGVD) becomes the second order nondiffereniabl version of Mond-Weir type vector dual [19] for (VP).

The following duality theorems can be proved on the similar lines to that of the proofs of Theorem 3.2 and Theorem 3.3, respectively therefore we omit the proofs of following theorems.

Theorem 4.1. (Weak Duality) *Let x be feasible for (VP), (u, λ, w, y, p) feasible for (NDHGVD) and $\lambda > 0$. Assume that $f_i(\cdot) + \cdot^T w_i - \sum_{i \in I_0} y_i g_i(\cdot)$ is higher order η-pseudo-invex with respect to $h_i(\cdot, p)$ and $-\sum_{j \in I_\alpha} y_j g_j(\cdot)$, $\alpha = 1, 2, ..., r$ is higher order η-quasi-invex with respect to $-k_j(\cdot, p)$,for $j \in I_{\alpha,}$ $\alpha = 1, 2, ..., r$. Then the following cannot hold:*

$$f_i(x) + s(x : C_i) \leqq f_i(u) + u^T w_i - \sum_{i \in I_0} y_i g_i(u) + h_i(u, p) - \sum_{i \in I_0} y_i k_i(u, p) - p^T \left(\nabla_p h_i(u, p) - \nabla_p \sum_{i \in I_0} y_i k_i(u, p) \right) \quad \forall i \in \{1, 2, ..., l\}$$

and

$$f_{i_0}(x) + s(x : C_{i_0}) < f_{i_0}(u) + u^T w_{i_0} - \sum_{i \in I_0} y_{i_0} g_{i_0}(u) + h_{i_0}(u, p) - \sum_{i \in I_0} y_{i_0} k_{i_0}(u, p) - p^T \left(\nabla_p h_{i_0}(u, p) - \nabla_p \sum_{i \in I_0} y_{i_0} k_{i_0}(u, p) \right) \quad \text{for } i_0 \in \{1, 2, ..., l\}.$$

Theorem 4.3. (Strong Duality) *Let x_0 be an efficient solution for (VP) at which a Kuhn-Tucker constraint qualification is satisfied and let*

$$h(x_0, 0) = 0,\ k(x_0, 0) = 0,\ \nabla_p h(x_0, 0) = \nabla f(x_0) \quad \text{and } \nabla_p k(x_0, 0) = \nabla g(x_0).$$

Then there exist $y \in R^m$ and $\lambda \in R^l$ such that $(x_0, \lambda, p = 0)$ is feasible for (VP) and $(x_0, y, w, \lambda, p = 0)$ is feasible for (NDHGVD) and the corresponding values of (VP) and (NDHGVD) are equal. If for all feasible (x_0, y, w, λ, p) the *assumptions of Theorem 4.2 are satisfied, then* $(x_0, y, w, \lambda, p = 0)$ *is efficient for (NDHGVD).*

5. Acknowledgements

The research was supported by the University Grants Commission of India, the National Natural Science Foundation of P.R. China and the Research Grants Council of Hong Kong.

References

[1] G. Bitran, Duality in nonlinear multiple criteria optimization problems, *Journal of Optimization Theory and Applications*, **35** (1981), 367-406.

[2] B.D. Craven, Strong vector minimization and duality, *Zeitschrift für Angewandte Mathematik und Mechanik*, **60** (1980), 1-5.

[3] R.R. Egudo, Efficiency and generalized convex duality for multiobjective programs, *Journal of Mathematical Analysis and Applications*, **138** (1989), 84-94.

[4] R.R. Egudo, M.A. Hanson, Second order duality inmultiobjective programming, *Opsearch*, **30** (1993), 223-230.

[5] T.R. Gulati, M.A. Islam, Sufficiency and duality in multiobjective programming involving generalized F-convex functions, *Journal of Mathematical Analysis and Applications*, **183** (1994), 181-195.

[6] E.H. Ivanov, R. Nehse, Some results on dual vector optimization problems, *Optimization*, **16** (1985), 505-517.

[7] V. Jeyakumar, B. Mond, On generalized convex mathematical programming, *Journal of Australian Mathematical Society Ser. B*, **34** (1992), 43-53.

[8] O.L. Mangasarian, Second and higher order duality in nonlinear programming, *Journal of Mathematical Analysis and Applications*, **51** (1975), 607-620.

[9] V. Preda, On efficiency and duality for multiobjective programs, *Journal of Mathematical Analysis and Applications*, **166** (1992), 365-377.

[10] S.K. Mishra, Second order generalized invexity and duality in mathematical programming, *Optimization*, **42** (1997), 51-69.

[11] S.K. Mishra, N.G. Rueda, Higher order generalized invexity and duality in nondifferentiable mathematical programming, *Journal of Mathematical Analysis and Applications*, **272** (2002), 496-506.

[12] B. Mond, A class of nondifferentiable mathematical programming problems, *Journal of Mathematical Analysis and Applications*, **46** (1974), 169-174.

[13] B. Mond, M. Schechter, Nondifferentiable symmetric duality, *Bulletin of Australian Mathematical Society*, **53** (1996), 177- 187.

[14] B. Mond, T. Weir, Generalized convexity and higher order duality, *Journal of Mathematical Sciences*, 16-18 (1981-1983), 74-94.

[15] B. Mond, J. Zhang, Duality for multiobjective programming involving second order V-invex functions, In: *Proceedings of Optimization Miniconference* (Ed-s: B.M. Glover, V. Jeyakumar), Universty of New South Wales, Sydney, Australia (1995), 89-100.

[16] T. Tanino, Y. Sawaragi, Duality theory in multiobjective programming, *Journal of Optimization Theory and Applications*, **27** (1979), 509-529.

[17] X.M. Yang, K.L. Teo, X.Q. Yang, Duality for a class of nondifferentiable multiobjective programming problems, *Journal of Mathematical Analysis and Applications*, **252** (2000), 999-1005.

[18] J. Zhang, Higher order convexity and duality in multiobjective programming problems, In: *Progress in Optimization, Contributions from Australasia*, (Ed-s: A. Eberhard, R. Hill, D. Ralph, B. M. Glover), Kluwer Academic Publishers Dordrecht-Boston-London, Applied Optimization, Volume **30** (1999), 101-116.

[19] J. Zhang, B. Mond, Second order duality for multiobjective non-linear programming involving generalized convexity, In: *Proceedings of the Optimization Miniconference III* (Ed-s: B.M. Glover, B.D. Craven, D. Ralph), The University of Melbourne, July 18 1996, University of Ballarat, Ballarat (1997), 79-95.